Convex Optimization-based Trajectory Planning and Control

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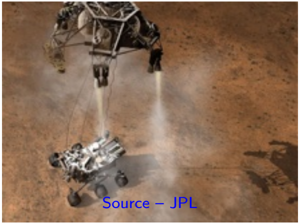
NASA – 05/2024

Autonomous Controls Lab

http://depts.washington.edu/uwacl/

Convex optimization based control for space vehicles



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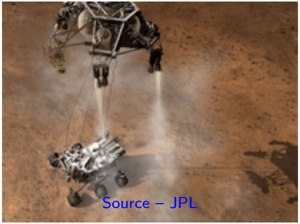

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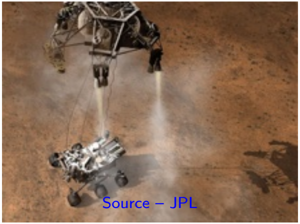

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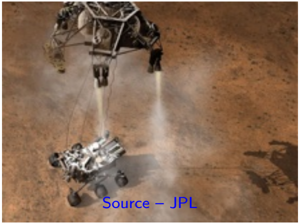
Source – JPL

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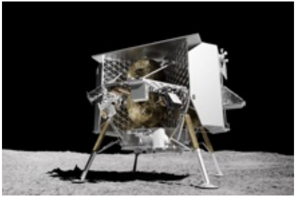
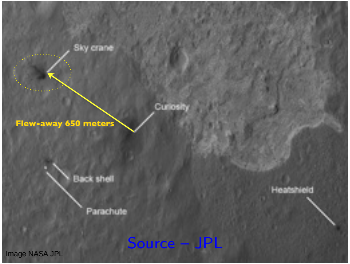
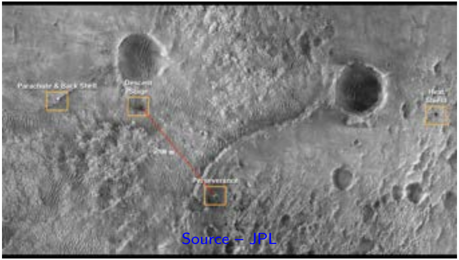


Upcoming Lunar Missions
https://www.astrobotic.com/lunar-delivery/landers

Perseverance Landing

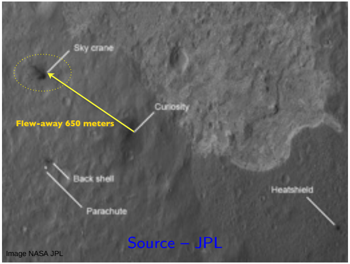


MSL Flyaway



Mars 2020 Flyaway

Perseverance Landing



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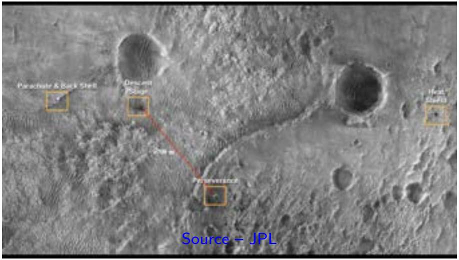

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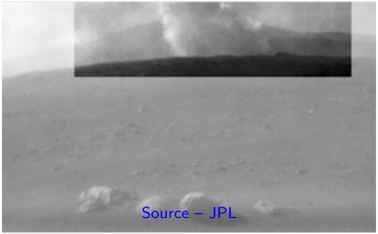
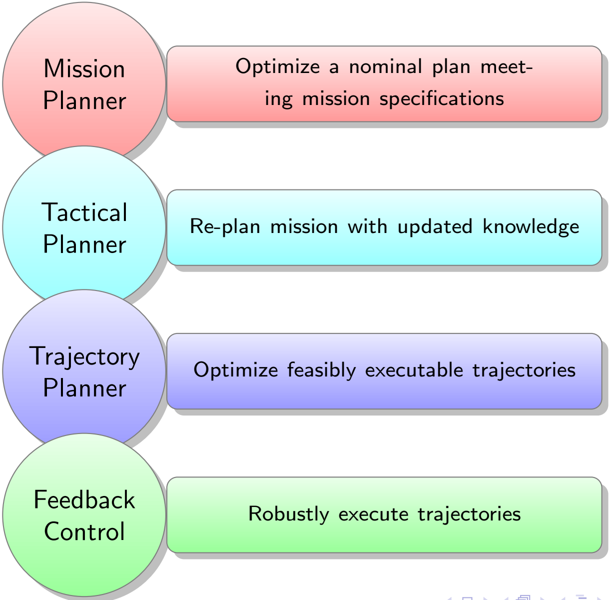
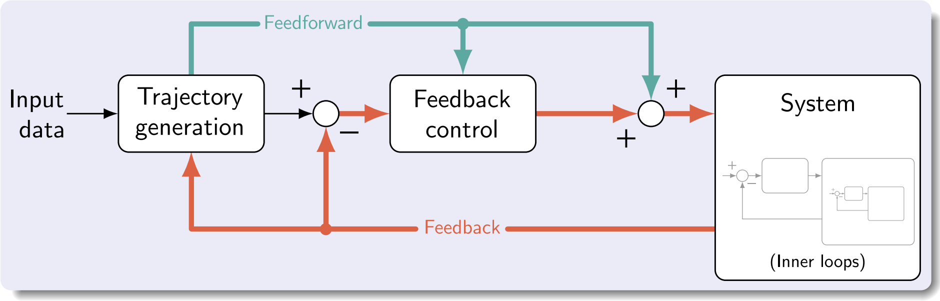


Mars 2020 crash site from Perseverance

A Hierarchy for Autonomous Control



Interplay of Trajectory Planning and Control



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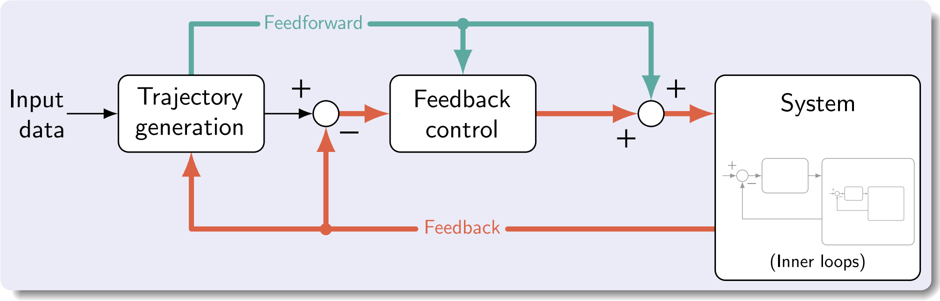
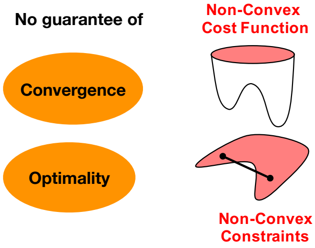


- Hierarchical, *decoupled*, synthesis
  - Optimality via optimizing the trajectory
  - Robustness via synthesizing feedback policy
- Joint, *coupled*, synthesis
  - Joint robust optimality via optimizing over all possible trajectories

Convexity Assures Optimal Solutions



Non-convex optimization

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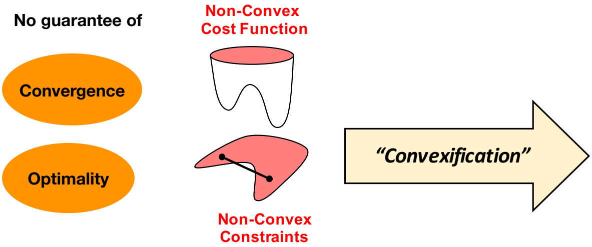

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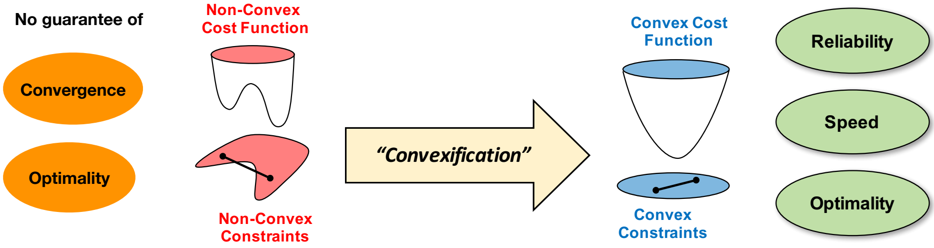

Non-convex optimization

Convex Optimization

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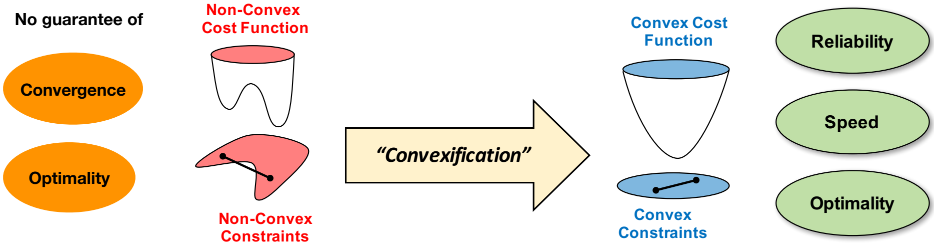

Non-convex optimization

Convex Optimization

Convexification = Conversion to convex problems

Convexification of Trajectory Planning Problems

1. Control problem

Convexification of Trajectory Planning Problems

1. Control problem
   - Nonlinear Dynamics
   - Non-convex State and Control constraints
   - Non-convex Costs/rewards

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   - Equivalent convex formulation for non-convex control constraints

     Lossless convexification – **LCvx** – for global optimality

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2022 CSM tutorial – Convex Optimization for Trajectory Generation

github.com/dmalyuta/scp_traj_opt/tree/csm

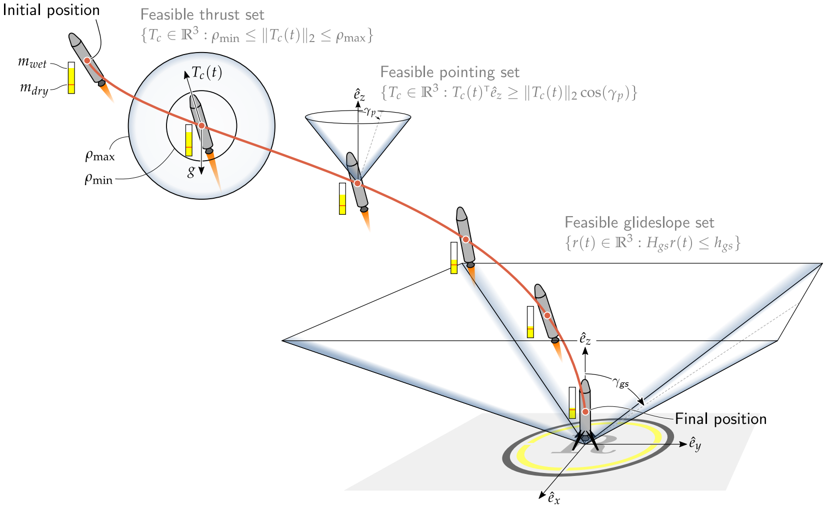
Malyuta* et al. [2022]

Lossless Convexification (LCvx)

Research question

Can we formulate equivalent convex relaxations/liftings for non-convex constraints?

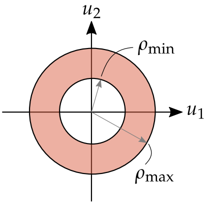
Planetary Softlanding

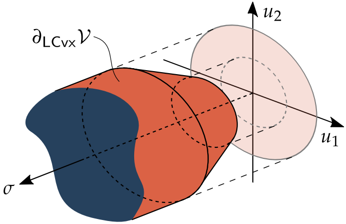


LCvx of Control Lower Bound

- Original *nonconvex* controls

$$\rho_{\min} \le \|u\|_2 \le \rho_{\max}$$





- Relaxed convex controls

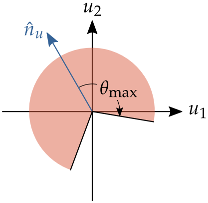
$$\mathcal{V} := \{(u,\sigma) \in \mathbb{R}^3 : \rho_{\min} \le \sigma, \|u\|_2 \le \min(\sigma, \rho_{\max})\}$$

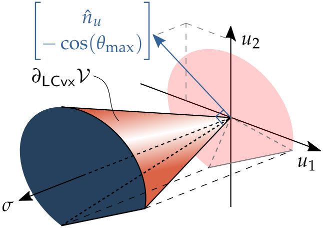
$$\partial_{\mathrm{LCvx}}\mathcal{V} := \{(u,\sigma) \in \mathbb{R}^3 : \rho_{\min} \le \sigma, \|u\|_2 = \min(\sigma, \rho_{\max})\}$$

LCvx of Control Pointing

- Original *non-convex* controls

$$\hat{n}_u^T u \geq \|u\|_2 \cos(\theta_{\max}),$$





- Relaxed convex controls

$$\mathcal{V} := \{(u, \sigma) \in \mathbb{R}^3 : \hat{n}_u^T u \geq \sigma \cos(\theta_{\max}),\ \|u\|_2 \leq \sigma\}$$

$$\partial_{\text{LCvx}}\mathcal{V} := \{(u, \sigma) \in \mathbb{R}^3 : \hat{n}_u^T u \geq \sigma \cos(\theta_{\max}),\ \|u\|_2 = \sigma\}$$

Example Theoretical Result

Original Problem

$$\min_{u,t_f} m(t_f, x(t_f)) + \zeta \int_0^{t_f} \ell(g(u(t)))\, dt$$

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + E(t)w(t) \quad \text{s.t.}$$

$$\rho_{\min} \le g(u(t)) \le \rho_{\max}$$

$$\hat{n}_u^T u(t) \ge \hat{n}_g g(u(t))$$

$$x(0) = x_0, \quad b(t_f, x(t_f)) = 0$$

Relaxed Problem

$$\min_{u,t_f,\sigma} m(t_f, x(t_f)) + \zeta \int_0^{t_f} \ell(\sigma(t))\, dt$$

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + E(t)w(t) \quad \text{s.t.}$$

$$\rho_{\min} \le \sigma(t) \le \rho_{\max}$$

$$g(u(t)) \le \sigma(t)$$

$$\hat{n}_u^T u(t) \ge \sigma(t)\hat{n}_g$$

$$x(0) = x_0, \quad b(t_f, x(t_f)) = 0$$

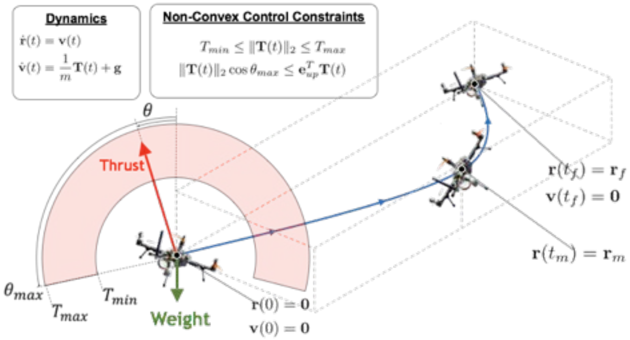
Theorem

Solutions of the relaxed problem define globally optimal solutions of the original problem if:

- $\{A(\cdot), B(\cdot)N\}$ pair is totally controllable, $\ker \hat{n}_u^T = \mathrm{im} N \in \mathbb{R}^{m\times(m-1)}$
- The following vector and the columns of the matrix are linearly independent

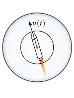
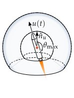
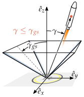
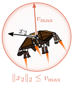
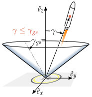
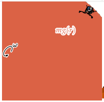
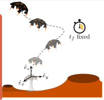
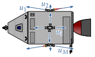
$$\begin{bmatrix} \nabla_x m[t_f] \\ \nabla_t m[t_f] + \zeta\ell(\sigma(t_f)) \end{bmatrix} \in \mathbb{R}^{n+1}, \quad \begin{bmatrix} \nabla_x b[t_f]^T \\ \nabla_t b[t_f]^T \end{bmatrix} \in \mathbb{R}^{(n+1)\times n_b}$$

LCvx Example: Non-Convex Control Constraints in Quad-rotors



Non-convex thrust constraints – LCvx applies as in **rocket landing**

LCvx - Summary of Results

	No State Constraints	Input Pointing Constraint	Affine State Constraints	Quadratic State Constraints	General Convex State Constraints	Nonlinear Dynamics	Fixed-final Time Problems	Hybrid System Problems
Terminal cost	$m(t_f, x(t_f))$	$m(t_f, x(t_f))$	$m(x(t_f))$	$m(x(t_f))$	$m(x(t_f))$	$m(t_f, x(t_f))$		
Running cost	$\zeta\ell(g_1(u(t)))$	$\ell(g(u(t)))$	$\zeta\ell(g_1(u(t)))$	$\zeta\ell(g_1(u(t)))$	$\zeta\ell(g_1(u(t)))$	$\zeta\ell(g(u(t)))$	$\ell(g(u(t)))$	$\sum_{i=1}^{M}\|u_i(t)\|_2$
Dynamics $\dot{x}(t) = \cdots$	$A(t)x(t) + B(t)u(t)$ $+ E(t)w(t)$	$A(t)x(t) + B(t)u(t)$ $+ E(t)w(t)$	$Ax(t) + Bu(t) + Ew$	$\dot{x}_1(t) = Ax_2(t)$ $\dot{x}_2(t) = Bu(t) + w$	$Ax(t) + Bu(t) + Ew$	$f(t, x(t), u(t),$ $g(u(t)))$	$Ax(t) + Bu(t)$	$Ax(t) + B\sum_{i=1}^{M} u_i(t)$
Input Norm Bounds	$g_1(u(t)) \geq \rho_{\min}$ $g_0(u(t)) \leq \rho_{\max}$	$g(u(t)) \geq \rho_{\min}$ $g(u(t)) \leq \rho_{\max}$	$g_1(u(t)) \geq \rho_{\min}$ $g_0(u(t)) \leq \rho_{\max}$	$g_1(u(t)) \geq \rho_{\min}$ $g_0(u(t)) \leq \rho_{\max}$	$g_1(u(t)) \geq \rho_{\min}$ $g_0(u(t)) \leq \rho_{\max}$	$g(u(t)) \geq \rho_{\min}$ $g(u(t)) \leq \rho_{\max}$	$g(u(t)) \geq \rho_{\min}$ $g(u(t)) \leq \rho_{\max}$	$\|u_i(t)\|_2 \geq \gamma_i(t)\rho_{\min,i}$ $\|u_i(t)\|_2 \leq \gamma_i(t)\rho_{\max,i}$ $\gamma_i(t) \in \{0,1\}$
Other Input Constraints		$\hat{n}_u^{\mathsf{T}} u(t) \geq \hat{n}_g g(u(t))$	$Cu(t) \leq c$					$C_i u_i(t) \leq 0$
Other State Constraints			$Hx(t) \leq h$	$x_2(t)^{\mathsf{T}} H x_2(t) \leq 1$	$x(t) \in \mathcal{X}$			
Boundary Conditions	$x(0) = x_0$ $b(t_f, x(t_f)) = 0$	$x(0) = x_0$ $b(t_f, x(t_f)) = 0$	$x(0) = x_0$ $b(x(t_f)) = 0$	$x_1(0) = x_{1,0}$ $x_2(0) = x_{2,0}$ $b(x(t_f)) = 0$	$x(0) = x_0$ $b(x(t_f)) = 0$	$x(0) = x_0$ $b(t_f, x(t_f)) = 0$	$x(0) = x_0$ $x(t_f) = x_f$	$x(0) = x_0$ $b(x(t_f)) = x_f$
Illustration*								

* The illustrations show a unique feature of the problem in the corresponding column. They do not show all of the constraints.

LCvx Enabling Large Diverts for Rockets

3 year test flight campaign (2012-2014) demonstrating LCvx for large diverts

- Flight campaign-1: *Offline* trajectories and GPS to assess feasibility
- Flight campaign-2: *Real-time onboard* trajectories and GPS
- Flight campaign-3: *Real-time onboard* trajectories and TRN

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Test Flight - G-FOLD(Guidance for Fuel Optimal Large Divert) 2013

3-D 800m divert video Video

JPL: G-FOLD testing public release

http://www.jpl.nasa.gov/video/details.php?id=1270

SpaceX Reusable Rocket Landings

- SpaceX has been using convex optimization-based control
- LCvx development is acknowledged to be a key idea leveraged in SpaceX reusable rockets

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Blackmore [2016] NAE Bridge Winter 2016

" The computation must be done autonomously, in a fraction of a second. Failure to find a feasible solution in time will crash the spacecraft into the ground. Failure to find the optimal solution may use up the available propellant, with the same result. Finally, a hardware failure may require replanning the trajectory multiple times. A general solution to such problems has existed in one dimension since the 1960s (Meditch 1964), but not in three dimensions. Over the past decade, research has shown how to use modern mathematical optimization techniques to solve this problem for a Mars landing, with guarantees that the best solution can be found in time (Açıkmeşe and Ploen 2007, Blackmore and Açıkmeşe 2010)"

LCvx in Other Applications

LCvx for Lunar Landings

New companies are baselining LCvx as their lunar landing guidance algorithm

- **Blue Origin** utilized LCvx formulations and theoretical results in **Açıkmeşe, Behçet** and Ploen† [2005, 2007], **Açıkmeşe, Behçet** et al. [2013] while incorporating several new features into their lunar landing guidance algorithm Berning et al. [2023].

Sequential Convex Programming (SCP)

Research question

Can we solve non-convex trajectory planning problems via solving a recursively generated sequence convex trajectory problems?

A Non-Convex Trajectory Planning Problem

$$\min_{u,p} J(x,u,p) \quad \text{s.t.}$$ Cost functional

$$\dot{x}(t) = f\big(t, x(t), u(t), p\big)$$ Nonlinear dynamics

$$(x(t), p) \in \mathcal{X}(t)$$ Convex state constraints

$$(u(t), p) \in \mathcal{U}(t)$$ Convex control constraints

$$s(t, x(t), u(t), p) \leq 0$$ Non-convex state & control constraints

$$g_{ic}(x(t_o), p) = 0$$ Initial boundary conditions

$$g_{tc}(x(t_f), p) = 0$$ Terminal boundary conditions

where we may have a Bolza type cost function

$$J(x,u,p) = \phi(x(t_f), p) + \int_{t_o}^{t_f} \Gamma(x(t), u(t), p)dt$$

Example: State-triggered Line-of-Sight (LoS) Constraint

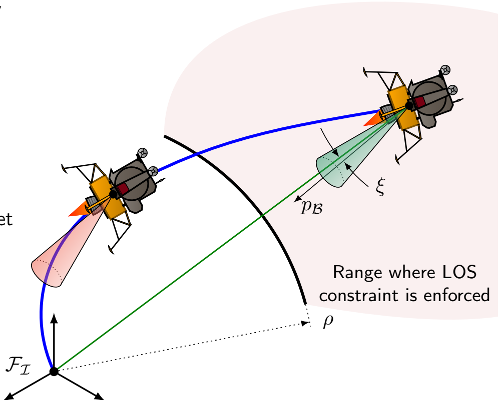
- Reconciles vehicle configuration with feasbility of optimal control problem

- Trigger: slant range larger than ρ

$$g(\tilde{q}) = \rho - \|2E_d\tilde{q}\|_2$$

- Constraint: line of sight angle to landing target

$$h(\tilde{q}) = \tilde{q}^\top M_\xi \tilde{q} + \|2E_d\tilde{q}\|_2 \cos\xi_{\max} - \varepsilon$$



Example: State-triggered Approach Angle Constraint

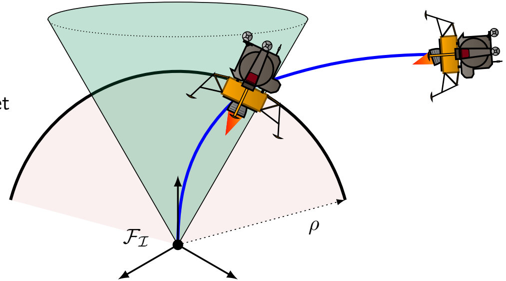
- Lunar descent orbits characteristically low altitude at large downrange distances

- Trigger: slant range small than ρ

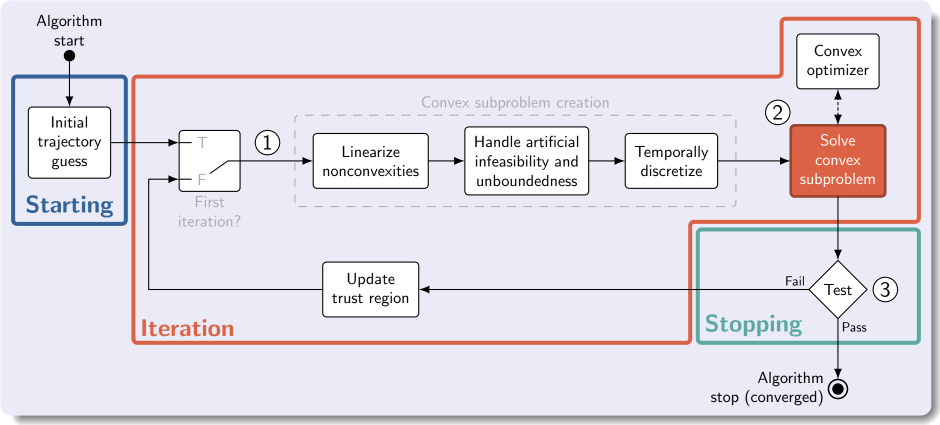
$$g(\tilde{q}) = \|2E_d\tilde{q}\|_2 - \rho$$

- Constraint: approach angle to landing target

$$h(\tilde{q}) = -\tilde{q}^\top M_\gamma \tilde{q} + \|2E_d\tilde{q}\|_2 \cos\gamma_{\max}$$



Successive Convexification (SCvx) - An SCP Algorithm



SCvx - Linearization for Convexification

$$
\begin{aligned}
\min_{u,p} J(x,u,p) \quad \text{s.t.} \\
\dot{x}(t) = A(t)x(t) + B(t)u(t) + F(t)p + r(t) \\
(x(t),p) \in \mathcal{X}(t) \\
(u(t),p) \in \mathcal{U}(t) \\
C(t)x(t) + D(t)u(t) + G(t)p + r(t)' \leq 0 \\
H_0 x(0) + K_0 p + \ell_0 = 0 \\
H_f x(1) + K_f p + \ell_f = 0
\end{aligned}
$$

SCvx - Linearization for Convexification

$$
\begin{aligned}
\min_{u,p} J(x,u,p) \quad \text{s.t.} \\
\dot{x}(t) = A(t)x(t) + B(t)u(t) + F(t)p + r(t) \\
(x(t),p) \in \mathcal{X}(t) \\
(u(t),p) \in \mathcal{U}(t) \\
C(t)x(t) + D(t)u(t) + G(t)p + r(t)' \leq 0 \\
H_0 x(0) + K_0 p + \ell_0 = 0 \\
H_f x(1) + K_f p + \ell_f = 0
\end{aligned}
$$

The above OCP is convex BUT it may suffer from

- Artificial unboundedness
- Artificial infeasibility

SCvx- Virtual Control for Artificial Infeasibility

Buffered linearized constraints ensuring feasibility

$$
\begin{aligned}
\dot{x} &= Ax + Bu + Fp + r + E\nu, \\
\nu_s &\geq Cx + Du + Gp + r', \\
0 &= H_0 x(t_o) + K_0 p + \ell_0 + \nu_{ic} \\
0 &= H_f x(t_f) + K_f p + \ell_f + \nu_{tc}
\end{aligned}
$$

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0 &= H_f x(t_f) + K_f p + \ell_f + \nu_{tc}
\end{aligned}
$$

Augmented cost penalizing virtual controls

$$
J_\lambda(x,u,p,\nu) := \phi_\lambda(x(t_f),p,\nu_{ic},\nu_{tc}) + \int_{t_o}^{t_f} \Gamma_\lambda(x,u,p,E\nu,\nu_s)dt,
$$

$$
\begin{aligned}
\phi_\lambda(x(t_f),p,\nu_{ic},\nu_{tc}) &= \phi(x(t_f),p) + \lambda P(0,\nu_{ic}) + \lambda P(0,\nu_{tc}) \\
\Gamma_\lambda(x,u,p,E\nu,\nu_s) &= \Gamma(x,u,p) + \lambda P\left(E\nu,\nu_s\right) \\
P(y,z) &:= \|y\|_1 + \|z\|_1
\end{aligned}
$$

SCvx - Convexification Accuracy

$$\rho := \frac{\mathcal{J}_\lambda(\bar{x}, \bar{u}, \bar{p}) - \mathcal{J}_\lambda(x^*, u^*, p^*)}{\mathcal{J}_\lambda(\bar{x}, \bar{u}, \bar{p}) - L_\lambda(x^*, u^*, p^*, \hat{\nu}^*)} = \frac{\text{actual improvement}}{\text{predicted improvement}}$$

where

$$\mathcal{J}_\lambda(x, u, p) := \phi_\lambda\left(x(t_f), p, g_{ic}(x(t_0), p), g_{tc}(x(t_f), p)\right) + trapz(\Gamma_\lambda^N)$$
$$\Gamma_{\lambda,k}^N = \Gamma_\lambda(x_k, u_k, p, \delta_k, [s(t_k, x_k, u_k, p)]^+)$$

$$L_\lambda(x, u, p, \hat{\nu}) := \phi_\lambda\left(x(t_f), p, \nu_{ic}, \nu_{tc}\right) + trapz(\Gamma_\lambda^N)$$
$$\Gamma_{\lambda,k}^N = \Gamma_\lambda(x_k, u_k, p, E\nu_k, \nu_{s,k})$$

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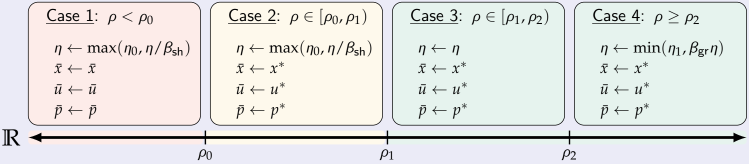
$$\Gamma_{\lambda,k}^N = \Gamma_\lambda(x_k, u_k, p, E\nu_k, \nu_{s,k})$$

- $\mathcal{J}_\lambda(\bar{x}, \bar{u}, \bar{p}) \geq L_\lambda(x^*, u^*, p^*, \hat{\nu}^*)$ always holds
- If $\mathcal{J}_\lambda(\bar{x}, \bar{u}, \bar{p}) = L_\lambda(x^*, u^*, p^*, \hat{\nu}^*)$ then $(\bar{x}, \bar{u}, \bar{p})$ is locally optimal solution – STOP SCvx!

SCvx - Thrust Region Update Rule



β_{sh}, $\beta_{gr} > 1$ user chosen constants

$0 < \eta_0 < \eta_1$ are min and max thrust region radii

$0 < \rho_0 < \rho_1 < \rho_2 \leq 1$ are user chosen

Thrust Region (TR) updates

- Case 1 – inaccurate model: Reject the solution, shrink TR and re-compute (does not continue indefinitely!)
- Case 2 – marginally accurate model: Accept the solution, shrink TR
- Case 3 – accurate model: Accept the solution, do not change the TR
- Case 4 – conservative model: Accept the solution, grow the TR

SCvx - Termination

Exit creteria

$$\|p^* - \bar{p}\|_{\hat{q}} + \max_{k=1,..,N} \|x_k^* - \bar{x}_k\|_{\hat{q}} \leq \epsilon \quad \text{OR} \quad \bar{\mathcal{J}}_\lambda - L_\lambda^* \leq \epsilon_r |\bar{\mathcal{J}}_\lambda|$$

where $\hat{q} = 1, 2, \infty$

SCvx - Termination

Exit creteria

$$\|p^* - \bar{p}\|_{\hat{q}} + \max_{k=1,..,N} \|x_k^* - \bar{x}_k\|_{\hat{q}} \leq \epsilon \quad \text{OR} \quad \bar{\mathcal{J}}_\lambda - L_\lambda^* \leq \epsilon_r |\bar{\mathcal{J}}_\lambda|$$

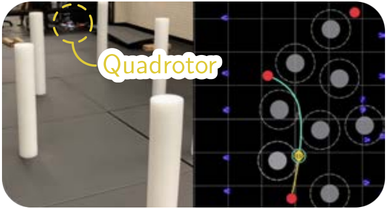
where $\hat{q} = 1, 2, \infty$

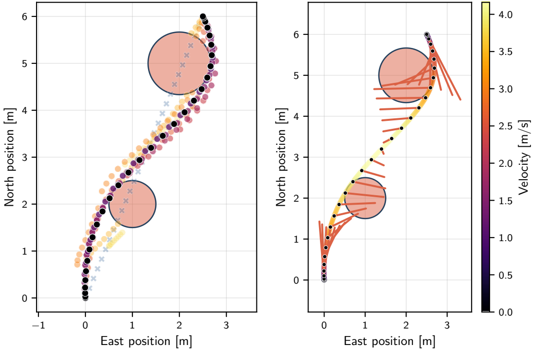
Convergence Theorem

Suppose that a standard constraint qualification holds and the weight λ in is large enough. For all initial reference trajectory:

- SCvx algorithm will always converge
- If the virtual controls are zero at convergence, then it is a stationary point of the non-convex OCP satisfying the KKT conditions.

SCvx - Quadrotor flying through obstacles





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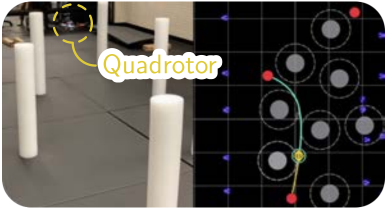

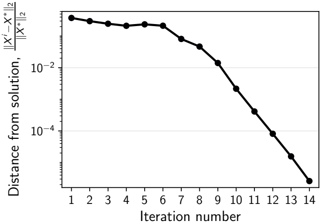

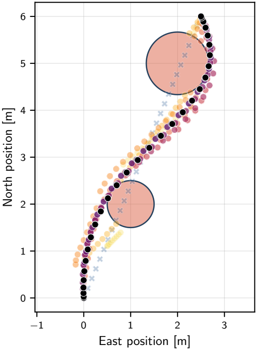

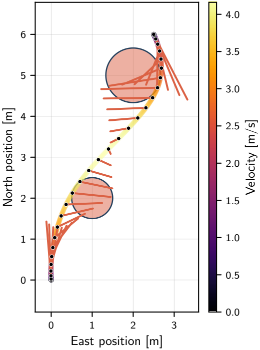

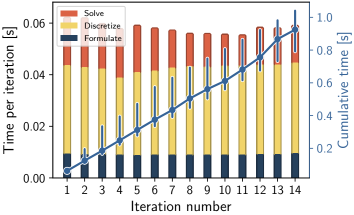

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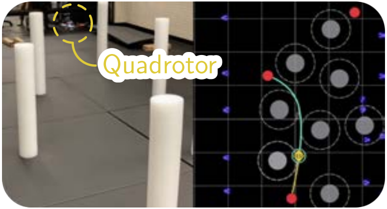
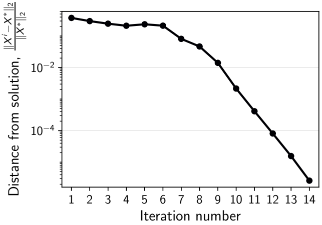
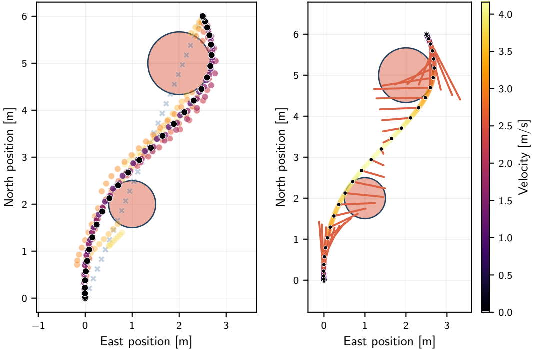
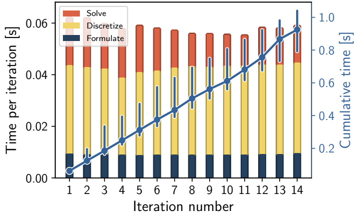






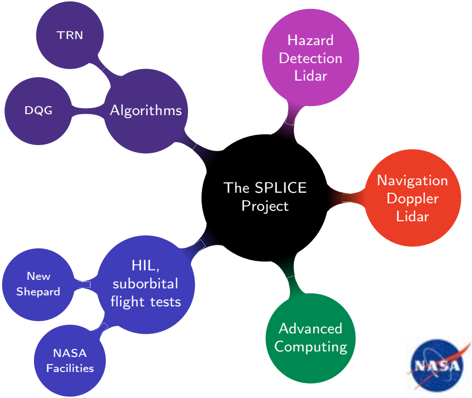


Flight Tests

Flight around 10 obstacles (Szmuk et al. [2017]). Video

SCvx Technology Transition – G-FOLD to SPLICE

SPLICE a game-changing NASA technology program for next-generation GN&C for safe precision planetary landing



B. Açıkmeşe is the "guidance project lead for SPLICE" – for a Moon mission

SCP-based trajectory planning algorithm for SPLICE is developed at UWashington

Embedded Convex Solvers

Research question

Can we develop real-time embedded convex solvers which are fast, guaranteed to converge, and easy to verify ?

Conic optimization for Trajectory Planning

$$\begin{aligned} \underset{z}{\text{minimize}} \quad & \tfrac{1}{2} z^\top P z \\ \text{subject to} \quad & Hz - g \in \mathbb{K}, \;\; z \in \mathbb{D} \end{aligned}$$

- Notation
  - P is a positive semidefinite matrix (better if positive definite)
  - $\mathbb{D}$ is a convex set with simple structure
    (box, ball, cone, or their cartesian products)
  - $\mathbb{K}$ is a convex cone
    (nonnegative orthant, second-order cone, or their cartesian products)
- LP, SOCP, and SDP are all examples of conic optimization

Proportional-Integral Projected Gradient (PIPG) Method

Conic optimization

$$\begin{array}{ll} \underset{z}{\text{minimize}} & \frac{1}{2} z^\top P z \\ \text{subject to} & Hz - g \in \mathbb{K}, \quad z \in \mathbb{D} \end{array}$$

PIPG itration

$$\begin{aligned} z^+ &\leftarrow \pi_{\mathbb{D}}[z - \alpha(Pz + H^\top w)] \\ w^+ &\leftarrow \pi_{\mathbb{K}^\circ}[w + \alpha H(2z^+ - z) - g] \end{aligned}$$

- Notation
  - α is a positive step size
  - $\pi_{\mathbb{X}}[\cdot]$ is the projection onto a set $\mathbb{X}$
  - $\mathbb{K}^\circ$ is the polar cone of cone $\mathbb{K}$

$$\mathbb{K}^\circ := \{w \in \mathbb{R}^m | \langle w, y \rangle \leq 0, \forall y \in \mathbb{K}\}.$$

Proportional-Integral Projected Gradient (PIPG) Method

Conic optimization

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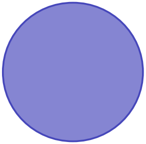
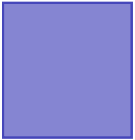
- Key features Yu* et al. [2020, 2022, 2023]
  - No matrix inverse, exploits sparsity pattern and parallel computation
  - Exploits problem structure to compute projections
  - Simple algorithm and code = Easy verification

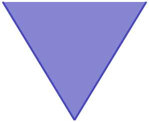
Simple Projections for OCPs in PIPG



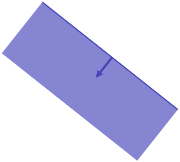
(a) Ball



(b) Box

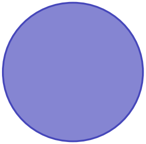


(c) Cone



(d) Halfspace

Simple Projections for OCPs in PIPG



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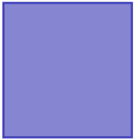
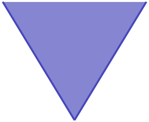


(b) Box



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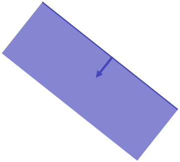
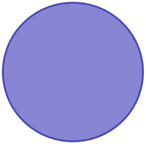


(d) Halfspace

- Euclidean projection on simple sets is fast

Simple Projections for OCPs in PIPG



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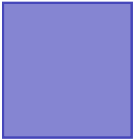

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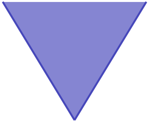

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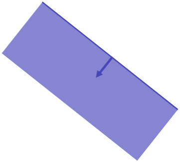
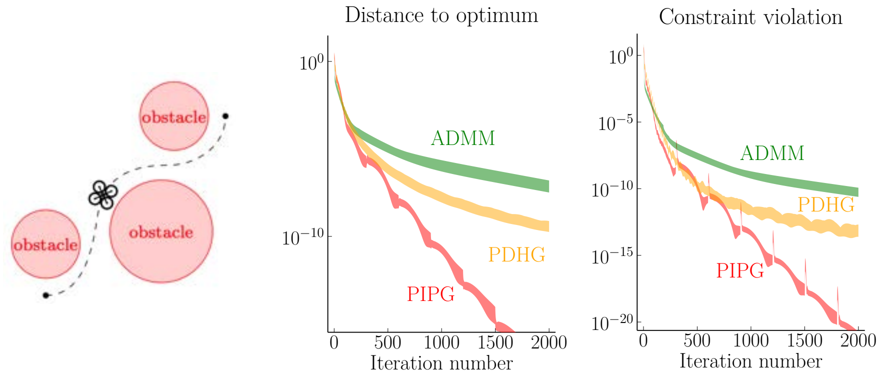


(d) Halfspace

- Euclidean projection on simple sets is fast
- Projection on intersections can be decomposed to projections on simple sets

PIPG Example – Quadrotor path planning

Optimally tracking a given reference trajectory while **avoid collision**.



PDHG: Primal-Dual Hybrid Gradient Method

Robust Planning and Control

Research question

Can we handle uncertanties in trajectory planning and control?

Uncertainties in Planning and Control

Typical sources of uncertainties

- External disturbance inputs
- Model uncertainties, e.g., state and/or control-dependent uncertainties
- Errors due to state estimation

Uncertainties in Planning and Control

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Modeling of disturbances is key step towards handling them

- Deterministic modeling
  - Bounded disturbance inputs
  - Bounded state estimation errors
  - Uncertainties constrained as a function of state and controls
- Probabilistic modeling aims to relax conservatism in deterministic modeling by
  - Relaxing boundedness to probability distributions

Uncertain Continuous-Time Dynamical System

$$\dot{x}(t) = f(t, x(t), u(t), w(t), p(t))$$

$$y(t) = h(t, x(t), u(t), w(t), p(t))$$

$$q(t) = g(t, x(t), u(t), w(t), p(t))$$

u: Control input
w: External input
p: Model uncertainty

y: Measured output
q: Uncertainty constraining output

An Uncertainty Model

- **External disturbance**

$$w(t) \in \mathcal{W}(t), \;\; t \geq t_0; \quad \mathcal{W}(t) \text{ compact set}$$

An Uncertainty Model

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$$w(t) \in \mathcal{W}(t),\ \ t \geq t_0;\quad \mathcal{W}(t) \text{ compact set}$$

- **State/control-dependent uncertainty**

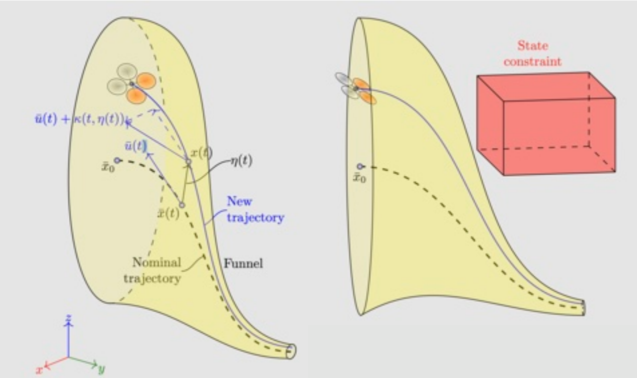
$$\Phi(q,p) \geq 0; \qquad \text{e.g.} \quad \Phi(q,p) = \langle M(q,p),\, (q,p)\rangle$$

M is a bounded linear operator

Includes uncertainties satisfying

- pointwise quadratic inequalities (pQIs): norm-bounded, polytopic, conic uncertainties
- pointwise incremental QIs (pIQIs): Lipschitz, monotonic uncertainties
- integral quadratic constraints (IQCs)
- incremental IQCs (iIQCs)
- . . .

Funnels/Tubes to Robustify Trajectories



Robustly feasible trajectory

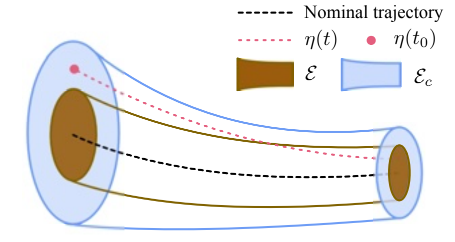
trajectory	funnel inclusion	Feasible trajectories
$(x(t), u(t)) \in$	$(\bar{x}(t), \bar{u}(t)) \oplus \mathcal{I}(t) \subset$	$\mathcal{F}(t)$

Funnel Design Objective

Funnel Attractivity and Invariance

Design two funnels:

- attractive set (brown): $\forall\ \eta(\cdot),\ \ d(\eta(t), \mathcal{E}(t)) \to 0\ \text{ as } t \to \infty$
- invariant set (blue): $\eta(t_0) \in \mathcal{E}_c(t_0)\ \Rightarrow\ \eta(t) \in \mathcal{E}_c(t)$



Basic Example - Funnels for LTV Systems

LTV System

$$\phi = 0 \qquad p = 0$$

- A Differential LMI (dLMI) formulation via a quadratic Lyapunov function for the incremental system

$$\begin{aligned} \dot{Q}(t) &= \overbrace{A(t)Q(t) + B(t)Y(t)}^{:=(*)} + (*)^\top + 2\alpha Q(t) + Z(t) \\ &:= F(Q, Y, Z) \\ Z(t) &\succeq 0 \end{aligned}$$

$$\text{Vectorize variables} \quad q := Q(:), \quad y := Y(:), \quad z := Z(:)$$

$$\Rightarrow \quad \dot{q}(t) = A_q(t)q(t) + B_q(t)y(t) + S_q(t)z(t)$$

A Simple dLMI Constrained Funnel Synthesis

Maximize Funnel Entry for LTV System Kim* et al. [2023]

$$
\begin{aligned}
\min_{Q,Y,Z} \quad & -\log\det Q(t_0) \\
& q = Q(:), \quad y = Y(:), \quad z = Z(:) \\
& \dot{q}(t) = A_q(t)q(t) + B_q(t)y(t) + S_q(t)z(t), \\
& Z(t) \succeq 0, \\
& 0 \prec Q(t) \preceq Q_{max}(t), \\
& \begin{bmatrix} Q(t) & Y(t)^\top \\ Y(t) & R_{max}(t) \end{bmatrix} \succeq 0 \qquad \equiv \quad KQK^\top \preceq R_{max}(t), \\
& \forall t \in [t_0, t_f].
\end{aligned}
$$

A Simple dLMI Constrained Funnel Synthesis

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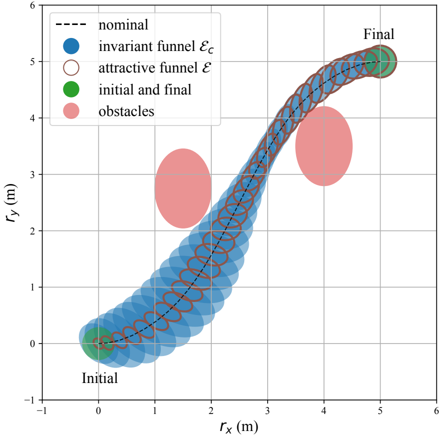
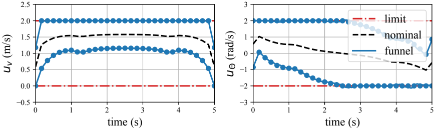
& \forall t \in [t_0, t_f].
\end{aligned}
$$

- This can be converted into a finite dimensional optimization problem with LMI constraints after a discretization
- Nominal trajectories and funnels can be **jointly synthesized via SCP**

Joint Funnel and Trajectory Synthesis for Dubin's car





- Constraints: obstacle avoidance and input constraints
- Cost: maximize the invariant set and minimize the attractive set

Towards Planning with Uncertainty

Research questions

- Can we handle uncertainties directly in trajectory planning?
- Can we develop convexify resulting problem formulations?

Towards Planning with Uncertainty

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- Can we handle uncertainties directly in trajectory planning?
- Can we develop convexify resulting problem formulations?

A First Linear Example

Consider the following discrete-time dynamical system:

$$x_{t+1} = A_t x_t + B_t u_t + F_t(g_t + w_t) + E_t p_t, \quad t = 0, 1, 2, \ldots$$

$$\Rightarrow x_t = \bar{A}_{t-1} x_0 + \bar{B}_{t-1} \bar{u}_{t-1} + \bar{F}_{t-1}(\bar{g}_{t-1} + \bar{w}_{t-1}) + \bar{E}_{t-1} \bar{p}_{t-1}$$

$$G_t w_t \leq g_t, \quad 0 \leq p_t \leq q(x_t^n, u_t)$$

g_t is known external input, $\bar{()}_t$ vectors are augmented vectors from $0 \rightarrow t$, and q is a convex function, x_t^n is TBD

Modeling State and Control Dependent Uncertainty

Consider nominal state, x_t^n, and state deviation, Δx_t, separately:

$$x_t^n = \bar{A}_{t-1} x_0 + \bar{B}_{t-1} \bar{u}_{t-1} + \bar{F}_{t-1} \bar{g}_{t-1},$$
$$\Delta x_t = \bar{F}_{t-1} \bar{w}_{t-1} + \bar{E}_{t-1} \bar{v}_{t-1}.$$

Robust Planning Problem

$$\min_{\bar{u}_{N-1}} J\left(\bar{x}_N^n, \bar{u}_{N-1}\right) \quad \text{s.t.}$$

Discrete-time dynamics as above

$u_t \in \mathcal{U}_t$ $\quad$ $\mathcal{U}_t$ is a convex set

$H_t x_t \leq h_t$ $\quad$ $\forall G_t w_t \leq g_t$ and $0 \leq p_t \leq q(x_t^n, u_t)$, $t = 1, \ldots, N$

Finite-dimensional Equivalent of Robust Planning Problem

The following result follows from convex duality theory Sheridan* and **Açıkmeşe, Behçet** [2022]

Lemma (Conversion to finite dimensional biconvex constraint)

$H_t x_t \leq h_t \ \forall G_t w_t \leq g_t, 0 \leq p_t \leq q(x_t^n, u_t)$ **IFF** $\exists Z_t \geq 0$ *and* $\Lambda_t \geq 0$ *s.t.*

$$Z_t \bar{G}_{t-1} = H_t \bar{F}_{t-1}$$

$$\Lambda_t \succeq H_t \bar{E}_{t-1}$$

$$\underbrace{Z_t \bar{g}_{t-1} + \Lambda_t \bar{q}(\bar{x}_{t-1}^n, \bar{u}_{t-1})}_{\text{uncertainty buffer}} \preceq \underbrace{h_t - H_t x_t^n}_{\text{nominal state constraint}}.$$

Convexification for Robust Planning Problem

Lemma (Convexification of I biconvex constraint)

Biconvex constraint term $\Lambda_t \bar{f}(\bar{x}^n_{t-1}, \bar{u}_{t-1})$ *can be equivalently replaced by* $\Gamma_t \bar{f}(\bar{x}^n_{t-1}, \bar{u}_{t-1})$, *where* $\Gamma_t = \max\left(0, H_t \bar{K}_{t-1}\right)$.

Convexified Robust Planning Problem

$$\min_{\bar{u}_{N-1}} J\left(\bar{x}^n_N, \bar{u}_{N-1}\right) \quad \text{s.t.}$$

Discrete-time dynamics

$$u_t \in \mathcal{U}_t$$

$$Z_t \bar{G}_{t-1} = H_t \bar{F}_{t-1}, \quad Z_t \geq 0$$

$$\Gamma_t \bar{q}(\bar{x}^n_{t-1}, \bar{u}_{t-1}) + Z_t \bar{g}_{t-1} \leq h_t - H_t x^n_t$$

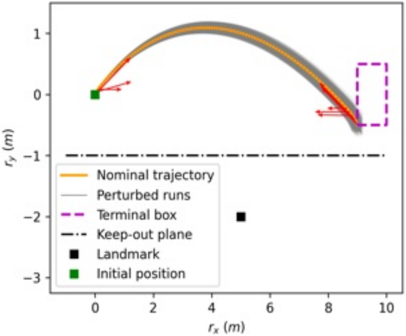
$$\text{where} \quad \Gamma_t = \max(0, H_t \bar{E}_{t-1})$$

Example – Relative Spacecraft Trajectory Planning in Orbit

- Orbital relative meneuvering problem with linearized Clohessy-Wiltshire dynamics
- Initial state at origin, final bounding box constraint on position and velocity
- Perturbation bound increases with increasing distance from "landmark"
- Keep-out plane constraint prevents collision with landmark

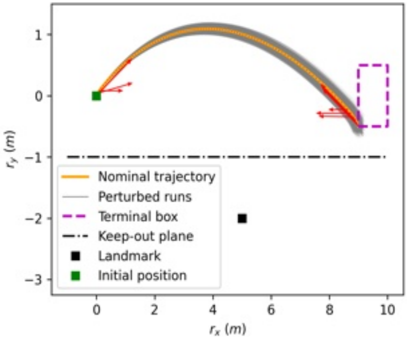
Example – "Monte-Carlo" Simulation Results

Planning without uncertainty model

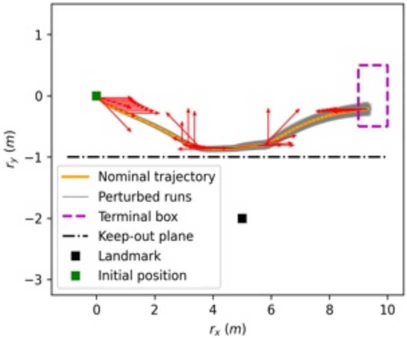


Example – "Monte-Carlo" Simulation Results

Planning without uncertainty model



Planning with uncertainty model



ACL Autonomous Vehicles

Research question

Can we validate our theoretical results?

Hardware and Software Infrastructure for Validation

Autonomous Control Lab (ACL)

- ACL: 1000sqft indoor flight arena with motion capture cameras
- In-house developed auto-pilot software and designed circuit board
- In-house developed generic and custom primal-dual IPM solver for SOCPs in C/C++
- In-house developed 10 indoor+10 outdoor quad-rotors, 30 omni-dir rovers

Hardware and Software Infrastructure for Validation

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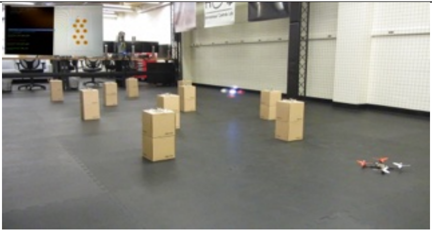

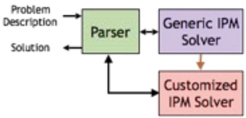

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Sponsors

Thank You!

More videos

- ACL website: http://depts.washington.edu/uwacl/media/
- Youtube channel: http://www.youtube.com/channel/UCZwV0cPCR3QeGn4dSfXxkKw

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