

Fundamentals of Electromagnetics

Transmission Lines

John McCloskey
Chief EMC Engineer
NASA/Goddard Space Flight Center
Greenbelt, MD
John.C.McCloskey@nasa.gov

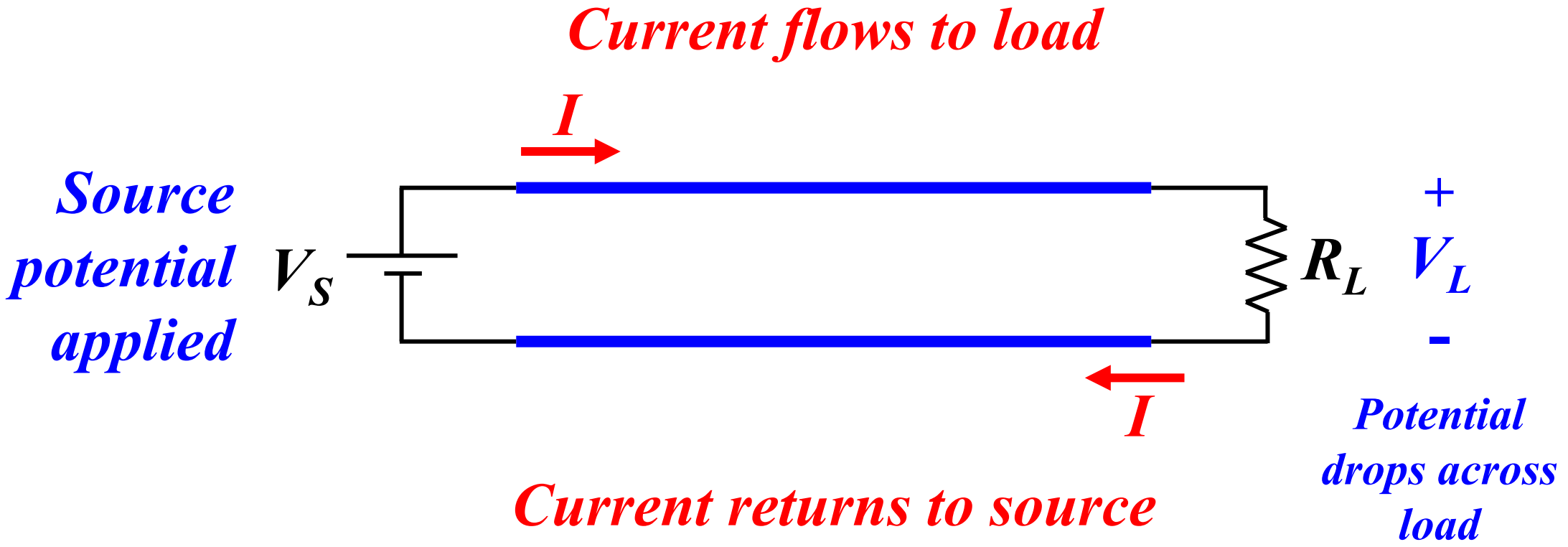


Topics

- **Traditional DC Circuit Model (Lumped Element)**
- **General Transmission Line Model**
  - Propagation Constant
  - Phase Constant (Wavenumber) and Velocity of Propagation
  - Characteristic Impedance
- **Examples: Coaxial Cable, Parallel Wires, Wire Above Ground Plane**
- **Impedance Mismatching and Reflections (with virtual demonstration)**
- **Input Impedance (with virtual demonstration)**
- **Summary**

- ***Shameless plug: "Fundamentals of Electromagnetics" video series***
  - *Publicly available on YouTube; search for above title*
  - *Direct playlist link:* https://youtube.com/playlist?list=PLtrpQ-gPvnJn2r9Mw49jij7Ky0mb6RJYF&si=UxEKqVRgsR9w6nZ7

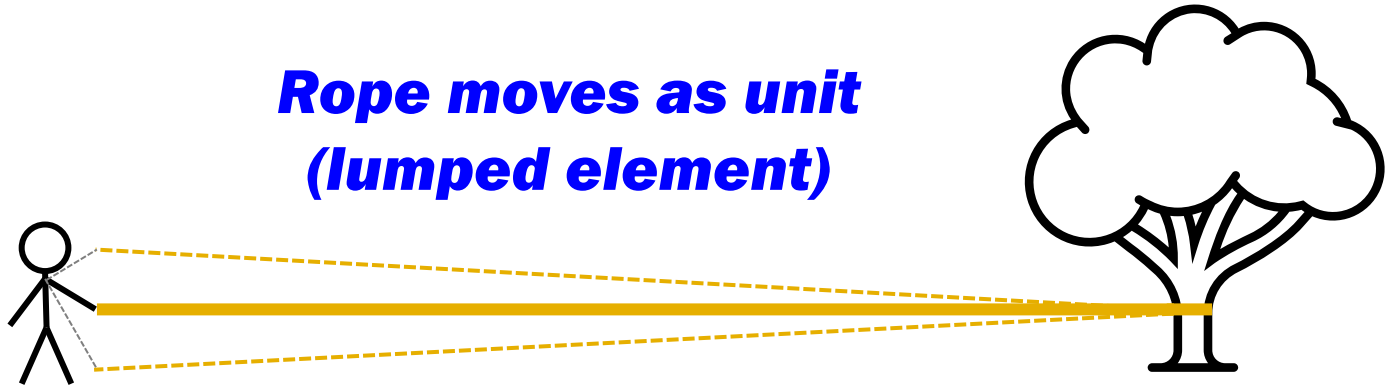
Traditional DC Circuit Model (Lumped Element)



Rope Analogy

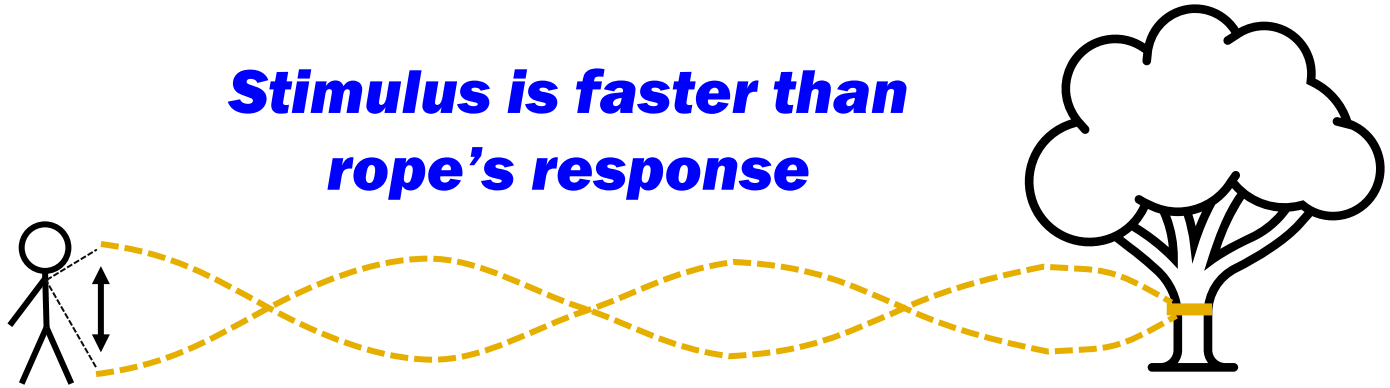
Rope moves as unit (lumped element)

Slow stimulus



Stimulus is faster than rope's response

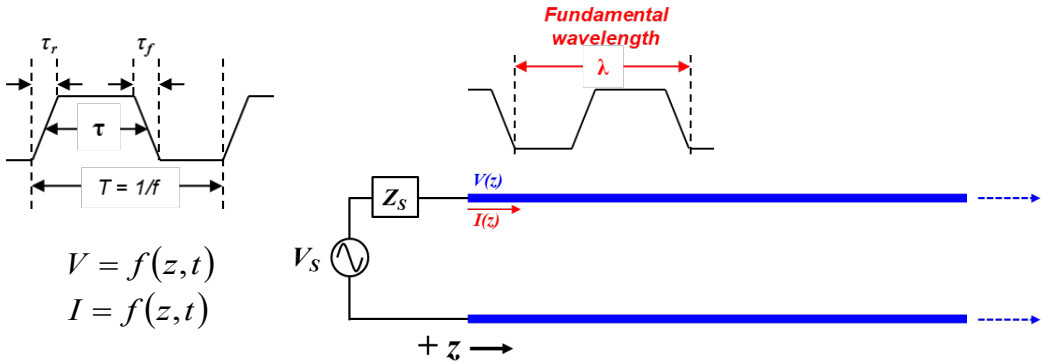
Fast stimulus



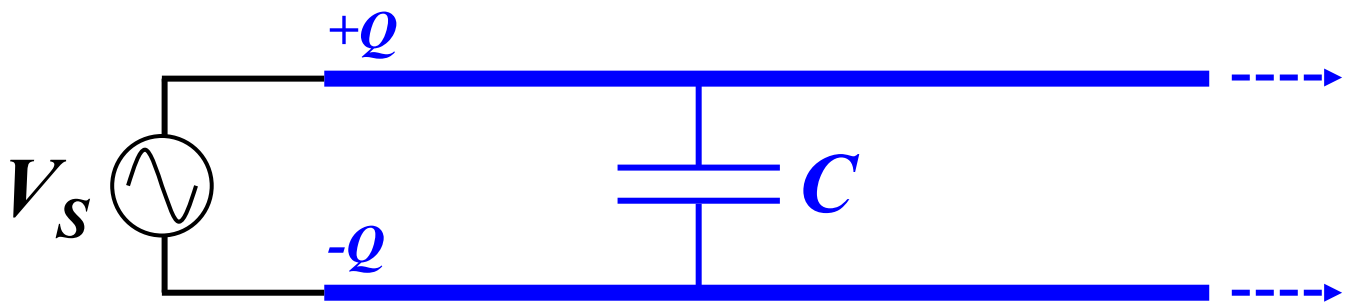
Energy propagates down rope as wave

General Transmission Line Model

- **Finite speed of electromagnetic waves means that it takes time for signal to travel down a cable**
- **When the voltage and/or current on a line changes in a time interval comparable to the time it takes for the signal to travel down the wire:**
  - Length becomes significant
  - Signal must be considered a wave propagating down line
  - Voilà, you have a transmission line
- **Rules of thumb:**
  - Treat cable, wire, or PCB trace as transmission line if length > λ/10 for highest frequency of interest
  - Must consider harmonics (rise/fall time), not just fundamental
  - Applies to digital as well as analog signals



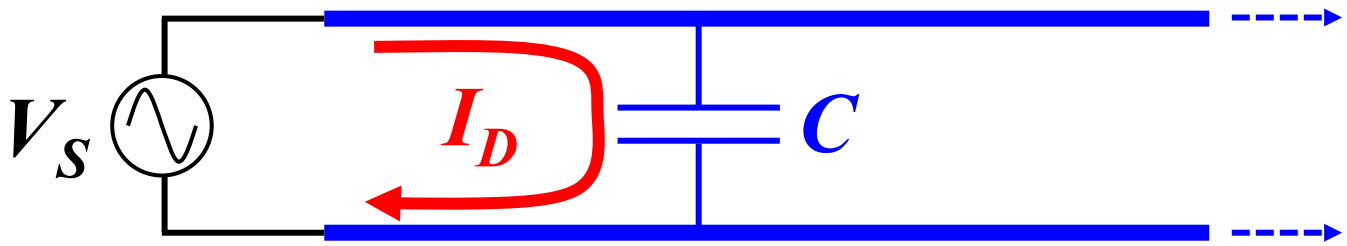
General Transmission Line Model (cont.)



$$C = \frac{Q}{V}$$

$$X_C = \frac{1}{j\omega C}$$

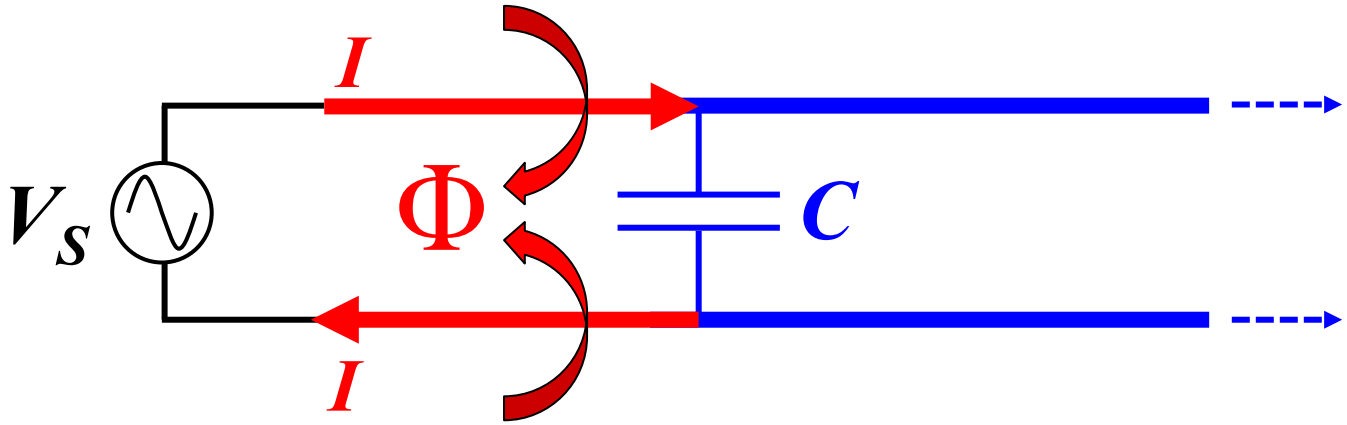
General Transmission Line Model (cont.)



Displacement current

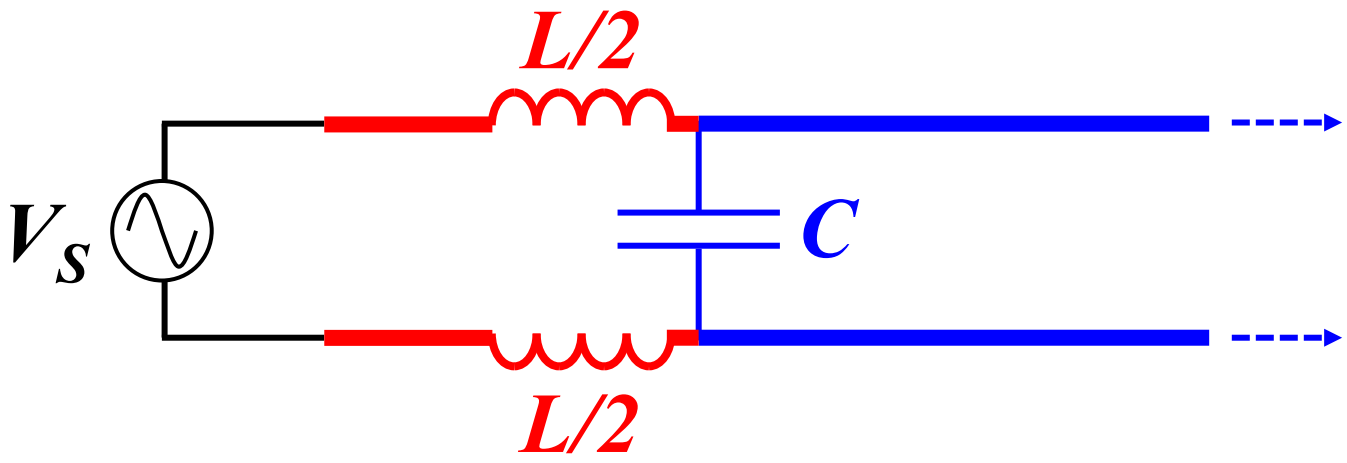
$$I_D = C\frac{dV_S}{dt}$$

General Transmission Line Model (cont.)



Current produces magnetic flux through loop between conductors

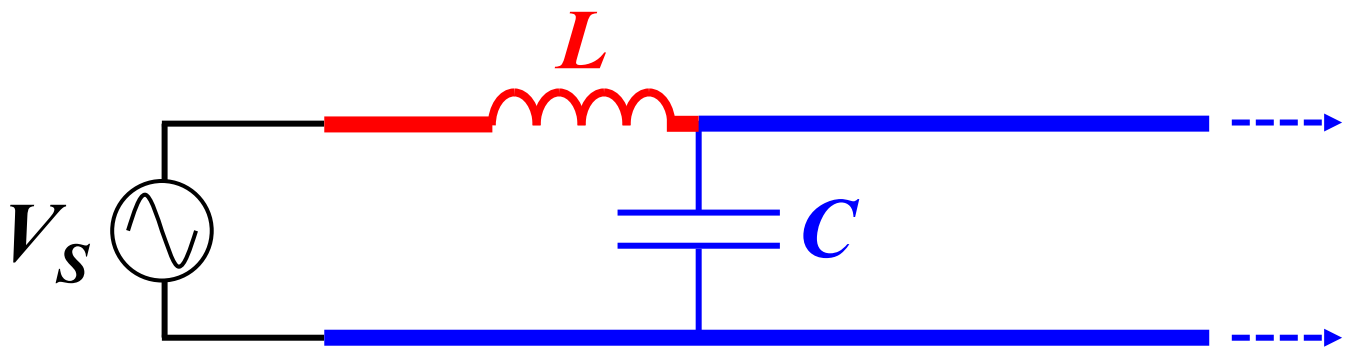
General Transmission Line Model (cont.)



Inductance = magnetic flux per unit current

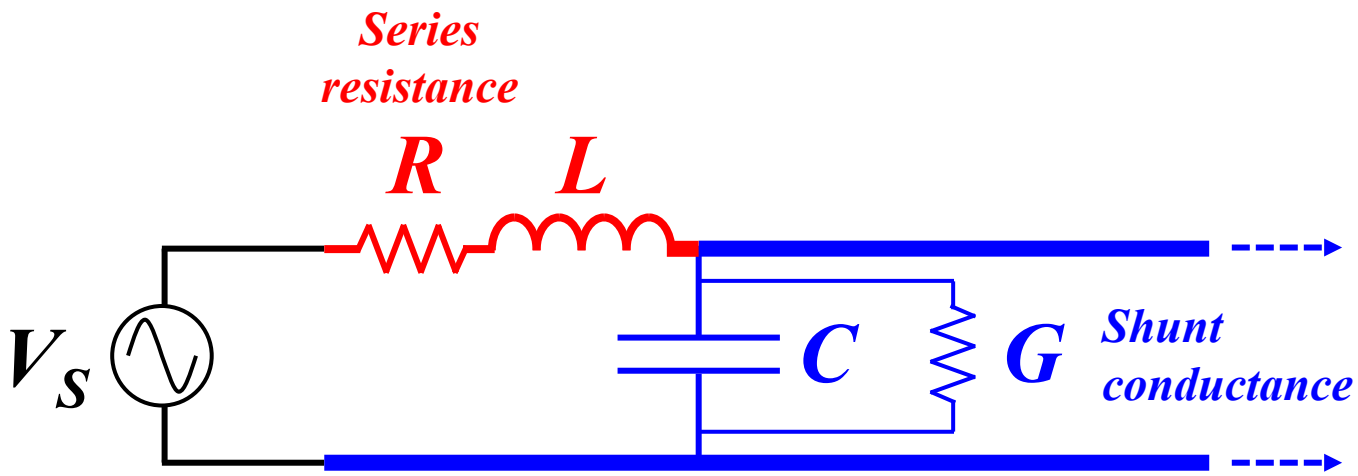
$$L = \frac{\Phi}{I} \qquad X_L = j\omega L$$

General Transmission Line Model (cont.)



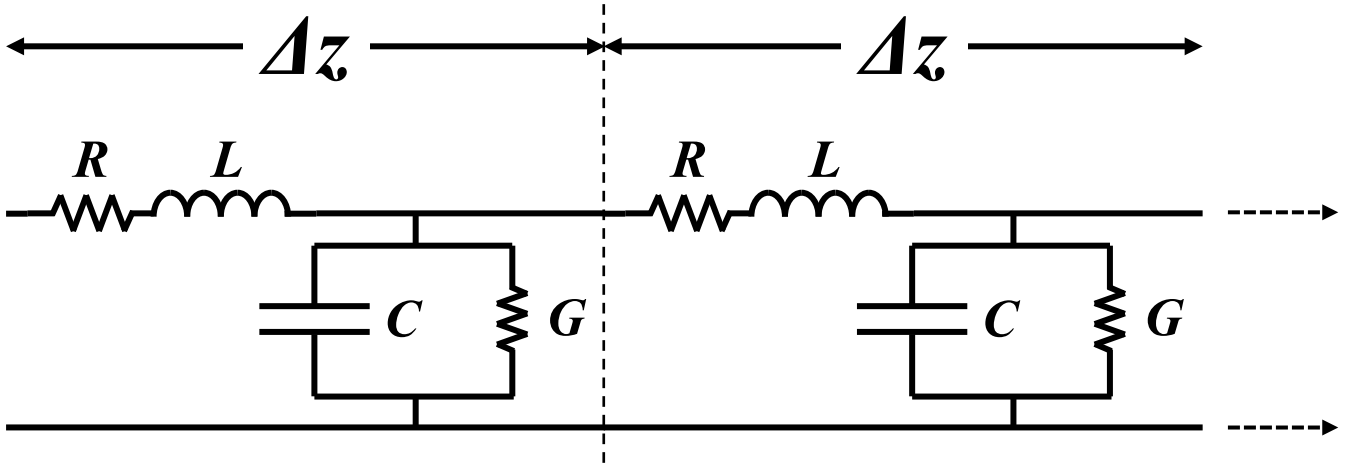
Lossless transmission line model

General Transmission Line Model (cont.)



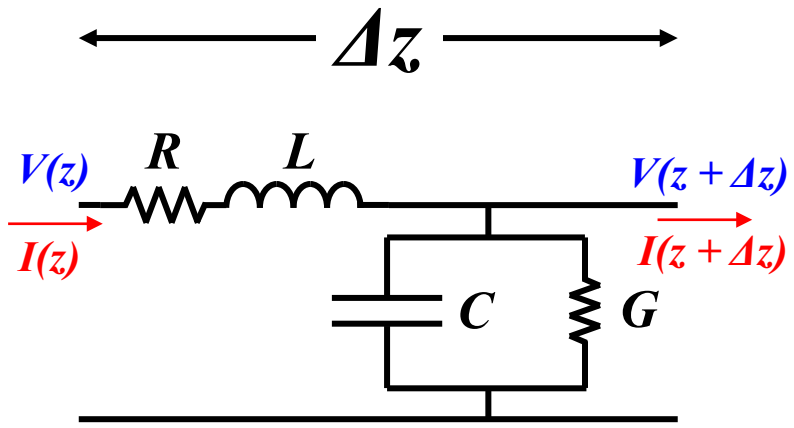
"Lossy" transmission line model

General Transmission Line Model (cont.)



R = series resistance per unit length (Ω/m)
L = series inductance per unit length (H/m)
C = shunt capacitance per unit length (F/m)
G = shunt conductance per unit length (Ω^{-1}/m)

General Transmission Line Model (cont.)



Series voltage drop:

$$V(z) - V(z + \Delta z) = I(z) \cdot (R + j\omega L)\Delta z$$

$$-\frac{V(z + \Delta z) - V(z)}{\Delta z} = I(z) \cdot (R + j\omega L)$$

$$\lim_{\Delta z \to 0} \rightarrow \quad -\frac{dV(z)}{dz} = I(z) \cdot (R + j\omega L)$$

Kirchoff's Current Law:

$$I(z) - I(z + \Delta z) = V(z) \cdot (G + j\omega C)\Delta z$$

$$-\frac{I(z + \Delta z) - I(z)}{\Delta z} = V(z) \cdot (G + j\omega C)$$

$$-\frac{dI(z)}{dz} = V(z) \cdot (G + j\omega C)$$

These will be a recurring theme...

General Transmission Line Model (cont.)

$$-\frac{dV(z)}{dz} = I(z)\cdot(R+j\omega L) \qquad -\frac{dI(z)}{dz} = V(z)\cdot(G+j\omega C)$$

$$\left(\frac{d}{dz}\right)$$

$$\frac{d^2V(z)}{dz^2} = -\frac{dI(z)}{dz}\cdot(R+j\omega L) \qquad \frac{d^2I(z)}{dz^2} = -\frac{dV(z)}{dz}\cdot(G+j\omega C)$$

$$\frac{d^2V(z)}{dz^2} = V(z)\cdot(G+j\omega C)(R+j\omega L) \qquad \frac{d^2I(z)}{dz^2} = I(z)\cdot(R+j\omega L)(G+j\omega C)$$

$$\gamma = \sqrt{(R+j\omega L)(G+j\omega C)}$$

$$\frac{d^2V(z)}{dz^2} = \gamma^2 V(z) \qquad \frac{d^2I(z)}{dz^2} = \gamma^2 I(z)$$

$$\frac{d^2V(z)}{dz^2} - \gamma^2 V(z) = 0 \qquad \frac{d^2I(z)}{dz^2} - \gamma^2 I(z) = 0$$

WAVE EQUATIONS

Propagation Constant

$$\frac{d^2V(z)}{dz^2} - \gamma^2 V(z) = 0$$

$$V(z) = V_0 e^{-\gamma z}$$

$$\frac{dV(z)}{dz} = -\gamma V_0 e^{-\gamma z}$$

$$\frac{d^2V(z)}{dz^2} = \gamma^2 V_0 e^{-\gamma z}$$

$$= \gamma^2 V(z)$$

GENERAL SOLUTIONS

$$\frac{d^2I(z)}{dz^2} - \gamma^2 I(z) = 0$$

$$I(z) = I_0 e^{-\gamma z}$$

$$\frac{dI(z)}{dz} = -\gamma I_0 e^{-\gamma z}$$

$$\frac{d^2I(z)}{dz^2} = \gamma^2 I_0 e^{-\gamma z}$$

$$= \gamma^2 I(z)$$

Propagation Constant

$$\frac{d^2V(z)}{dz^2} - \gamma^2 V(z) = 0 \qquad \frac{d^2I(z)}{dz^2} - \gamma^2 I(z) = 0$$

$$V(z) = V_0 e^{-\gamma z} \qquad \textbf{GENERAL SOLUTIONS} \qquad I(z) = I_0 e^{-\gamma z}$$

Propagation constant

$$\gamma = \alpha + j\beta$$

α = attenuation constant

β = phase constant (a.k.a. wavenumber)

$$V(z) = V_0 e^{-(\alpha + j\beta)z} \qquad I(z) = I_0 e^{-(\alpha + j\beta)z}$$

$$V(z) = V_0 e^{-\alpha z} e^{-j\beta z} \qquad I(z) = I_0 e^{-\alpha z} e^{-j\beta z}$$

Euler's Formula: $e^{jx} = \cos x + j \sin x$

Propagation Constant

$$\frac{d^2V(z)}{dz^2} - \gamma^2 V(z) = 0 \qquad \frac{d^2I(z)}{dz^2} - \gamma^2 I(z) = 0$$

GENERAL SOLUTIONS

$$V(z) = V_0 e^{-\gamma z} \qquad I(z) = I_0 e^{-\gamma z}$$

Propagation constant

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α = attenuation constant

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$$V(z) = V_0 e^{-(\alpha + j\beta)z} \qquad I(z) = I_0 e^{-(\alpha + j\beta)z}$$

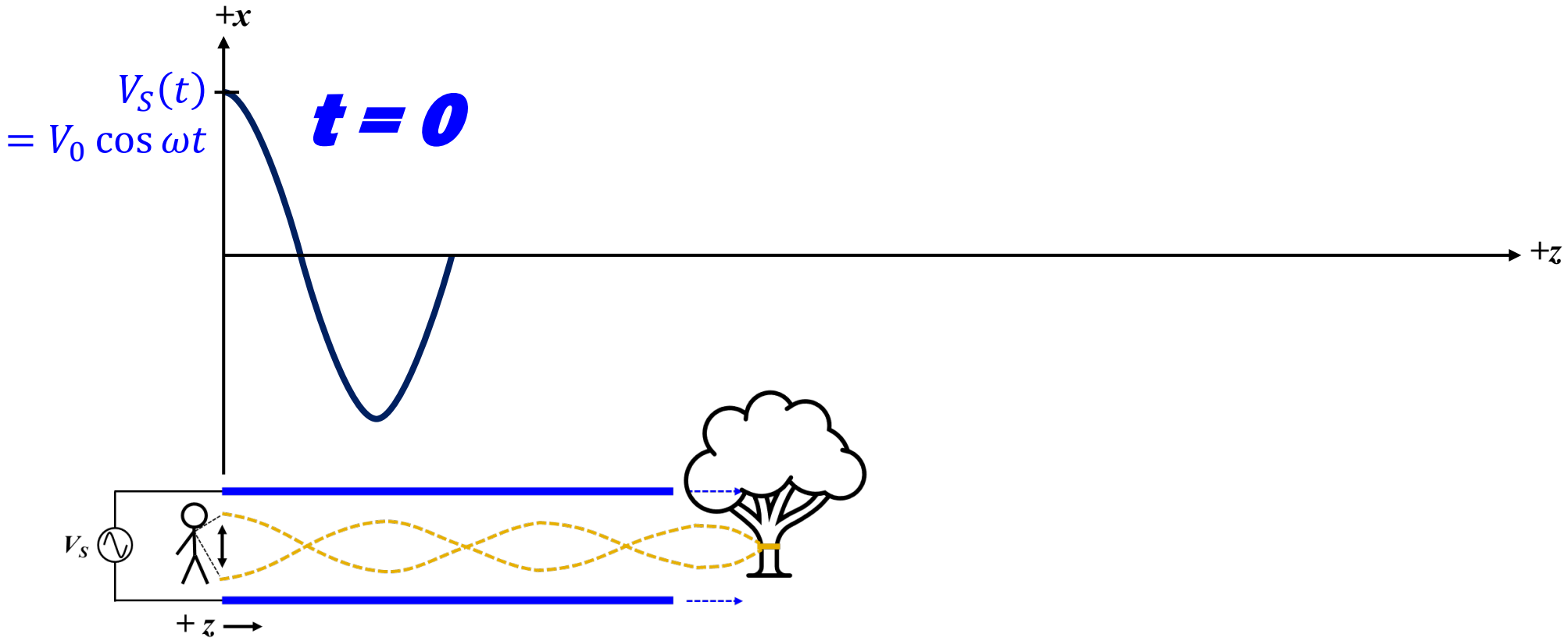
$$V(z) = V_0 e^{-\alpha z} e^{-j\beta z} \qquad I(z) = I_0 e^{-\alpha z} e^{-j\beta z}$$

$$Re[V(z)] = V_0 e^{-\alpha z} \cos(\beta z) \qquad Re[I(z)] = I_0 e^{-\alpha z} \cos(\beta z)$$

Wave Propagation on Transmission Line

$$V(z,t) = V_0 e^{j(\omega t - \beta z)}$$
$$\Re e[V(z,t)] = V_0 \cos(\omega t - \beta z)$$

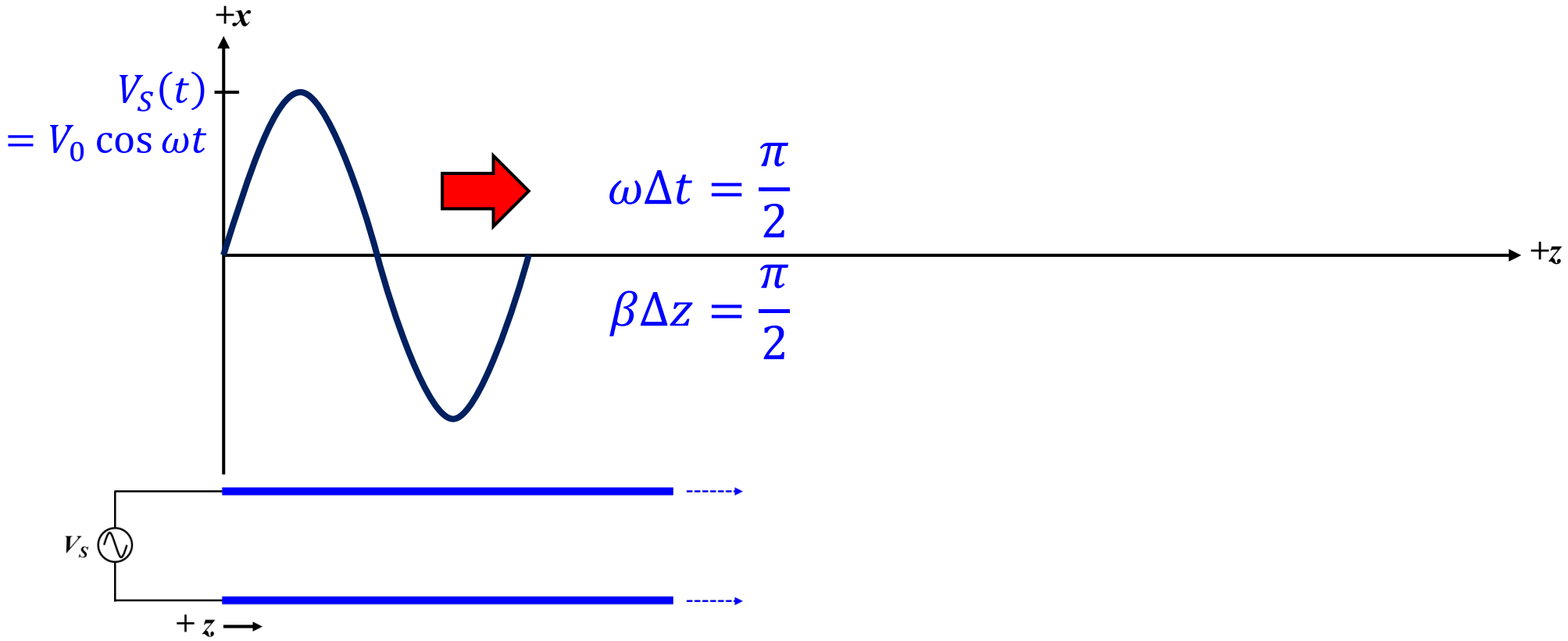
Including time variance
α = 0 for simplicity
("lossless" line)



Wave Propagation on Transmission Line (cont.)

$$V(z,t) = V_0 e^{j(\omega t - \beta z)}$$

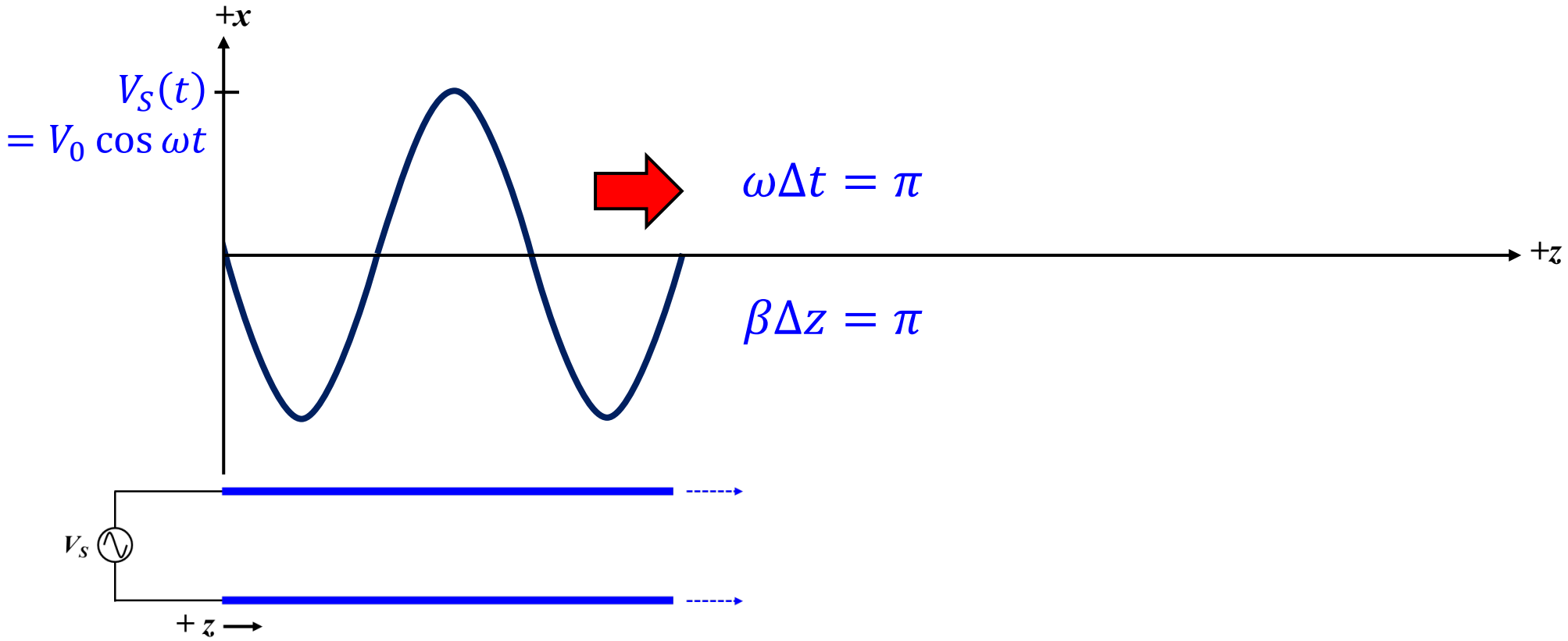
$$\Re e[V(z,t)] = V_0 \cos(\omega t - \beta z)$$



Wave Propagation on Transmission Line (cont.)

$$V(z,t) = V_0 e^{j(\omega t - \beta z)}$$

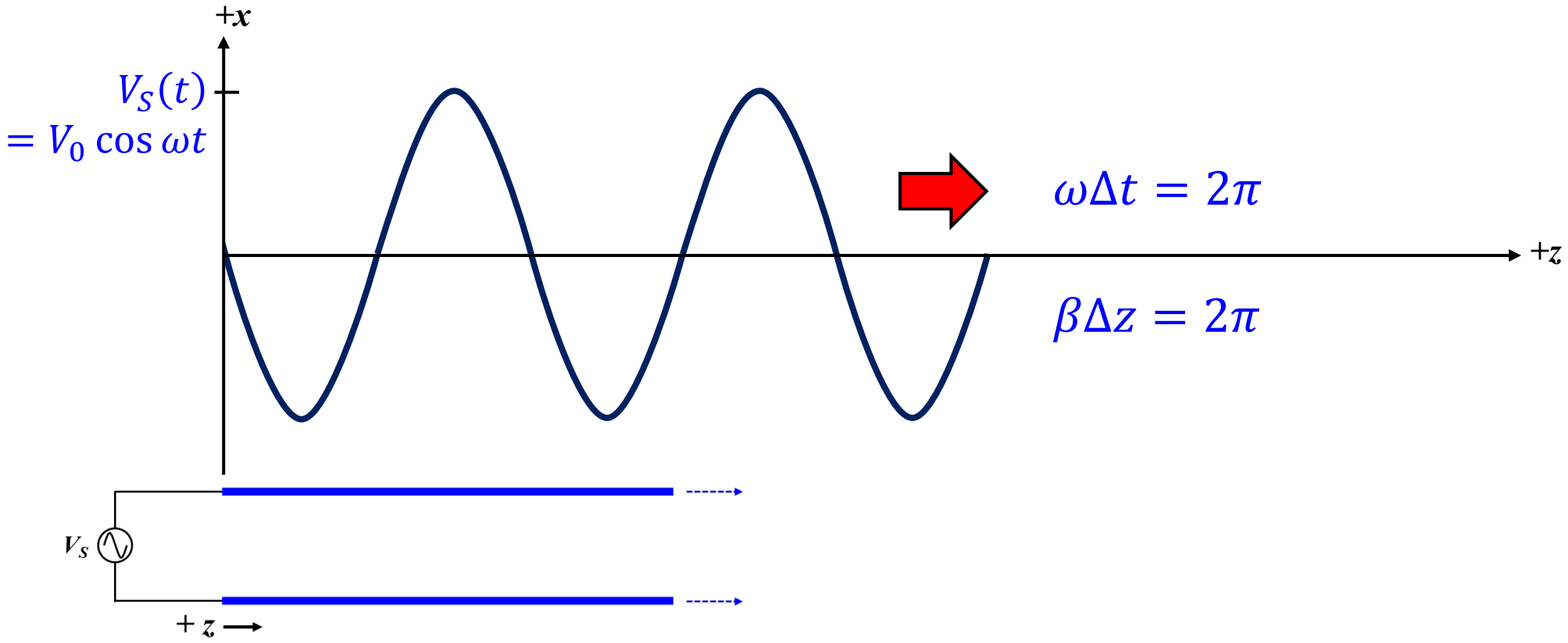
$$\Re e[V(z,t)] = V_0 \cos(\omega t - \beta z)$$



Wave Propagation on Transmission Line (cont.)

$$V(z,t) = V_0 e^{j(\omega t - \beta z)}$$

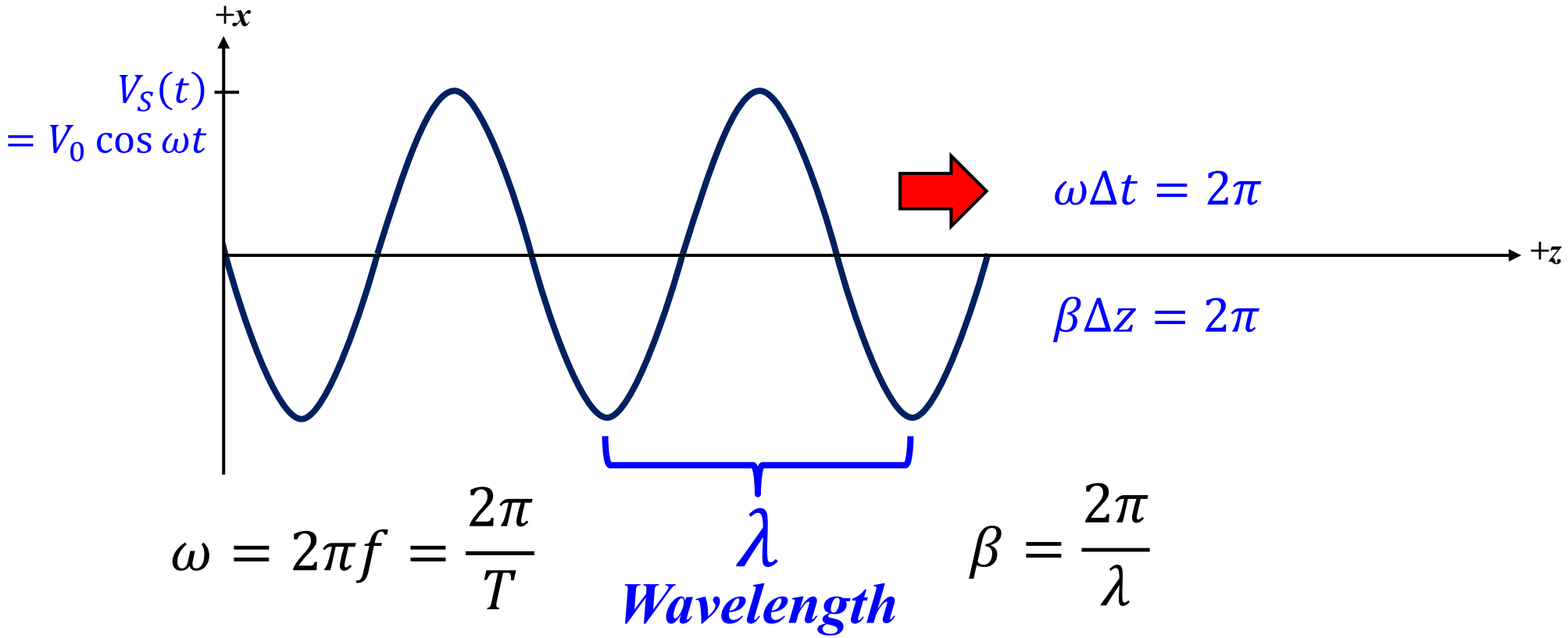
$$\Re e[V(z,t)] = V_0 \cos(\omega t - \beta z)$$



Wave Propagation on Transmission Line (cont.)

$$V(z,t) = V_0 e^{j(\omega t - \beta z)}$$

$$\Re e[V(z,t)] = V_0 \cos(\omega t - \beta z)$$

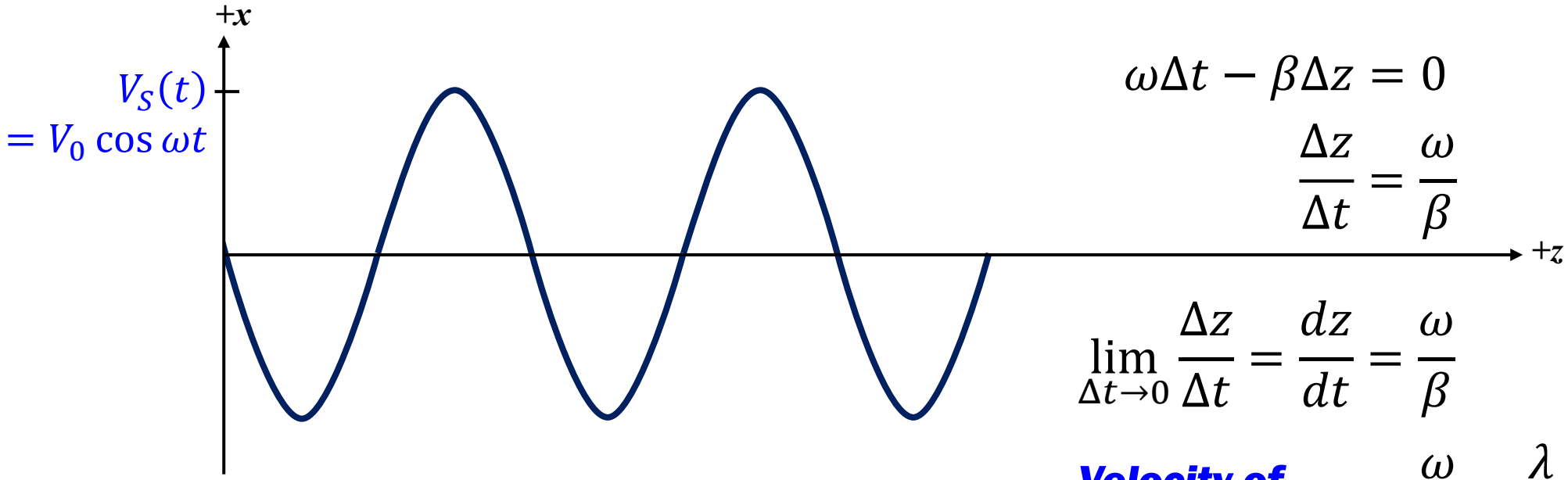


$$\omega = 2\pi f = \frac{2\pi}{T} \qquad \beta = \frac{2\pi}{\lambda}$$

Wave Propagation on Transmission Line (cont.)

$$V(z,t) = V_0 e^{j(\omega t - \beta z)}$$

$$\Re e[V(z,t)] = V_0 \cos(\omega t - \beta z)$$



$$\omega \Delta t - \beta \Delta z = 0$$

$$\frac{\Delta z}{\Delta t} = \frac{\omega}{\beta}$$

$$\lim_{\Delta t \to 0} \frac{\Delta z}{\Delta t} = \frac{dz}{dt} = \frac{\omega}{\beta}$$

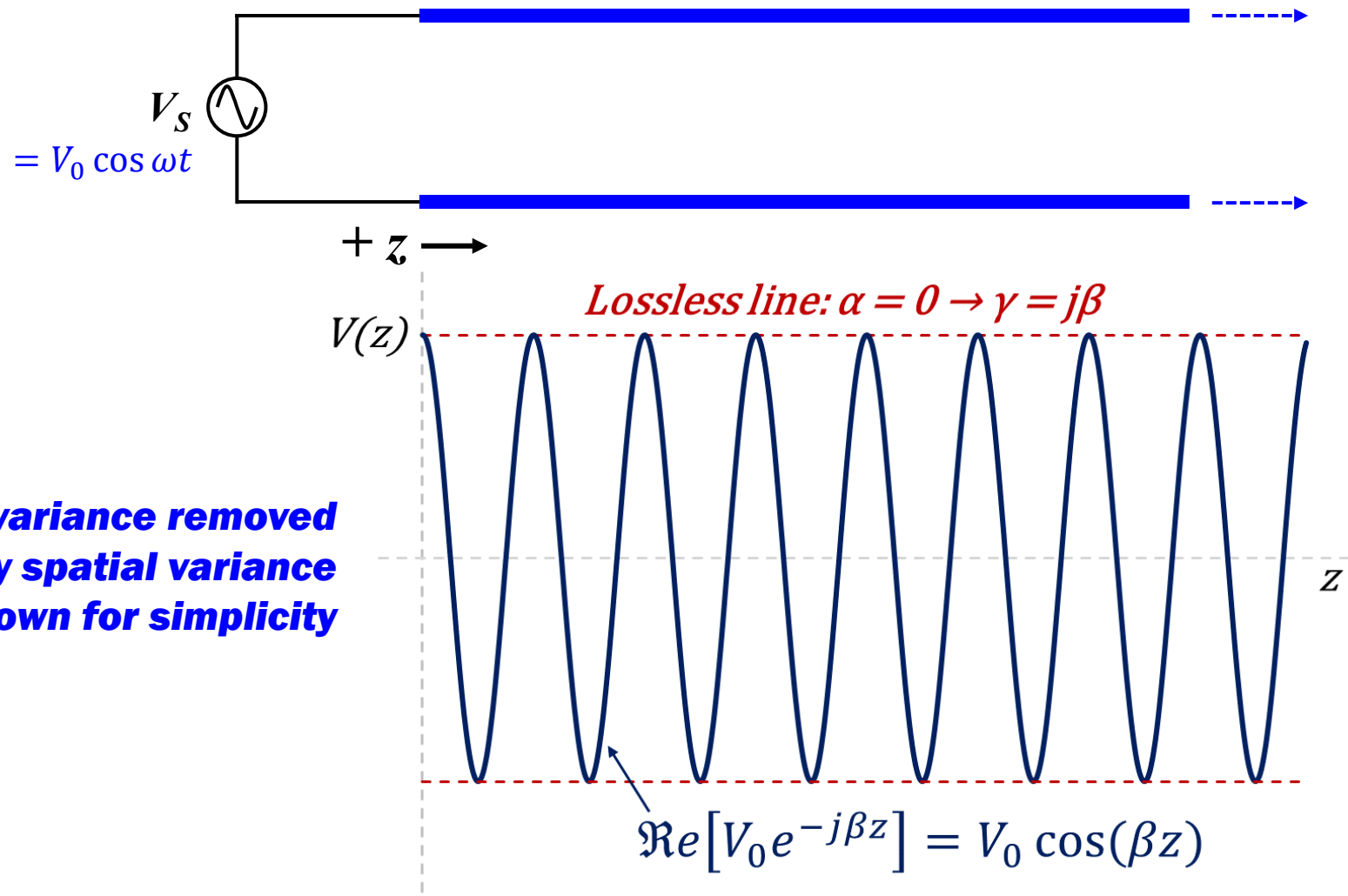
$$\omega = 2\pi f = \frac{2\pi}{T} \qquad \beta = \frac{2\pi}{\lambda}$$

Velocity of propagation

$$u = \frac{\omega}{\beta} = \frac{\lambda}{T} = \lambda f$$

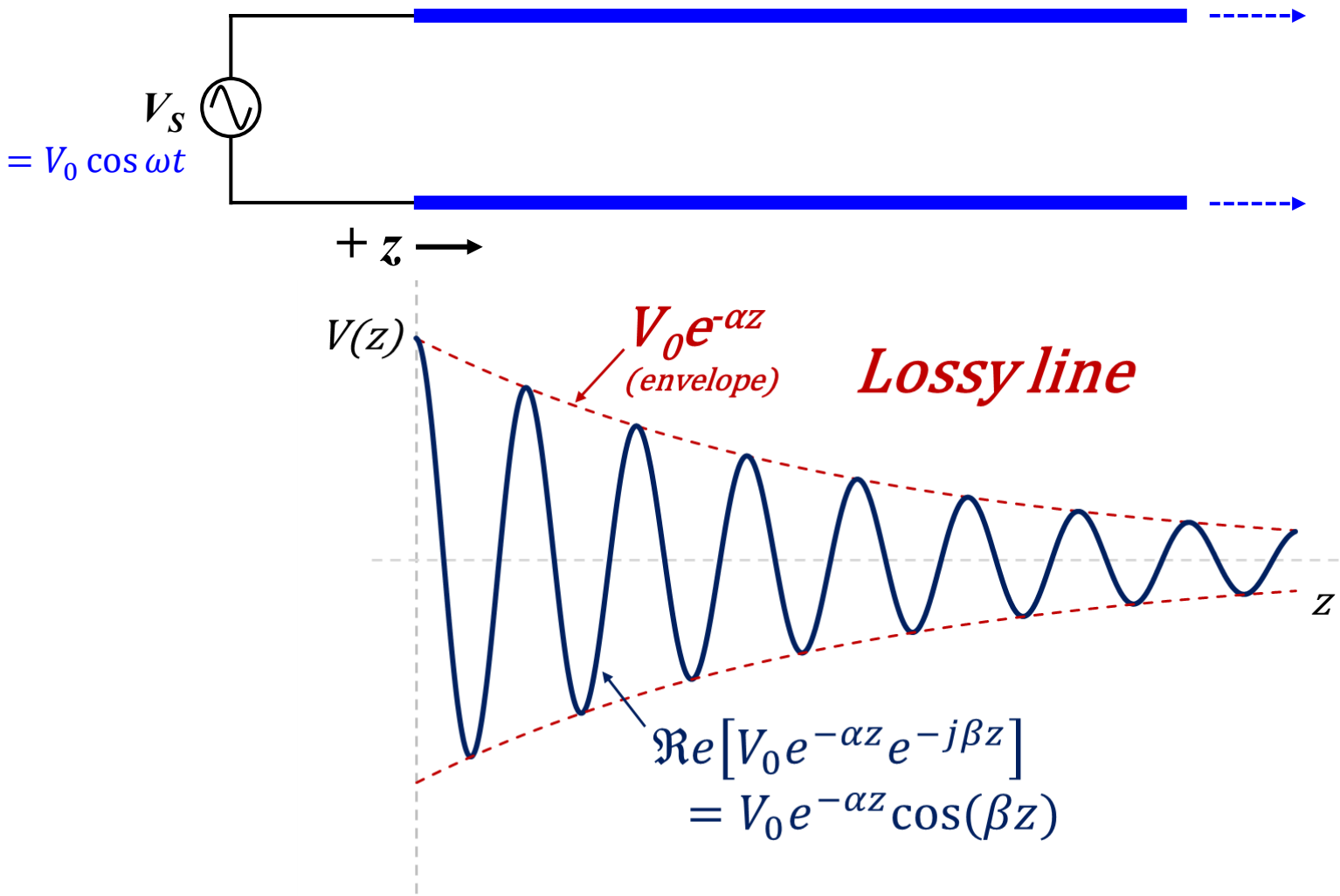
$$\lambda = \frac{u}{f}$$

Propagation Constant (cont.)



Time variance removed
Only spatial variance shown for simplicity

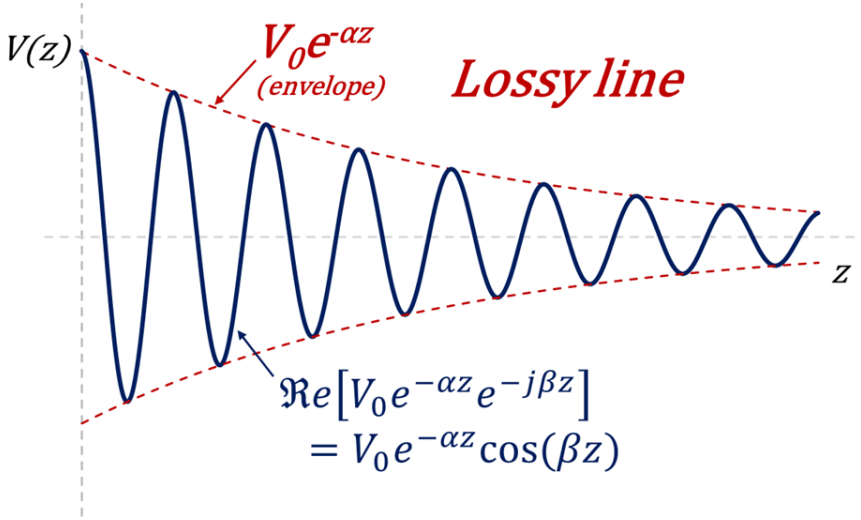
Propagation Constant (cont.)



Propagation Constant (cont.)

$$\gamma = \alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)}$$

For lossy transmission line (R, G > 0):





Extracting α and β terms is non-trivial

Both terms are frequency dependent

β addressed on next slide, but...

α frequency dependence → expect attenuation to increase with frequency

Digital waveform with fast rise/fall times:

→ acts as low-pass filter

→ rounds edges, distorts waveform

Phase Constant (Wavenumber) and Velocity of Propagation

$$\gamma = \alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)}$$

For lossless transmission line (R, G → 0):

$$\gamma = \alpha + j\beta = \sqrt{(j\omega L)(j\omega C)}$$
$$= \sqrt{-\omega^2 LC}$$
$$= j\omega\sqrt{LC}$$
$$\alpha = 0 \qquad \beta = \omega\sqrt{LC}$$
$$\frac{\omega}{\beta} = \frac{1}{\sqrt{LC}}$$

Angular temporal frequency:

$$\omega = 2\pi f = \frac{2\pi}{T}$$ *T = period (temporal)*

Wavenumber: $$\beta = \frac{2\pi}{\lambda}$$ *λ = wavelength (spatial)*

$$\frac{\omega}{\beta} = \frac{\lambda}{T} = \frac{\textbf{\textit{distance}}}{\textbf{\textit{time}}}$$

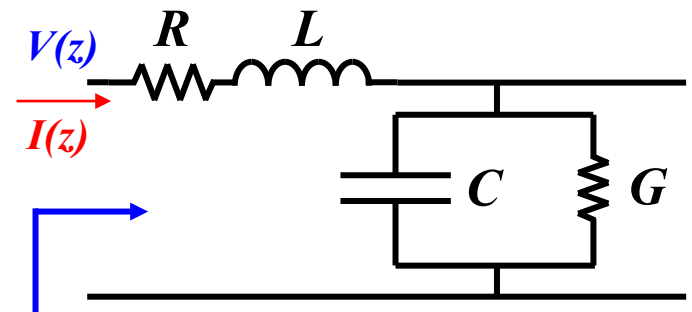
Velocity of propagation:

$$u = \frac{1}{\sqrt{LC}}$$

Recall for uniform plane wave:

$$u = \frac{1}{\sqrt{\mu\varepsilon}}$$

Characteristic Impedance



$$Z_0 = \frac{V(z)}{I(z)}$$

$$V(z) = V_0 e^{-\gamma z} \qquad I(z) = I_0 e^{-\gamma z}$$

$$-\frac{dV(z)}{dz} = I(z)\cdot(R + j\omega L) \quad \leftarrow \textit{Recall} \rightarrow \quad -\frac{dI(z)}{dz} = V(z)\cdot(G + j\omega C)$$

$$-\frac{dV(z)}{dz} = -(-\gamma V_0 e^{-\gamma z}) = \gamma V(z) \qquad -\frac{dI(z)}{dz} = -(-\gamma I_0 e^{-\gamma z}) = \gamma I(z)$$

$$\gamma V(z) = I(z)\cdot(R + j\omega L) \qquad \gamma I(z) = V(z)\cdot(G + j\omega C)$$

$$\frac{V(z)}{I(z)} = \frac{R + j\omega L}{\gamma} = Z_0 \qquad \frac{V(z)}{I(z)} = \frac{\gamma}{G + j\omega C} = Z_0$$

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)}$$

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

Characteristic impedance of lossy transmission line

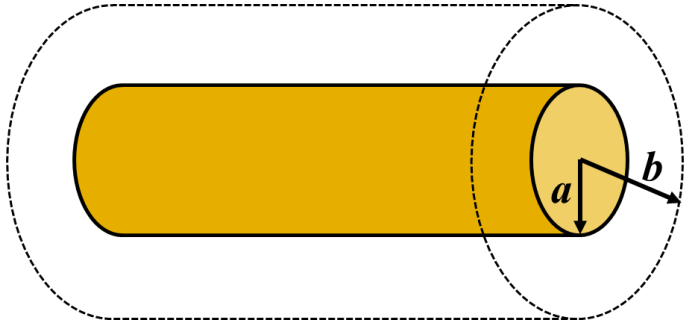
Characteristic impedance of lossless transmission line (R, G → 0)

$$Z_0 = \sqrt{\frac{L}{C}}$$

Recall for uniform plane wave:

$$\eta = \sqrt{\frac{\mu}{\varepsilon}}$$

Example: Coaxial Cable



From "Magnetic Fields" session:

$$L = \frac{\mu}{2\pi}\ln\left(\frac{b}{a}\right)$$

From "Electric Fields" session:

$$C = \frac{2\pi\varepsilon}{\ln\left(\frac{b}{a}\right)}$$

$$u = \frac{1}{\sqrt{LC}}$$

$$LC = \frac{\mu}{2\pi}\ln\left(\frac{b}{a}\right)\left[\frac{2\pi\varepsilon}{\ln\left(\frac{b}{a}\right)}\right] = \mu\varepsilon$$

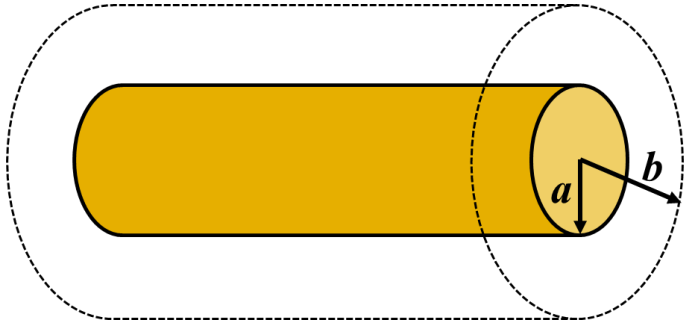
Geometric dependence cancels

$$u = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{\mu_r\mu_0\varepsilon_r\varepsilon_0}} = \frac{1}{\sqrt{\mu_r\varepsilon_r}}\cdot\frac{1}{\sqrt{\mu_0\varepsilon_0}} = \frac{c}{\sqrt{\mu_r\varepsilon_r}}$$

Back to this presently...

Speed of light in free space (c)

Example: Coaxial Cable



From "Magnetic Fields" session:

$$L = \frac{\mu}{2\pi}\ln\left(\frac{b}{a}\right)$$

$$Z_0 = \sqrt{\frac{L}{C}}$$

From "Electric Fields" session:

$$C = \frac{2\pi\varepsilon}{\ln\left(\frac{b}{a}\right)}$$

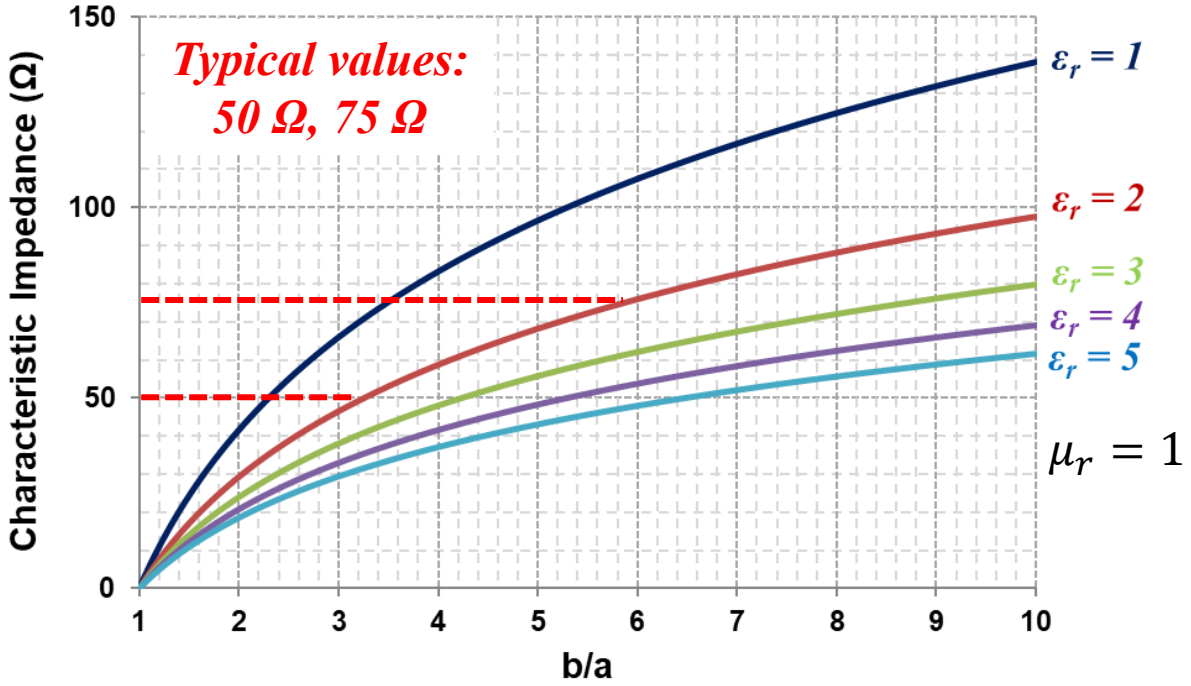
$$\frac{L}{C} = \frac{\mu}{\varepsilon}\left(\frac{1}{2\pi}\right)^2\left[\ln\left(\frac{b}{a}\right)\right]^2$$

$$Z_0 = \sqrt{\frac{\mu}{\varepsilon}}\cdot\frac{1}{2\pi}\ln\left(\frac{b}{a}\right) = \sqrt{\frac{\mu_r\mu_0}{\varepsilon_r\varepsilon_0}}\cdot\frac{1}{2\pi}\ln\left(\frac{b}{a}\right)$$

$$= \sqrt{\frac{\mu_r}{\varepsilon_r}}\sqrt{\frac{\mu_0}{\varepsilon_0}}\cdot\frac{1}{2\pi}\ln\left(\frac{b}{a}\right)$$

Characteristic impedance of free space

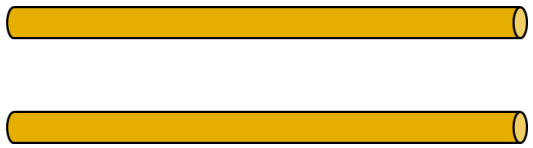
$$\approx 120\pi \approx 377\ \Omega$$

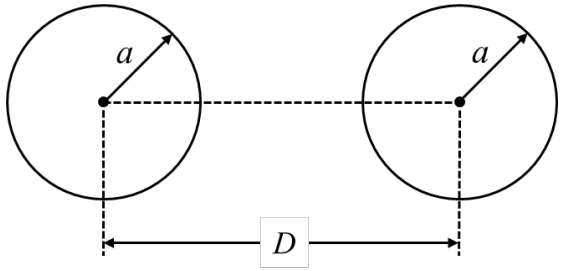


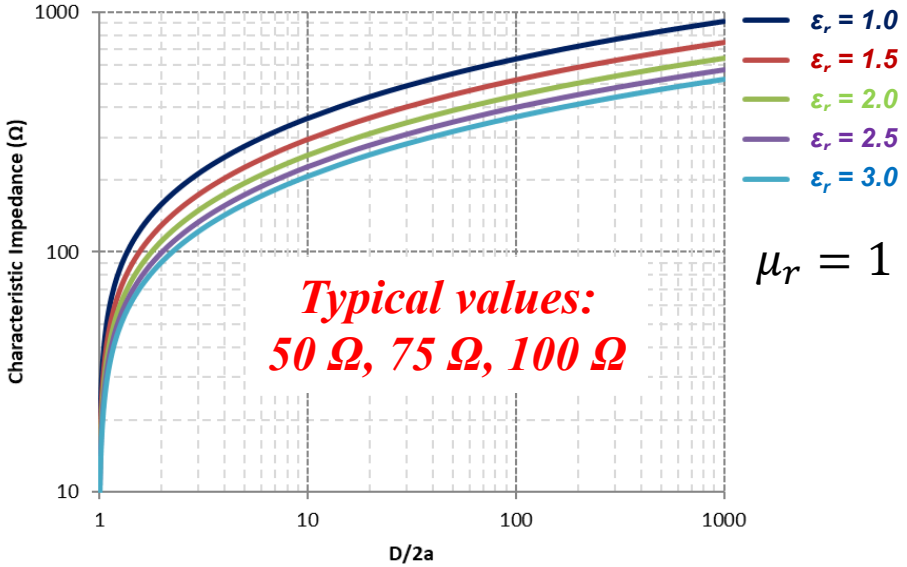
$$Z_0 = \sqrt{\frac{\mu_r}{\varepsilon_r}}\cdot 60\ln\left(\frac{b}{a}\right)$$

Characteristic impedance of coaxial transmission line

Example: Parallel Wires







$\mu_r = 1$

From "Magnetic Fields" session:

$$L = \frac{\mu}{\pi} \cdot \ln\left[\left(\frac{D}{2a}\right) + \sqrt{\left(\frac{D}{2a}\right)^2 - 1}\right]$$

From "Electric Fields" session:

$$C = \frac{\pi\varepsilon}{\ln\left[\left(\frac{D}{2a}\right) + \sqrt{\left(\frac{D}{2a}\right)^2 - 1}\right]}$$

$$u = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{c}{\sqrt{\mu_r \varepsilon_r}}$$

Geometric dependence cancels (Sensing a pattern?)

$$Z_0 = \sqrt{\frac{L}{C}} = \sqrt{\frac{\mu}{\varepsilon}} \cdot \frac{1}{\pi} \cdot \ln\left[\left(\frac{D}{2a}\right) + \sqrt{\left(\frac{D}{2a}\right)^2 - 1}\right]$$

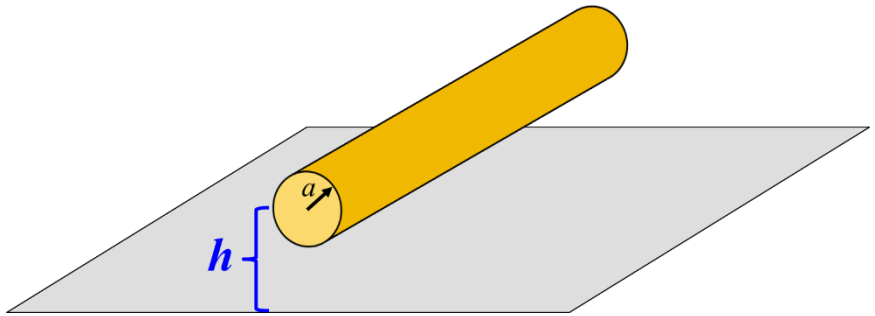
$$= \sqrt{\frac{\mu_r}{\varepsilon_r}} \cdot \sqrt{\frac{\mu_0}{\varepsilon_0}} \cdot \frac{1}{\pi} \cdot \ln\left[\left(\frac{D}{2a}\right) + \sqrt{\left(\frac{D}{2a}\right)^2 - 1}\right]$$

$\sqrt{\frac{\mu_0}{\varepsilon_0}} = 120\pi$

$$Z_0 = \sqrt{\frac{\mu_r}{\varepsilon_r}} \cdot 120 \ln\left[\left(\frac{D}{2a}\right) + \sqrt{\left(\frac{D}{2a}\right)^2 - 1}\right]$$

Characteristic impedance of parallel wire transmission line

Example: Wire Above Ground Plane



From "Magnetic Fields" session:

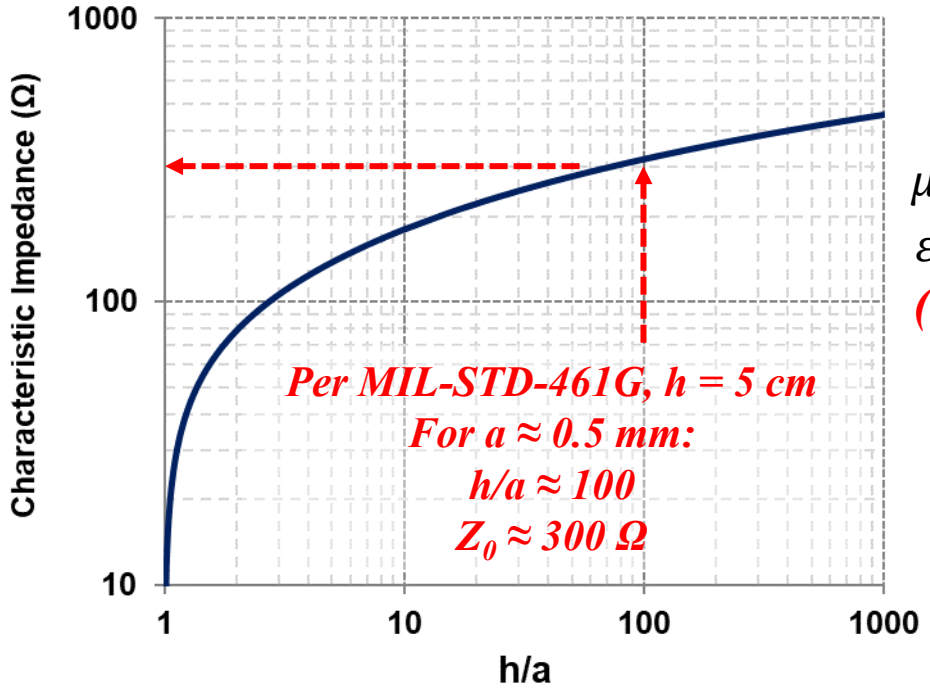
$$L = \frac{\mu}{2\pi} \cdot \ln\left[\left(\frac{h}{a}\right) + \sqrt{\left(\frac{h}{a}\right)^2 - 1}\right]$$

From "Electric Fields" session:

$$C = \frac{2\pi\varepsilon}{\ln\left[\left(\frac{h}{a}\right) + \sqrt{\left(\frac{h}{a}\right)^2 - 1}\right]}$$

$$u = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{c}{\sqrt{\mu_r \varepsilon_r}}$$

3rd time's the charm...



$\mu_r = 1$

$\varepsilon_r = 1$

(air)

$$Z_0 = \sqrt{\frac{L}{C}} = \sqrt{\frac{\mu}{\varepsilon}} \cdot \frac{1}{2\pi} \cdot \ln\left[\left(\frac{h}{a}\right) + \sqrt{\left(\frac{h}{a}\right)^2 - 1}\right]$$

$$= \sqrt{\frac{\mu_r}{\varepsilon_r}} \cdot \underbrace{\sqrt{\frac{\mu_0}{\varepsilon_0}}}_{120\pi} \cdot \frac{1}{2\pi} \cdot \ln\left[\left(\frac{h}{a}\right) + \sqrt{\left(\frac{h}{a}\right)^2 - 1}\right]$$

$$Z_0 = \sqrt{\frac{\mu_r}{\varepsilon_r}} \cdot 60 \ln\left[\left(\frac{h}{a}\right) + \sqrt{\left(\frac{h}{a}\right)^2 - 1}\right]$$

Characteristic impedance of wire-above-ground transmission line

Velocity of Propagation

Velocity of propagation for lossless transmission line:

$$u = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\mu\varepsilon}}$$

Independent of geometry

Depends only on material properties of dielectric

$$= \frac{1}{\sqrt{\mu_r \mu_0 \varepsilon_r \varepsilon_0}} = \frac{1}{\sqrt{\mu_r \varepsilon_r}} \cdot \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

$$u = \frac{c}{\sqrt{\mu_r \varepsilon_r}}$$

For most dielectrics:

$$\mu_r = 1 \qquad u = \frac{c}{\sqrt{\varepsilon_r}} \qquad \frac{u}{c} = \frac{1}{\sqrt{\varepsilon_r}}$$

$$c = 299{,}792{,}458\ m/sec$$
$$c \approx 3 \times 10^8\ m/sec$$
$$\approx 30\ cm/nsec$$
$$\approx 1\ foot/nsec$$

Wavelength:

$$\lambda = \frac{u}{f} = \frac{1}{\sqrt{\varepsilon_r}} \cdot \frac{c}{f} \approx \frac{1}{\sqrt{\varepsilon_r}} \cdot \frac{3 \times 10^8\ m/sec}{f}$$

$$\lambda \approx \frac{1}{\sqrt{\varepsilon_r}} \cdot \frac{300}{f_{MHz}}\ (m)$$

Velocity of Propagation

Velocity of propagation for lossless transmission line:

$$u = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\mu\varepsilon}}$$

Independent of geometry

Depends only on material properties of dielectric

$$= \frac{1}{\sqrt{\mu_r \mu_0 \varepsilon_r \varepsilon_0}} = \frac{1}{\sqrt{\mu_r \varepsilon_r}} \cdot \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

$$\lambda \approx \frac{1}{\sqrt{\varepsilon_r}} \cdot \frac{300}{f_{MHz}} \ (m)$$

For air dielectric (e.g. wire-above-ground):

$$\varepsilon_r = 1 \qquad u = c$$

$$\lambda \approx \frac{300}{f_{MHz}} \ (m)$$

Same as plane wave in free space (air)

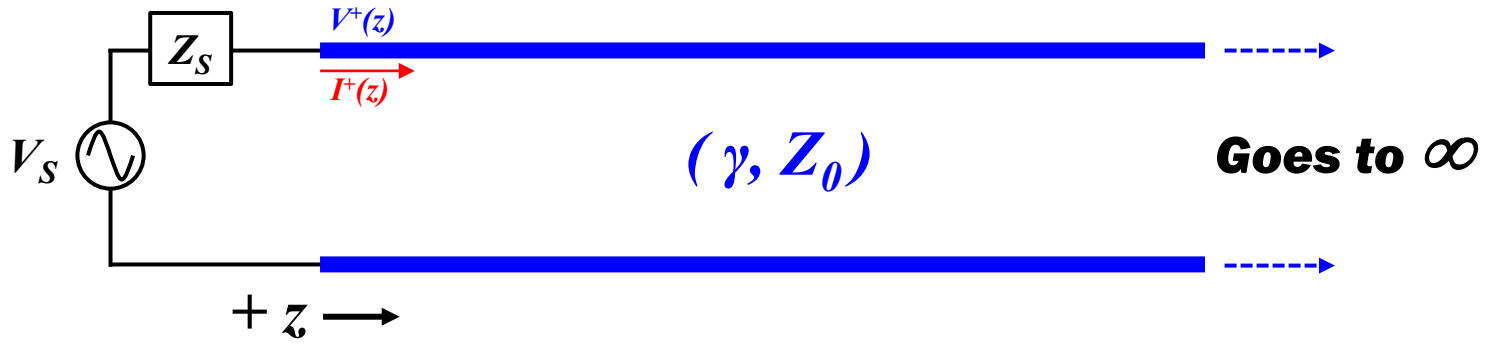
For typical dielectric (e.g Teflon):

$$\varepsilon_r \approx 2 - 2.5 \qquad u \approx \frac{2}{3} c$$

$$\lambda \approx \frac{200}{f_{MHz}} \ (m)$$

Good rule of thumb for many cables

Infinite Transmission Line



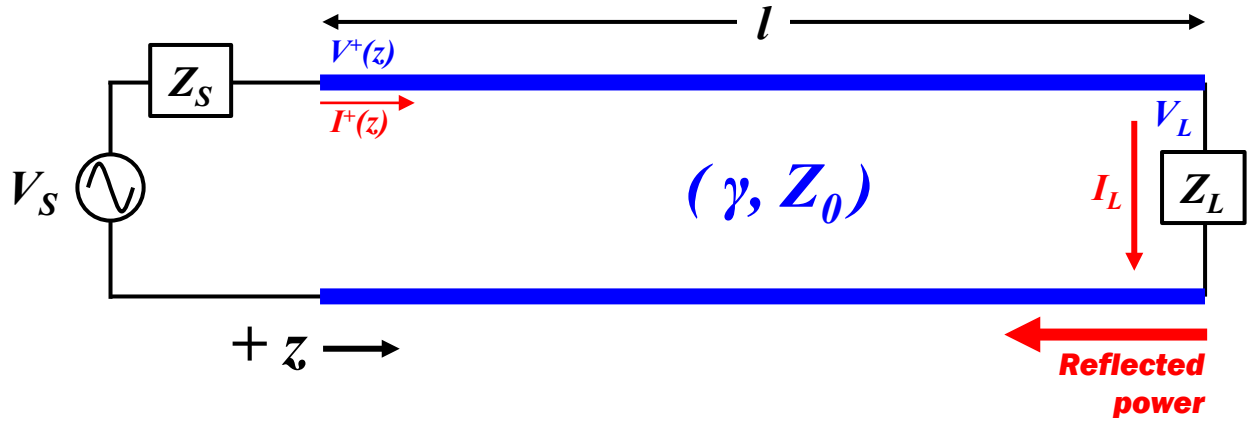
Transmitted voltage and current waves travelling in +z direction:

$$V^+(z) = V_0^+ e^{-\gamma z}$$

$$I^+(z) = I_0^+ e^{-\gamma z}$$

Not terribly useful...

Finite Transmission Line



Real-world transmission line has finite length l and terminates in load impedance Z_L

Transmitted power:

$$P(z) = I^+(z)V^+(z)$$
$$= [I^+(z)]^2 Z_0$$
$$= \frac{[V^+(z)]^2}{Z_0}$$

Load potential:

$$V_L = I_L Z_L$$

Received power (energy) at load:

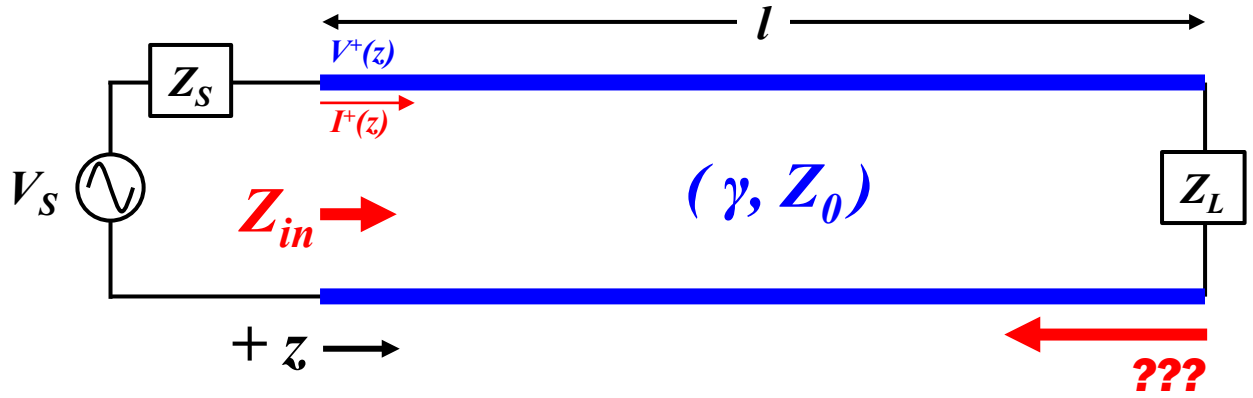
$$P_L = I_L V_L = I_L^2 Z_L = \frac{V_L^2}{Z_L}$$

$Z_L > Z_0$: Can achieve desired V but not I
$Z_L < Z_0$: Can achieve desired I but not V

In either case:

- ***Full transmitted power not transferred to load***
- ***Some power (energy) is reflected back toward source***

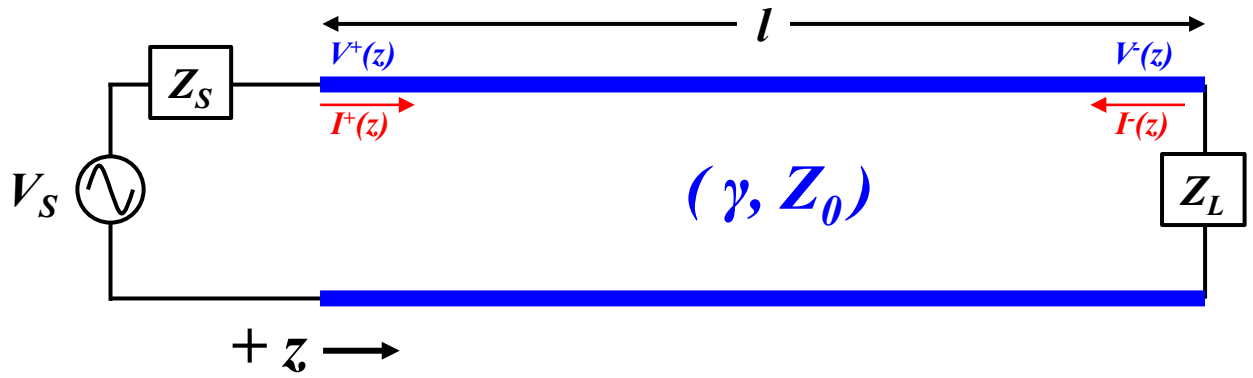
Impedance Matching (or not)



- Terminating with matched impedance ($Z_L = Z_0$) ensures maximum power transfer to load
- For mismatched load ($Z_L \neq Z_0$), we have two (2) basic questions:
  - What fraction of the incident (transmitted) wave is reflected?
  - What is the apparent input impedance Z_{in} observed at a distance from the load?

MATH (stick with me)…

Transmitted and Reflected Waves



Transmitted voltage and current waves travelling in +z direction:

$$V^+(z) = V_0^+ e^{-\gamma z}$$

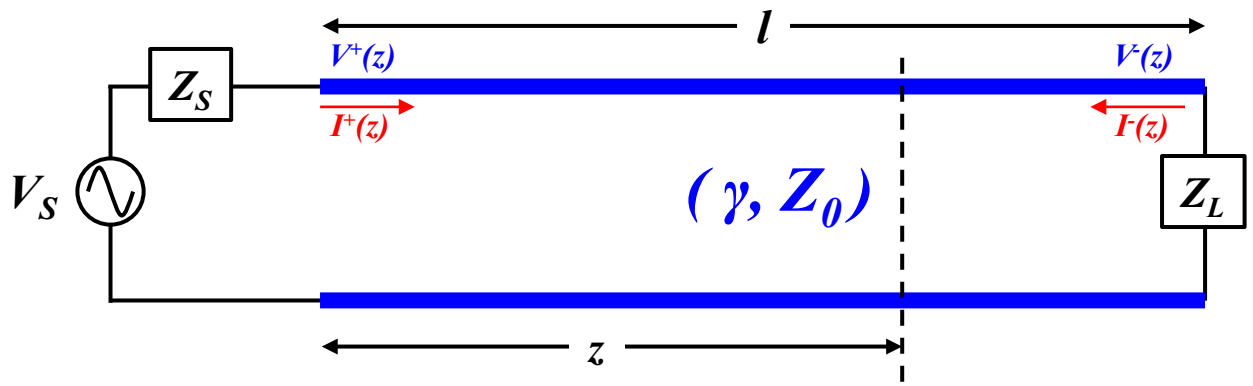
$$I^+(z) = I_0^+ e^{-\gamma z}$$

Reflected voltage and current waves travelling in -z direction:

$$V^-(z) = V_0^- e^{\gamma z}$$

$$I^-(z) = I_0^- e^{\gamma z}$$

Transmitted and Reflected Waves (cont.)



Total transmitted and reflected potential at distance z from source:

$$V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}$$

$$-\frac{dV(z)}{dz} = I(z) \cdot (R + j\omega L)$$

$$\gamma V_0^+ e^{-\gamma z} - \gamma V_0^- e^{\gamma z} = (I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z}) \cdot (R + j\omega L)$$

$$= I_0^+ (R+j\omega L) e^{-\gamma z} + I_0^- (R + j\omega L) e^{\gamma z}$$

Recall

Total transmitted and reflected current at distance z from source:

$$I(z) = I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z}$$

$$-\frac{dI(z)}{dz} = V(z) \cdot (G + j\omega C)$$

$$\gamma I_0^+ e^{-\gamma z} - \gamma I_0^- e^{\gamma z} = (V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}) \cdot (G + j\omega C)$$

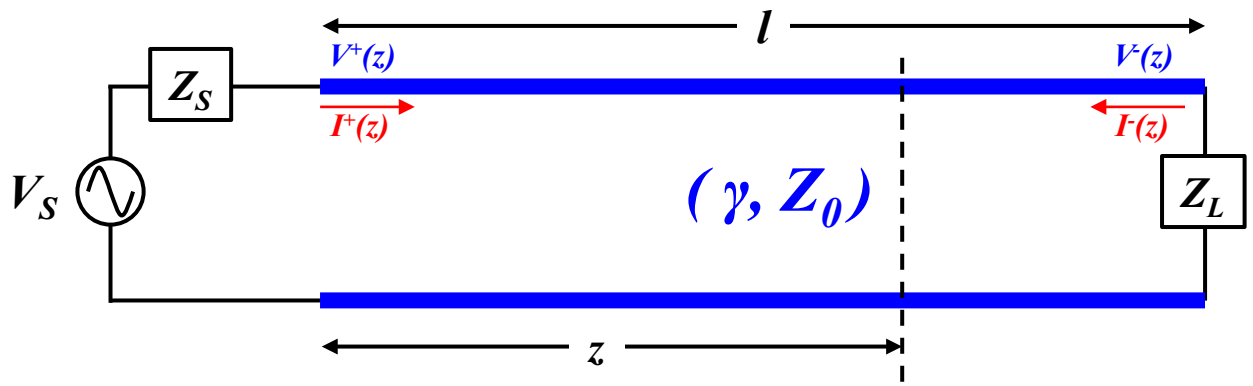
$$= V_0^+ (G + j\omega C) e^{-\gamma z} + V_0^- (G+j\omega C) e^{\gamma z}$$

$$\gamma V_0^+ = I_0^+ (R+j\omega L) \qquad \gamma V_0^- = I_0^- (R+j\omega L)$$

$$\frac{V_0^+}{I_0^+} = \frac{R + j\omega L}{\gamma} = Z_0 \qquad -\frac{V_0^-}{I_0^-} = \frac{R + j\omega L}{\gamma} = Z_0$$

$$\frac{V_0^+}{I_0^+} = \frac{\gamma}{G + j\omega C} = Z_0 \qquad -\frac{V_0^-}{I_0^-} = \frac{\gamma}{G + j\omega C} = Z_0$$

Transmitted and Reflected Waves (cont.)



Total transmitted and reflected potential at distance z from source:

$$V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}$$

$$-\frac{dV(z)}{dz} = I(z) \cdot (R + j\omega L)$$

$$\gamma V_0^+ e^{-\gamma z} - \gamma V_0^- e^{\gamma z} = (I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z}) \cdot (R + j\omega L)$$

$$= I_0^+ (R + j\omega L) e^{-\gamma z} + I_0^- (R + j\omega L) e^{\gamma z}$$

Recall

Total transmitted and reflected current at distance z from source:

$$I(z) = I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z}$$

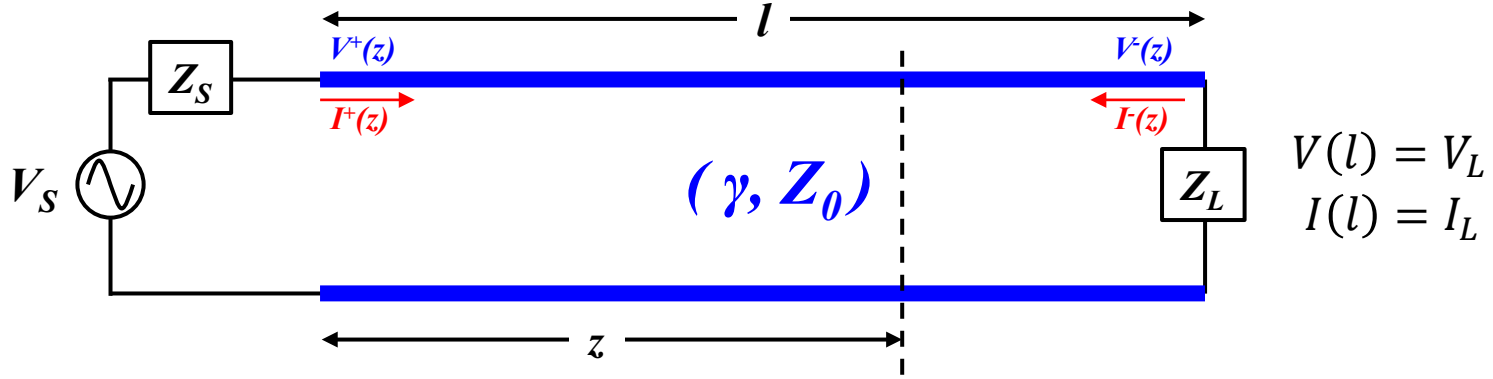
$$-\frac{dI(z)}{dz} = V(z) \cdot (G + j\omega C)$$

$$\gamma I_0^+ e^{-\gamma z} - \gamma I_0^- e^{\gamma z} = (V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}) \cdot (G + j\omega C)$$

$$= V_0^+ (G + j\omega C) e^{-\gamma z} + V_0^- (G + j\omega C) e^{\gamma z}$$

$$\frac{V_0^+}{I_0^+} = -\frac{V_0^-}{I_0^-} = Z_0$$

Transmitted and Reflected Waves (cont.)



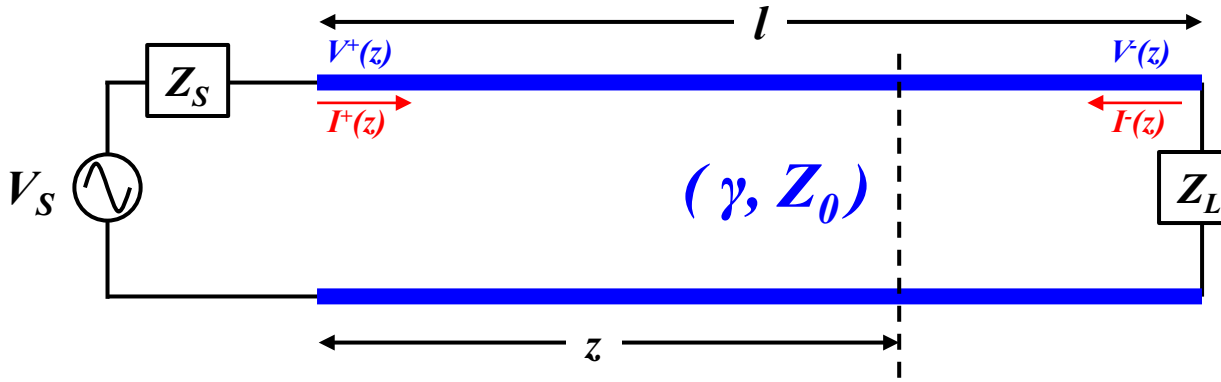
$$V(l) = V_L$$
$$I(l) = I_L$$

$$V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}$$
$$V(l) = V_L = V_0^+ e^{-\gamma l} + V_0^- e^{\gamma l}$$

$$\frac{V_0^+}{I_0^+} = -\frac{V_0^-}{I_0^-} = Z_0$$
$$I_0^+ = \frac{V_0^+}{Z_0} \qquad I_0^- = \frac{V_0^-}{Z_0}$$

$$I(z) = I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z}$$
$$I(z) = \frac{V_0^+}{Z_0} e^{-\gamma z} - \frac{V_0^-}{Z_0} e^{\gamma z}$$
$$I(l) = I_L = \frac{V_0^+}{Z_0} e^{-\gamma l} - \frac{V_0^-}{Z_0} e^{\gamma l}$$
$$I_L Z_0 = V_0^+ e^{-\gamma l} - V_0^- e^{\gamma l}$$

Transmitted and Reflected Waves (cont.)



$$V(l) = V_L$$
$$I(l) = I_L$$
$$\frac{V_L}{I_L} = Z_L$$

$$V_L = V_0^+ e^{-\gamma l} + V_0^- e^{\gamma l}$$
$$I_L Z_0 = V_0^+ e^{-\gamma l} - V_0^- e^{\gamma l}$$

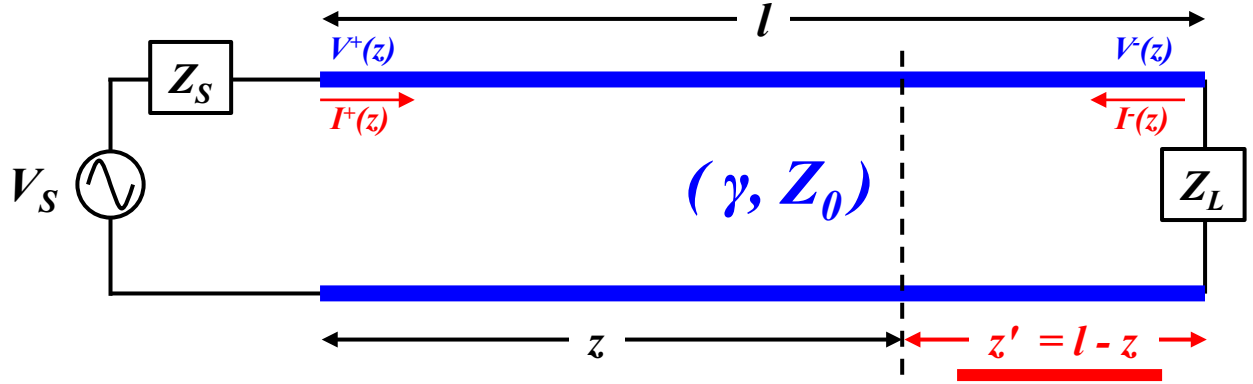
(+)

$$V_L + I_L Z_0 = 2V_0^+ e^{-\gamma l}$$
$$V_0^+ = \tfrac{1}{2}(V_L + I_L Z_0) e^{\gamma l}$$
$$V_0^+ = \frac{I_L}{2}\left(\frac{V_L}{I_L} + Z_0\right) e^{\gamma l}$$
$$V_0^+ = \frac{I_L}{2}(Z_L + Z_0) e^{\gamma l}$$

(-)

$$V_L - I_L Z_0 = 2V_0^- e^{\gamma l}$$
$$V_0^- = \tfrac{1}{2}(V_L - I_L Z_0) e^{-\gamma l}$$
$$V_0^- = \frac{I_L}{2}\left(\frac{V_L}{I_L} - Z_0\right) e^{-\gamma l}$$
$$V_0^- = \frac{I_L}{2}(Z_L - Z_0) e^{-\gamma l}$$

Transmitted and Reflected Waves (cont.)



$$V_0^+ = \frac{I_L}{2}(Z_L + Z_0)e^{\gamma l} \qquad V_0^- = \frac{I_L}{2}(Z_L - Z_0)e^{-\gamma l}$$

$$I_0^+ = \frac{V_0^+}{Z_0} \qquad I_0^- = \frac{V_0^-}{Z_0}$$

$$V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}$$

$$I(z) = I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z}$$

$$V(z) = \frac{I_L}{2}\left[(Z_L + Z_0)e^{\gamma(l-z)} + (Z_L - Z_0)e^{-\gamma(l-z)}\right]$$

$$I(z) = \frac{I_L}{2Z_0}\left[(Z_L + Z_0)e^{\gamma(l-z)} - (Z_L - Z_0)e^{-\gamma(l-z)}\right]$$

$z' = l - z$ (distance from load)

$$V(z') = \frac{I_L}{2}\left[(Z_L + Z_0)e^{\gamma z'} + (Z_L - Z_0)e^{-\gamma z'}\right]$$

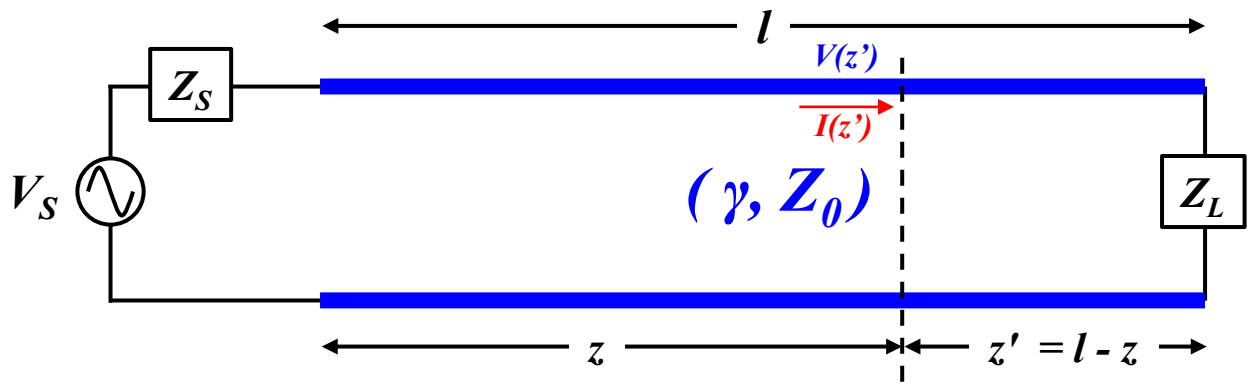
We'll come back to these...

$$I(z') = \frac{I_L}{2Z_0}\left[(Z_L + Z_0)e^{\gamma z'} - (Z_L - Z_0)e^{-\gamma z'}\right]$$

$$V(z') = \frac{I_L}{2}\underbrace{(Z_L + Z_0)e^{\gamma z'}}_{\text{Factored out}}\left[1 + \frac{Z_L - Z_0}{Z_L + Z_0}e^{-2\gamma z'}\right]$$

$$I(z') = \frac{I_L}{2Z_0}\underbrace{(Z_L + Z_0)e^{\gamma z'}}_{\text{Factored out}}\left[1 - \frac{Z_L - Z_0}{Z_L + Z_0}e^{-2\gamma z'}\right]$$

Reflection Coefficients



$$V(z') = \frac{I_L}{2}(Z_L + Z_0)e^{\gamma z'}\left[1 + \frac{Z_L - Z_0}{Z_L + Z_0}e^{-2\gamma z'}\right]$$

$$I(z') = \frac{I_L}{2Z_0}(Z_L + Z_0)e^{\gamma z'}\left[1 - \frac{Z_L - Z_0}{Z_L + Z_0}e^{-2\gamma z'}\right]$$

Voltage reflection coefficient Load end $(\Gamma_L)_V = \dfrac{Z_L - Z_0}{Z_L + Z_0}$

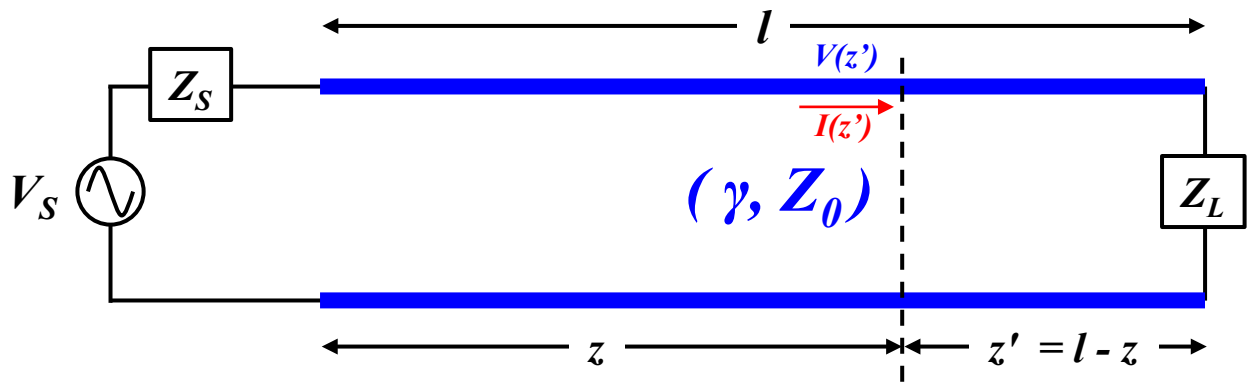
Current reflection coefficient Load end $(\Gamma_L)_I = -\dfrac{Z_L - Z_0}{Z_L + Z_0}$

Similarly:

Voltage reflection coefficient Source end $(\Gamma_S)_V = \dfrac{Z_S - Z_0}{Z_S + Z_0}$

Current reflection coefficient Load end $(\Gamma_S)_I = -\dfrac{Z_S - Z_0}{Z_L + Z_0}$

Reflection Coefficient Examples



Voltage reflection coefficient Load end

$$(\Gamma_L)_V = \frac{Z_L - Z_0}{Z_L + Z_0}$$

$Z_L = 0\ (short)$: $(\Gamma_L)_V = -1$ $V_L = 0$

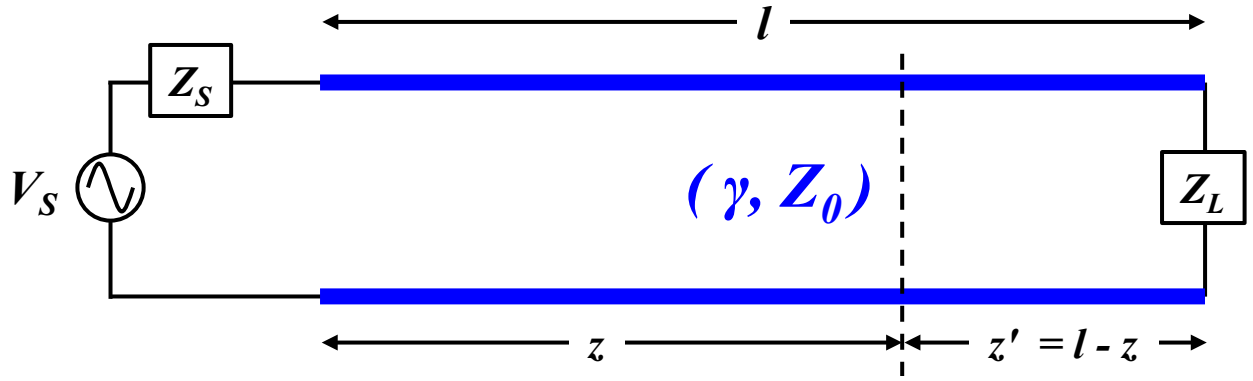
Reflected wave cancels transmitted potential at shorted load

$Z_L = \infty\ (open)$: $(\Gamma_L)_V = 1$ $V_L = 2V_0^+$

Reflected wave doubles transmitted potential at open load

(Opposite relationship for current reflection coefficients)

Voltage Standing Wave Ratio (VSWR, a.k.a. "Vizz-Whirr")

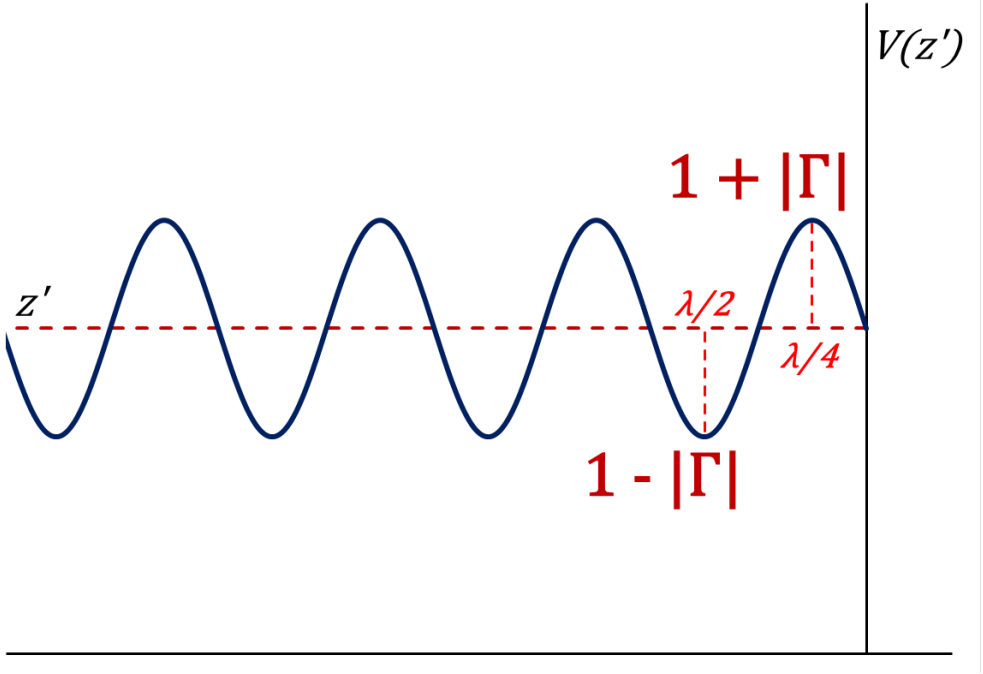


General:

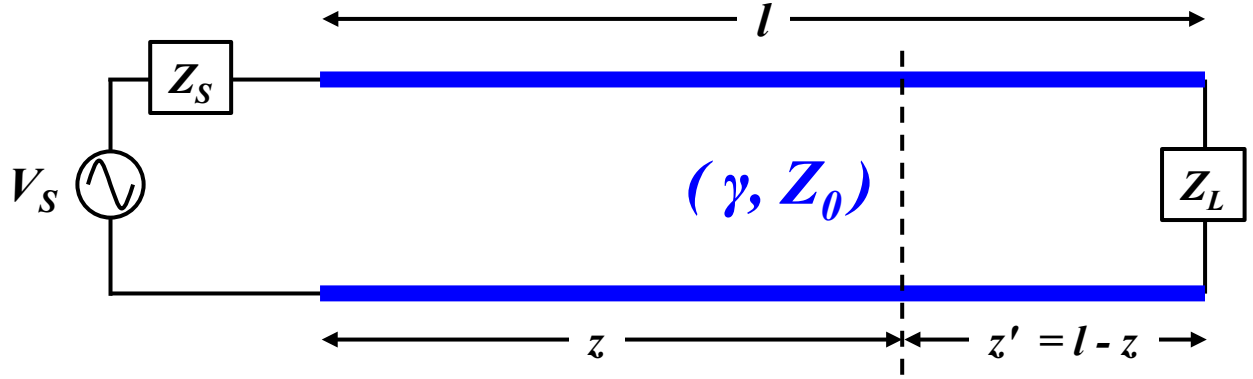
$$\Gamma = |\Gamma| e^{j\theta}$$

Voltage Standing Wave Ratio:

$$VSWR = \frac{1 + |\Gamma|}{1 - |\Gamma|}$$



Vizz-Whirr (cont.)



$$VSWR = \frac{1 + |\Gamma|}{1 - |\Gamma|}$$

$$VSWR(1 - |\Gamma|) = 1 + |\Gamma|$$

$$VSWR - |\Gamma| \cdot VSWR = 1 + |\Gamma|$$

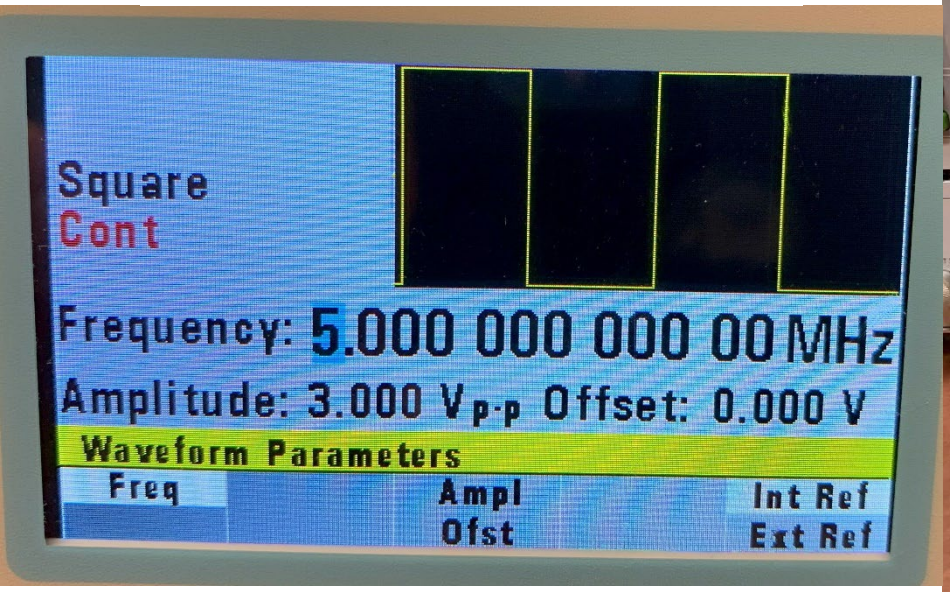
$$VSWR - 1 = |\Gamma| \cdot VSWR + |\Gamma|$$

$$VSWR - 1 = |\Gamma| \cdot (VSWR + 1)$$

$$|\Gamma| = \frac{VSWR - 1}{VSWR + 1}$$

Virtual Demonstration #1: Impedance Mismatching and Reflections

5 MHz square wave
3 Vp-p (into 50 Ω load)
Rise/fall times ≈ 4 nsec

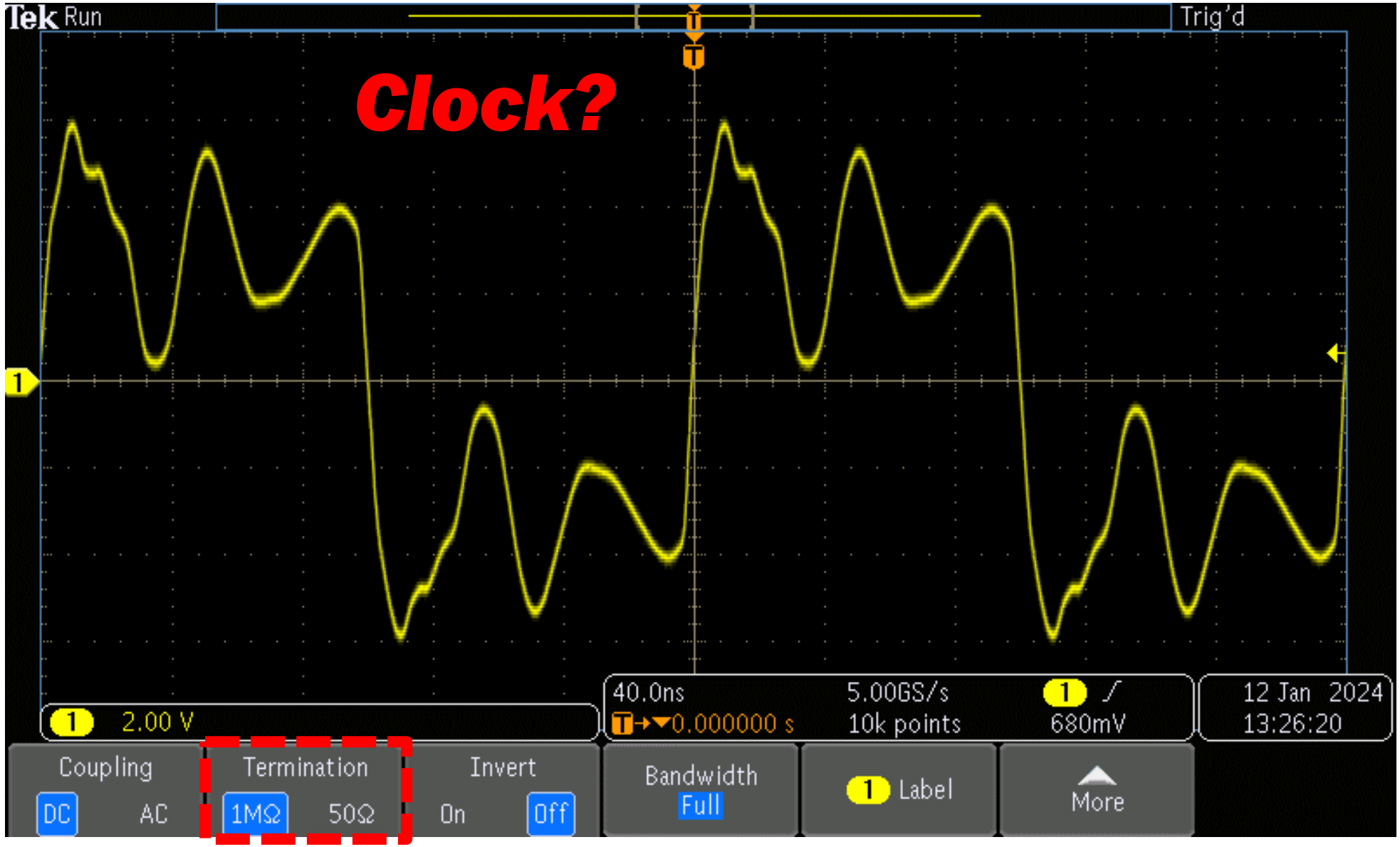


Oscilloscope

Signal generator

Uncontrolled transmission line

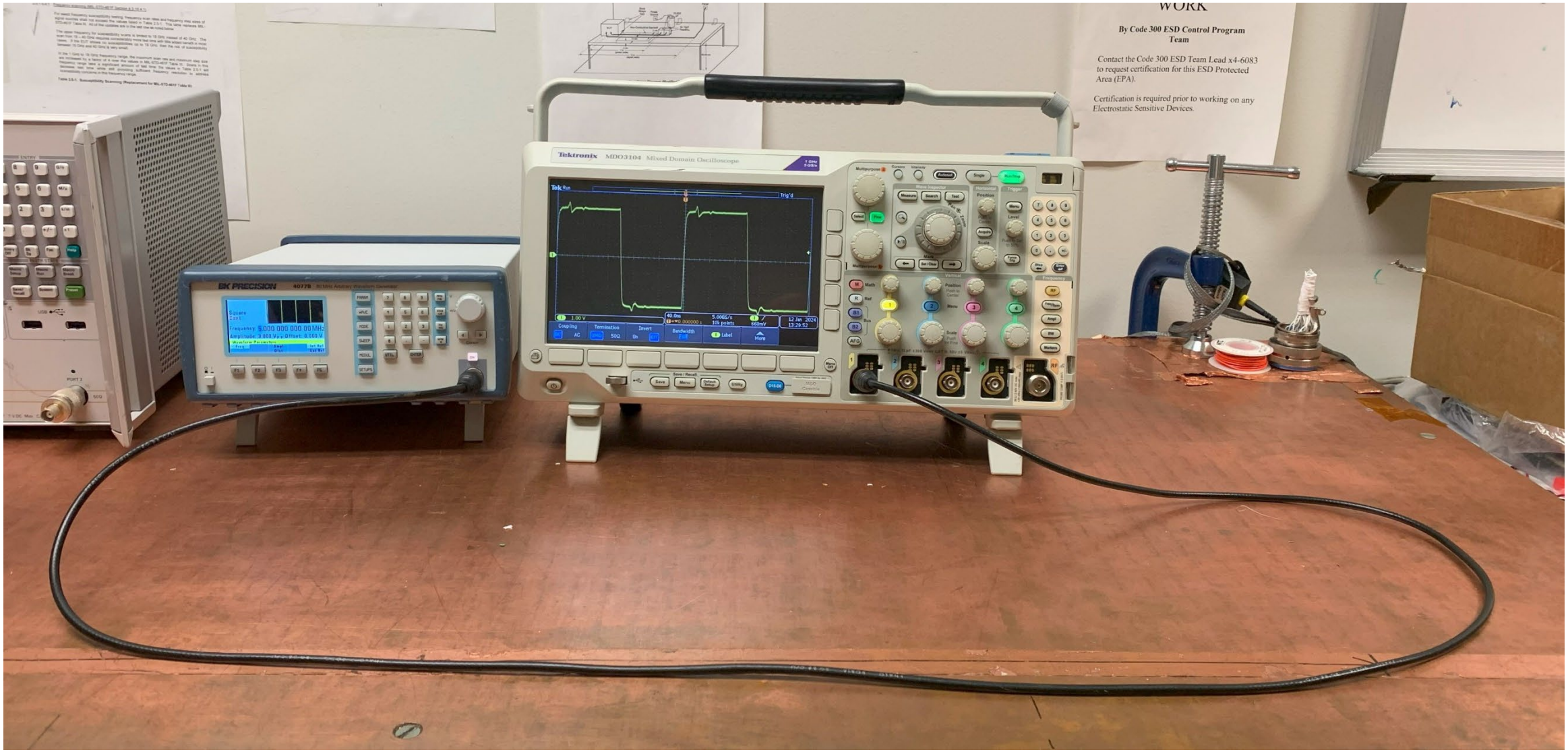
Uncontrolled Transmission Line – 1 MΩ Termination



Uncontrolled Transmission Line – 50 Ω Termination



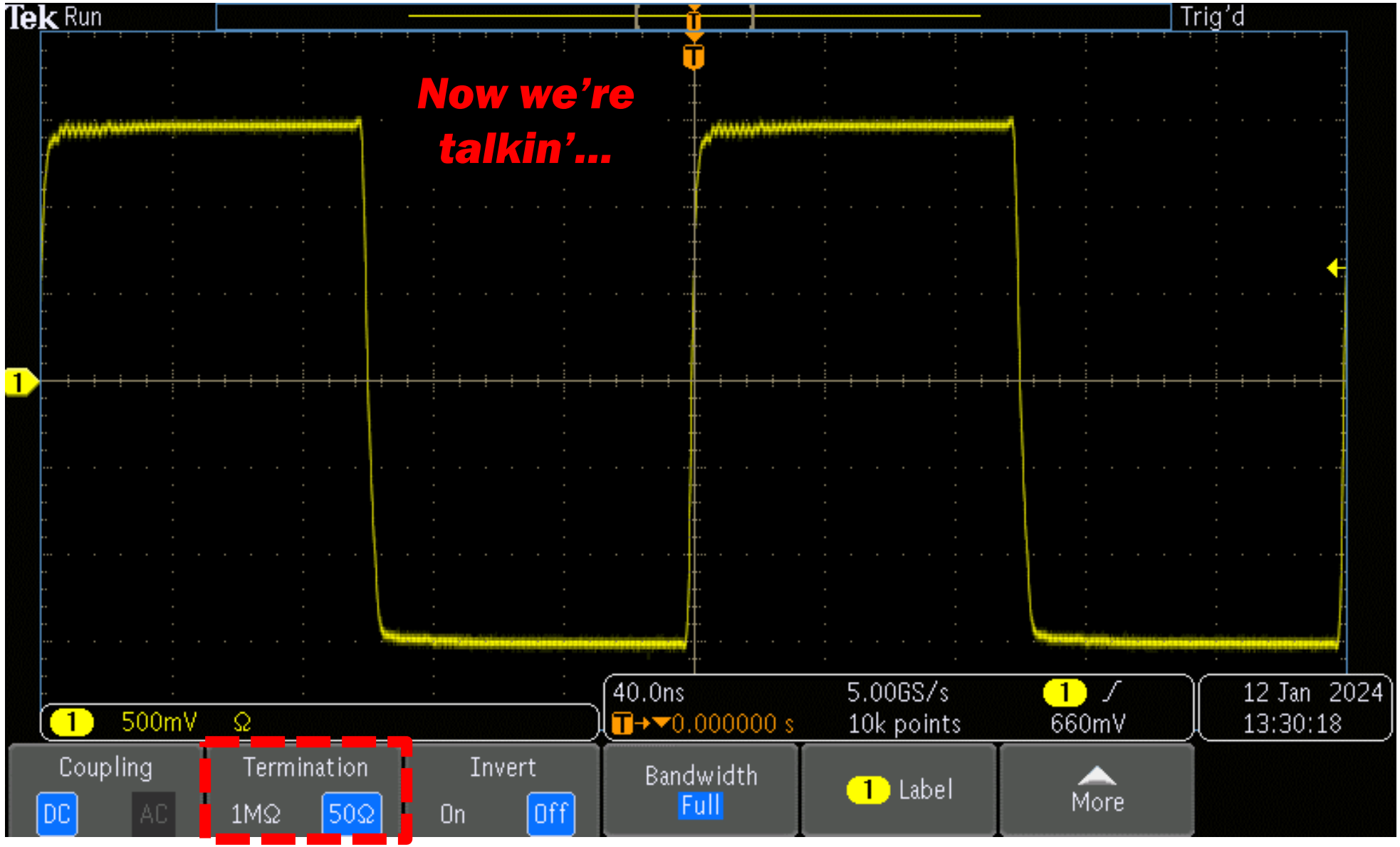
RG-58 Coaxial Cable (Z_0 = 50 Ω)



RG-58 Coaxial Cable – 1 MΩ Termination



RG-58 Coaxial Cable – 50 Ω Termination

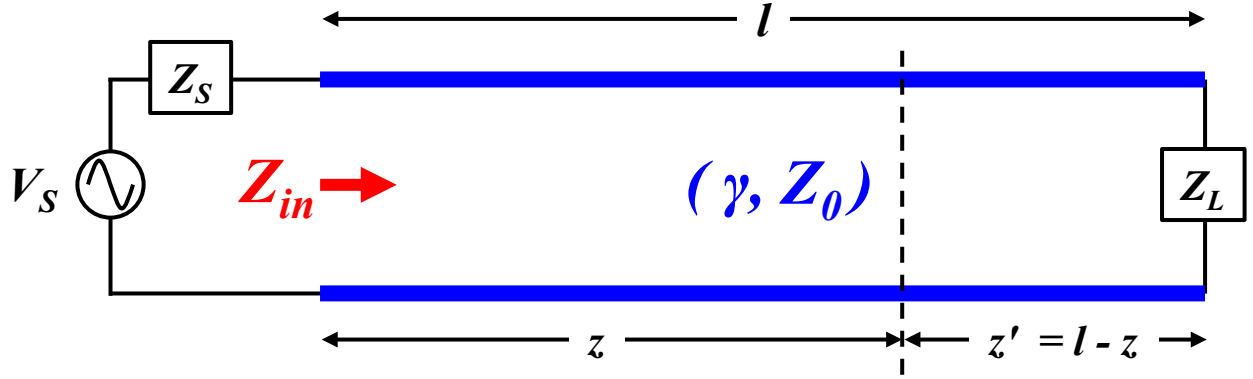




Impedance Mismatching and Reflections Summary

- **Impedance mismatches cause reflections and distortions of desired waveform**
- **Reflections can occur both at load and source ends**
- **In digital systems:**
  - Can cause bit errors and spurious clock transitions
  - Can completely freeze state machine
  - Rise/fall times much greater concern than base frequency
- **For interconnections whose length is significant fraction of wavelength ($l > 0.1\lambda$):**
  - Connections behave as transmission lines
  - Use controlled characteristic impedance Z_0
  - Terminate with matched impedance

Transmitted and Reflected Waves (cont.)



$$V(z') = \frac{I_L}{2}[(Z_L + Z_0)e^{\gamma z'} + (Z_L - Z_0)e^{-\gamma z'}]$$ ← **Getting back to these...** → $$I(z') = \frac{I_L}{2Z_0}[(Z_L + Z_0)e^{\gamma z'} - (Z_L - Z_0)e^{-\gamma z'}]$$

$$V(z') = \frac{I_L}{2}[Z_L(e^{\gamma z'} + e^{-\gamma z'}) + Z_0(e^{\gamma z'} - e^{-\gamma z'})]$$ **Vas dis?** $$I(z') = \frac{I_L}{2Z_0}[Z_L(e^{\gamma z'} - e^{-\gamma z'}) + Z_0(e^{\gamma z'} + e^{-\gamma z'})]$$

Recall hyperbolic trig functions: $\cosh x = \frac{1}{2}(e^x + e^{-x})$ $\sinh x = \frac{1}{2}(e^x - e^{-x})$ $\tanh x = \frac{\sinh x}{\cosh x}$

$$V(z') = I_L[Z_L \cosh(\gamma z') + Z_0 \sinh(\gamma z')]$$

$$I(z') = \frac{I_L}{Z_0}[Z_L \sinh(\gamma z') + Z_0 \cosh(\gamma z')]$$

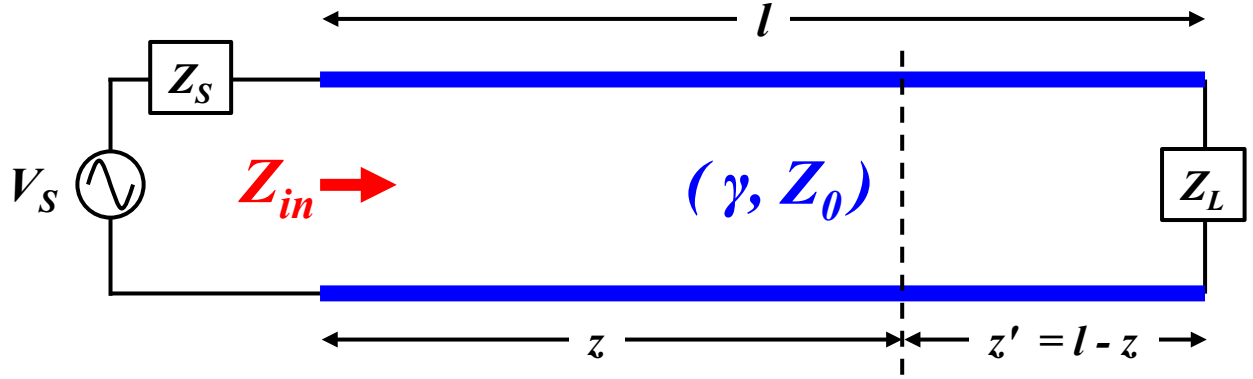
$$Z(z') = \frac{V(z')}{I(z')} = Z_0 \frac{Z_L \cosh(\gamma z') + Z_0 \sinh(\gamma z')}{Z_L \sinh(\gamma z') + Z_0 \cosh(\gamma z')}$$

$$Z(z') = Z_0 \frac{Z_L + Z_0 \tanh(\gamma z')}{Z_0 + Z_L \tanh(\gamma z')}$$

$$Z_{in} = Z(l) = Z_0 \frac{Z_L + Z_0 \tanh(\gamma l)}{Z_0 + Z_L \tanh(\gamma l)}$$

INPUT IMPEDANCE OF LOSSY TRANSMISSION LINE

Input Impedance



$$Z(z') = Z_0 \frac{Z_L + Z_0 \tanh(\gamma z')}{Z_0 + Z_L \tanh(\gamma z')}$$

$$\gamma = \alpha + j\beta$$

$$\cosh(\gamma z') = \cosh(j\beta z') = \tfrac{1}{2}\left(e^{j\beta z'} + e^{-j\beta z'}\right) = \cos(\beta z')$$

$$\sinh(\gamma z') = \sinh(j\beta z') = \tfrac{1}{2}\left(e^{j\beta z'} - e^{-j\beta z'}\right) = j\sin(\beta z')$$

Lossless line:

$$\alpha = 0$$

$$\gamma = j\beta$$

$$\tanh(\gamma z') = \tanh(j\beta z') = \frac{\sinh(j\beta z')}{\cosh(j\beta z')} = \frac{j\sin(\beta z')}{\cos(\beta z')} = j\tan(\beta z')$$

$$Z(z') = Z_0 \frac{Z_L + jZ_0 \tan(\beta z')}{Z_0 + jZ_L \tan(\beta z')}$$

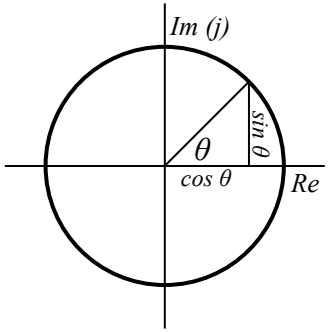
Standard trig identities:

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$e^{-j\theta} = \cos\theta - j\sin\theta$$

$$\cos\theta = \tfrac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right)$$

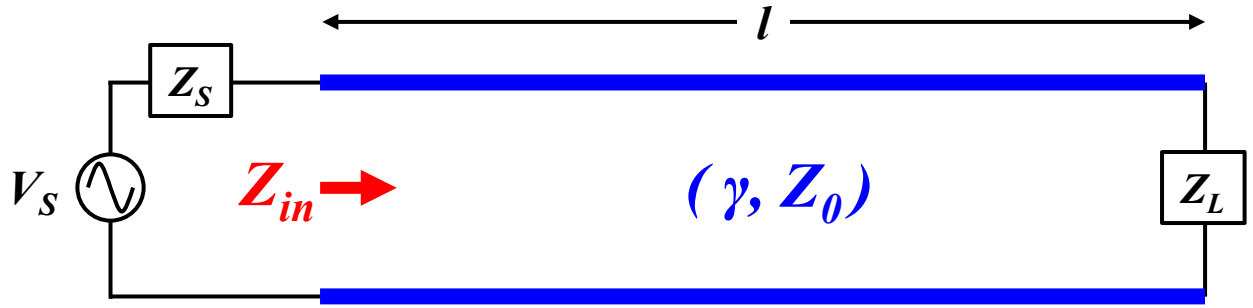
$$j\sin\theta = \tfrac{1}{2}\left(e^{j\theta} - e^{-j\theta}\right)$$



$$Z_{in} = Z_0 \frac{Z_L + jZ_0 \tan(\beta l)}{Z_0 + jZ_L \tan(\beta l)}$$

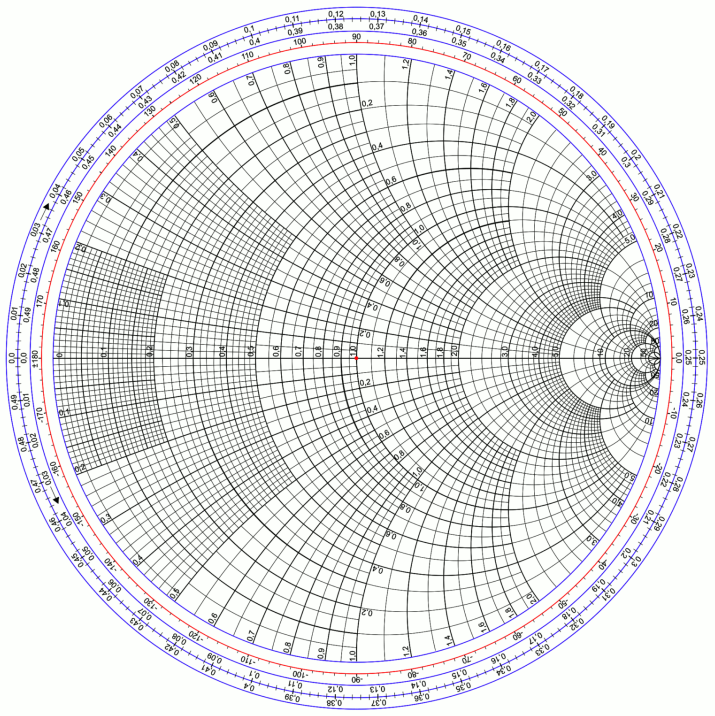
INPUT IMPEDANCE OF LOSSLESS TRANSMISSION LINE

Input Impedance (cont.)

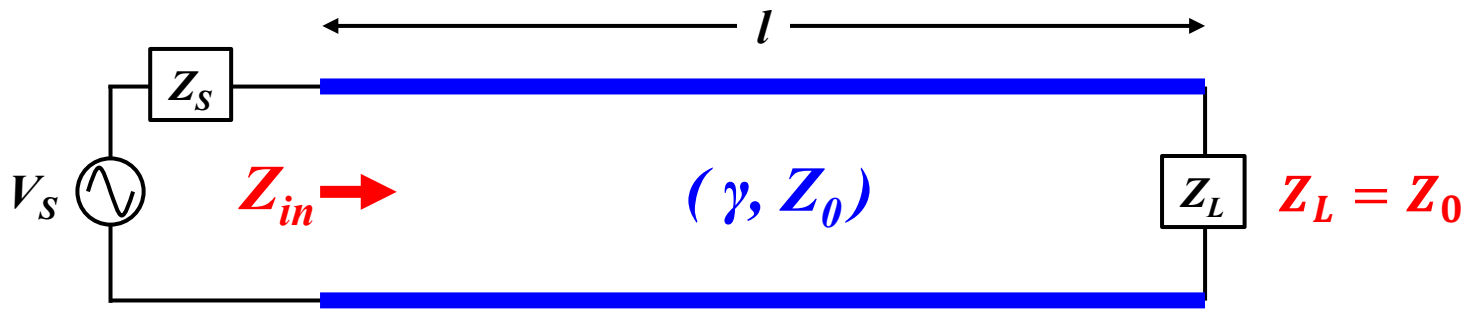


$$Z_{in} = Z_0 \frac{Z_L + jZ_0 \tan(\beta l)}{Z_0 + jZ_L \tan(\beta l)}$$

Basis of Smith chart



Input Impedance (cont.)



$$Z_{in} = Z_0 \frac{Z_L + jZ_0 \tan(\beta l)}{Z_0 + jZ_L \tan(\beta l)}$$

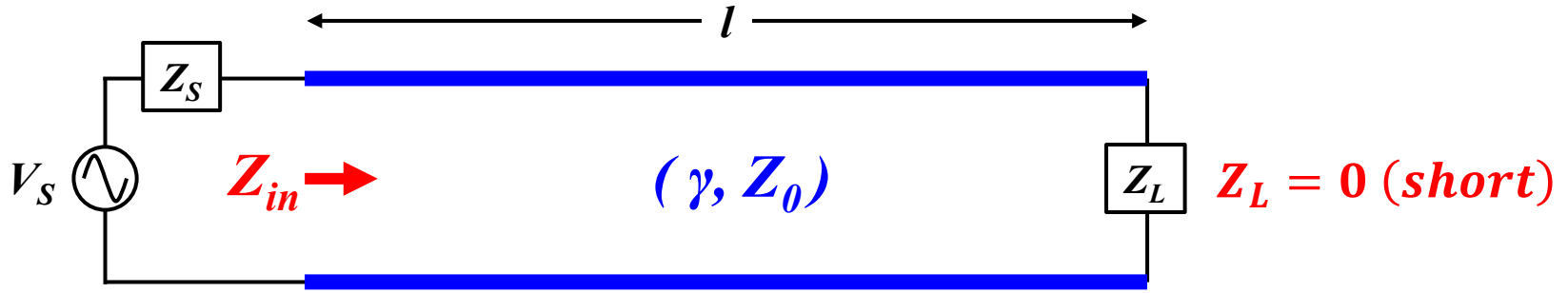
$Z_L = Z_0$:

$$Z_{in} = Z_0 \frac{Z_0 + jZ_0 \tan(\beta l)}{Z_0 + jZ_0 \tan(\beta l)}$$

$$\beta = \frac{2\pi}{\lambda} = \omega\sqrt{LC}$$

$$Z_{in} = Z_0$$

Input Impedance (cont.)



$$Z_{in} = Z_0 \frac{\cancel{Z_L} + jZ_0 \tan(\beta l)}{Z_0 + \cancel{jZ_L \tan(\beta l)}}$$

$$\beta = \frac{2\pi}{\lambda} = \omega\sqrt{LC}$$

$Z_L = 0\ (short)$: $\quad Z_{sc} = jZ_0 \tan(\beta l)$

For $l << \lambda$ ("electrically short"):

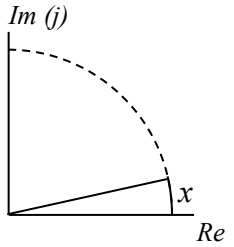
$$Z_{sc} \approx jZ_0\beta l$$

$$Z_{sc} \approx j\sqrt{\frac{L}{C}} \cdot \omega\sqrt{LC} \cdot l \qquad Z_0 = \sqrt{\frac{L}{C}}$$

$$Z_{sc} \approx j\omega L l$$

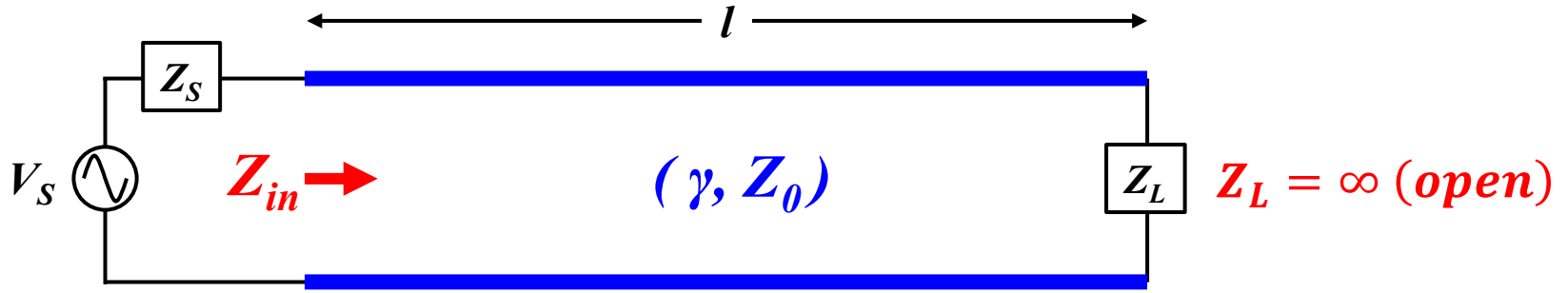
L = inductance per unit length

"Short-circuit" input impedance equals total inductive reactance



For small x:
$\tan x \approx x$

Input Impedance (cont.)



$$Z_{in} = Z_0 \frac{Z_L + \cancel{jZ_0 \tan(\beta l)}}{\cancel{Z_0} + jZ_L \tan(\beta l)}$$

$$\beta = \frac{2\pi}{\lambda} = \omega\sqrt{LC}$$

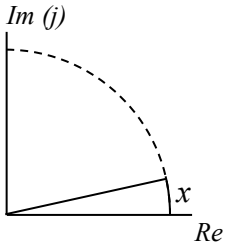
$Z_L = \infty\ (open)$: $\quad Z_{oc} = \dfrac{Z_0}{j\tan(\beta l)}$

For $l << \lambda$ ("electrically short"):

$$Z_{oc} \approx \frac{Z_0}{j\beta l} \qquad Z_0 = \sqrt{\frac{L}{C}}$$

$$Z_{oc} \approx \sqrt{\frac{L}{C}} \cdot \frac{1}{j\omega\sqrt{LC} \cdot l}$$

$$Z_{oc} \approx \frac{1}{j\omega C l}$$

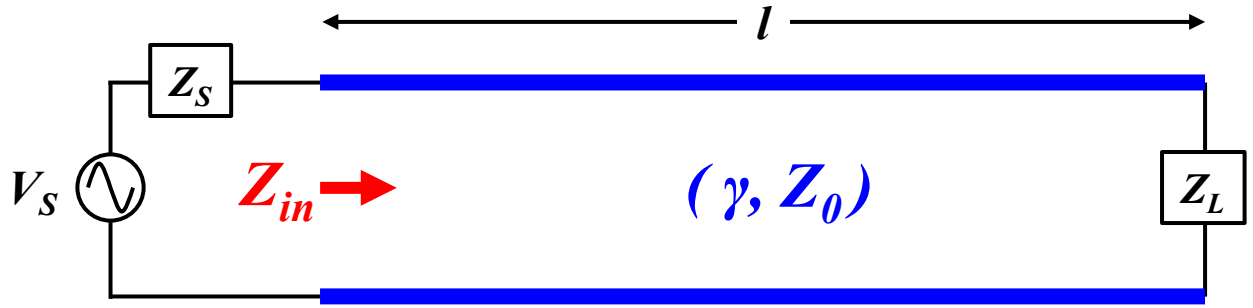


For small x:
$\tan x \approx x$

C = capacitance per unit length

"Open-circuit" input impedance equals total capacitive reactance

Input Impedance (cont.)



$$Z_{in} = Z_0 \frac{Z_L + jZ_0 \tan(\beta l)}{Z_0 + jZ_L \tan(\beta l)} \qquad Z_{sc} = jZ_0 \tan(\beta l) \qquad Z_{oc} = \frac{Z_0}{j \tan(\beta l)}$$

$$\beta = \frac{2\pi}{\lambda} = \omega\sqrt{LC} \qquad \sqrt{Z_{sc} Z_{oc}} = Z_0$$

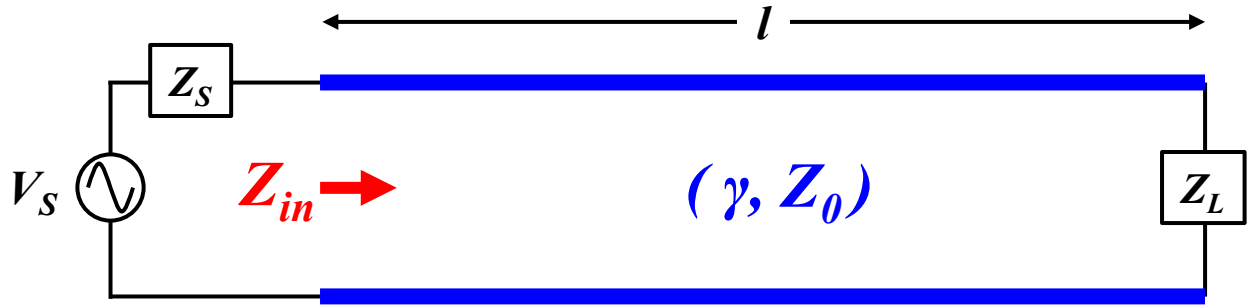
Z_0 can be obtained from measurements of Z_{SC} and Z_{OC}

For $l \ll \lambda$ ("electrically short"):

$$Z_{sc} \approx j\omega L l \qquad Z_{oc} \approx \frac{1}{j\omega C l} \qquad \sqrt{Z_{sc} Z_{oc}} = \sqrt{\frac{L}{C}}$$

The pieces all fit together (as they must…)

Input Impedance (cont.)

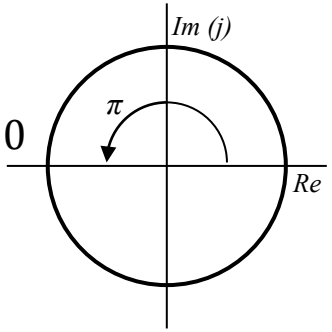


$$Z_{in} = Z_0 \frac{Z_L + jZ_0 \tan(\beta l)}{Z_0 + jZ_L \tan(\beta l)}$$

$$\beta = \frac{2\pi}{\lambda} = \omega\sqrt{LC}$$

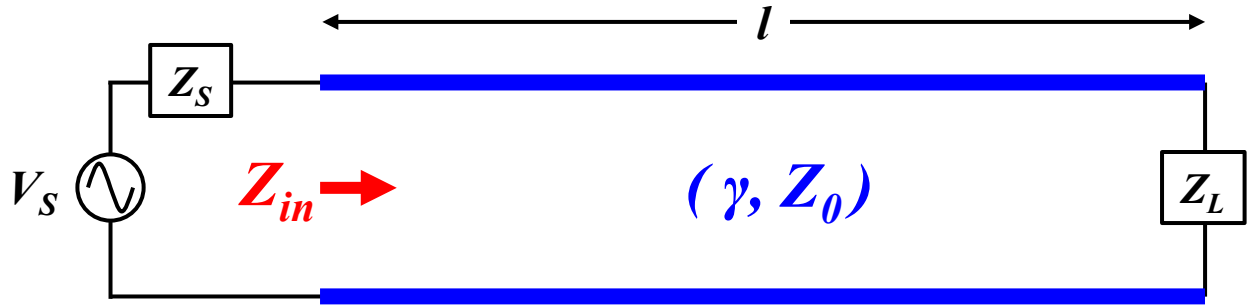
$$l = \frac{\lambda}{2}: \quad Z_{in} = \cancel{Z_0} \frac{Z_L + \cancel{jZ_0 \tan(\pi)}}{\cancel{Z_0} + \cancel{jZ_L \tan(\pi)}}$$

$$\tan(\pi) = 0$$



$$Z_{in} = Z_L$$ *Also true for l = multiples of λ/2*

Input Impedance (cont.)

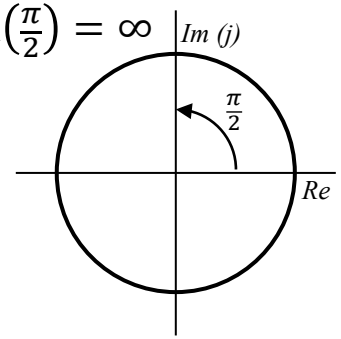


$$Z_{in} = Z_0 \frac{Z_L + jZ_0 \tan(\beta l)}{Z_0 + jZ_L \tan(\beta l)}$$

$$\beta = \frac{2\pi}{\lambda} = \omega\sqrt{LC}$$

$$l = \frac{\lambda}{4}: \quad Z_{in} = Z_0 \frac{\cancel{Z_L} + \cancel{jZ_0} \cancel{\tan\left(\frac{\pi}{2}\right)}}{\cancel{Z_0} + \cancel{jZ_L} \cancel{\tan\left(\frac{\pi}{2}\right)}}$$

$\tan\left(\frac{\pi}{2}\right) = \infty$



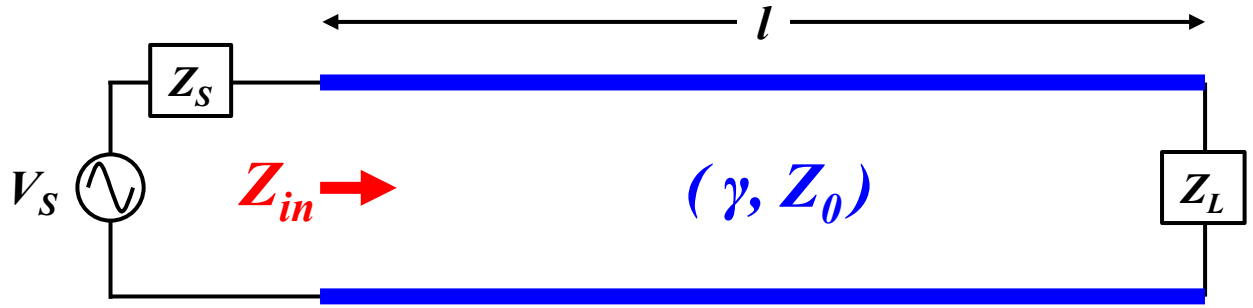
$$Z_{in} = \frac{Z_0^2}{Z_L}$$

Z_L = 0 (short) → Z_{in} = ∞ (open)

Z_L = ∞ (open) → Z_{in} = 0 (short)

Also true for l = odd multiples of λ/4

Input Impedance (cont.)

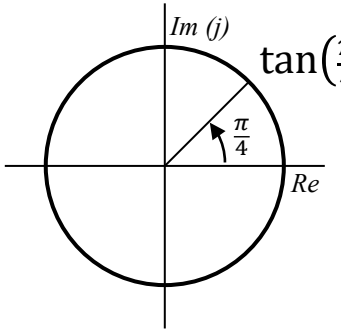


$$Z_{in} = Z_0 \frac{Z_L + jZ_0 \tan(\beta l)}{Z_0 + jZ_L \tan(\beta l)}$$

$$\beta = \frac{2\pi}{\lambda} = \omega\sqrt{LC}$$

$$l = \frac{\lambda}{8}: \quad Z_{in} = Z_0 \frac{Z_L + jZ_0 \tan\left(\frac{\pi}{4}\right)}{Z_0 + jZ_L \tan\left(\frac{\pi}{4}\right)}$$

$$Z_{in} = Z_0 \frac{Z_L + jZ_0}{Z_0 + jZ_L} \qquad |Z_{in}| = Z_0 \frac{\sqrt{Z_L^2 + Z_0^2}}{\sqrt{Z_0^2 + Z_L^2}}$$



$$\tan\left(\frac{\pi}{4}\right) = 1$$

$$|Z_{in}| = Z_0$$ ***Independent of Z_L***

Also true for l = odd multiples of λ/8

Input Impedance – Shorted Termination

$$Z_{in} = Z_0 \frac{Z_L + jZ_0 \tan(\beta l)}{Z_0 + jZ_L \tan(\beta l)}$$

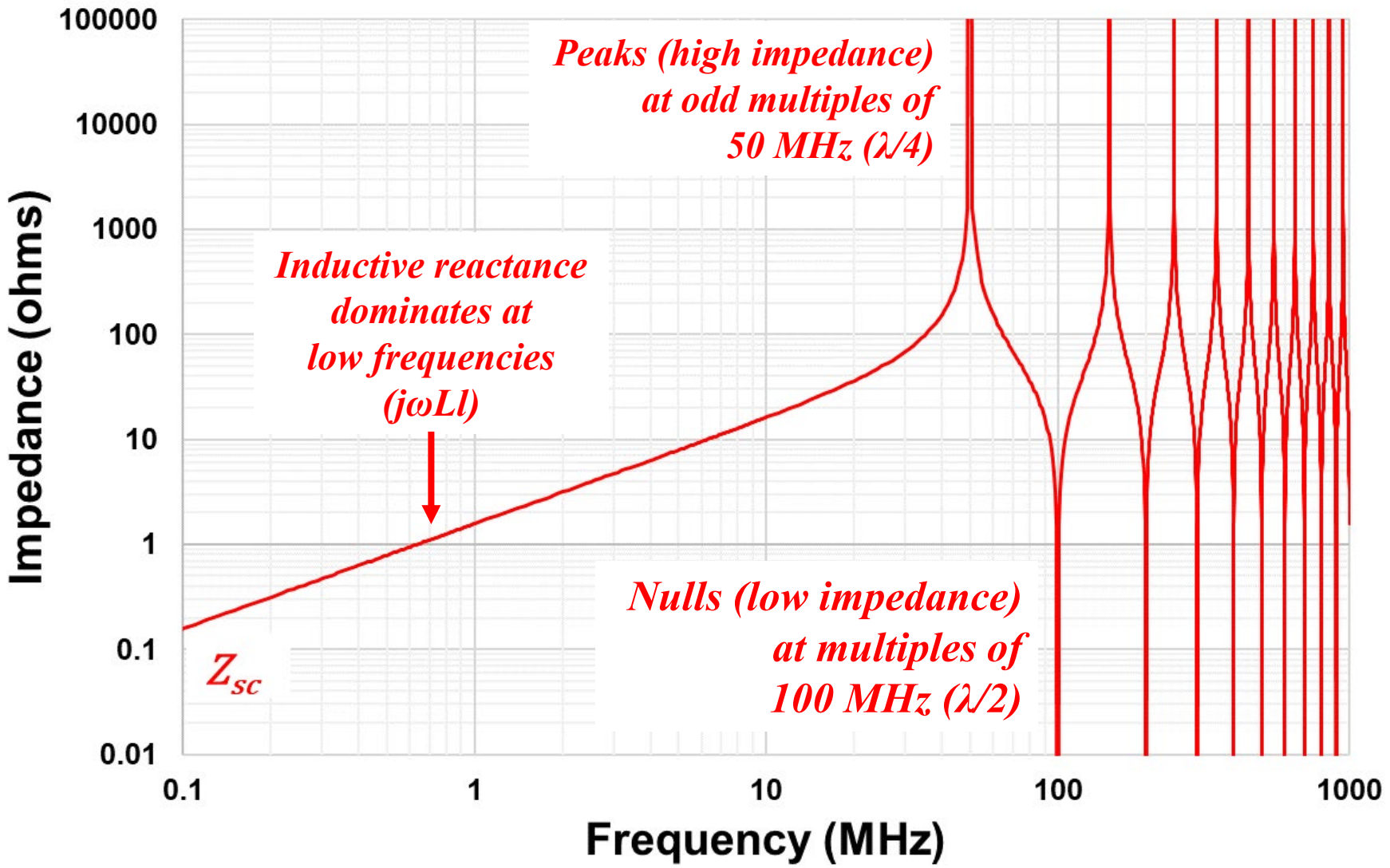
Example:

$l = 1$ *meter*

$Z_0 = 50\ \Omega$

$$\lambda \approx \frac{200}{f_{MHz}}\ (m)$$

1 m ≈ λ @ 200 MHz
≈ λ/2 @ 100 MHz
≈ λ/4 @ 50 MHz
≈ λ/8 @ 25 MHz



Input Impedance – Open Termination

$$Z_{in} = Z_0 \frac{Z_L + jZ_0 \tan(\beta l)}{Z_0 + jZ_L \tan(\beta l)}$$

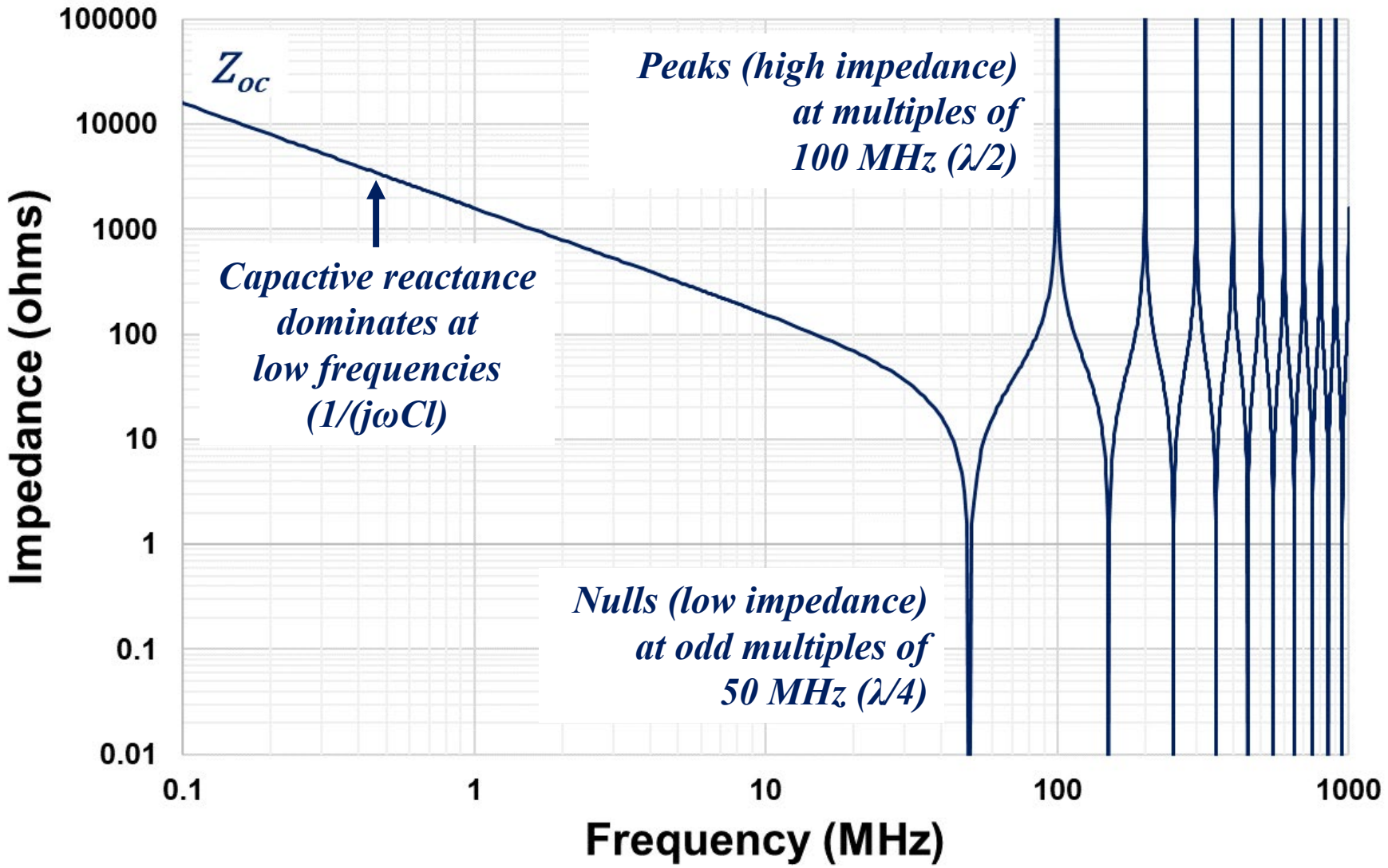
Example:

$l = 1$ *meter*

$Z_0 = 50\ \Omega$

$$\lambda \approx \frac{200}{f_{MHz}}\ (m)$$

1 m ≈ λ @ 200 MHz
≈ λ/2 @ 100 MHz
≈ λ/4 @ 50 MHz
≈ λ/8 @ 25 MHz



Input Impedance (cont.)

$$Z_{in} = Z_0 \frac{Z_L + jZ_0 \tan(\beta l)}{Z_0 + jZ_L \tan(\beta l)}$$

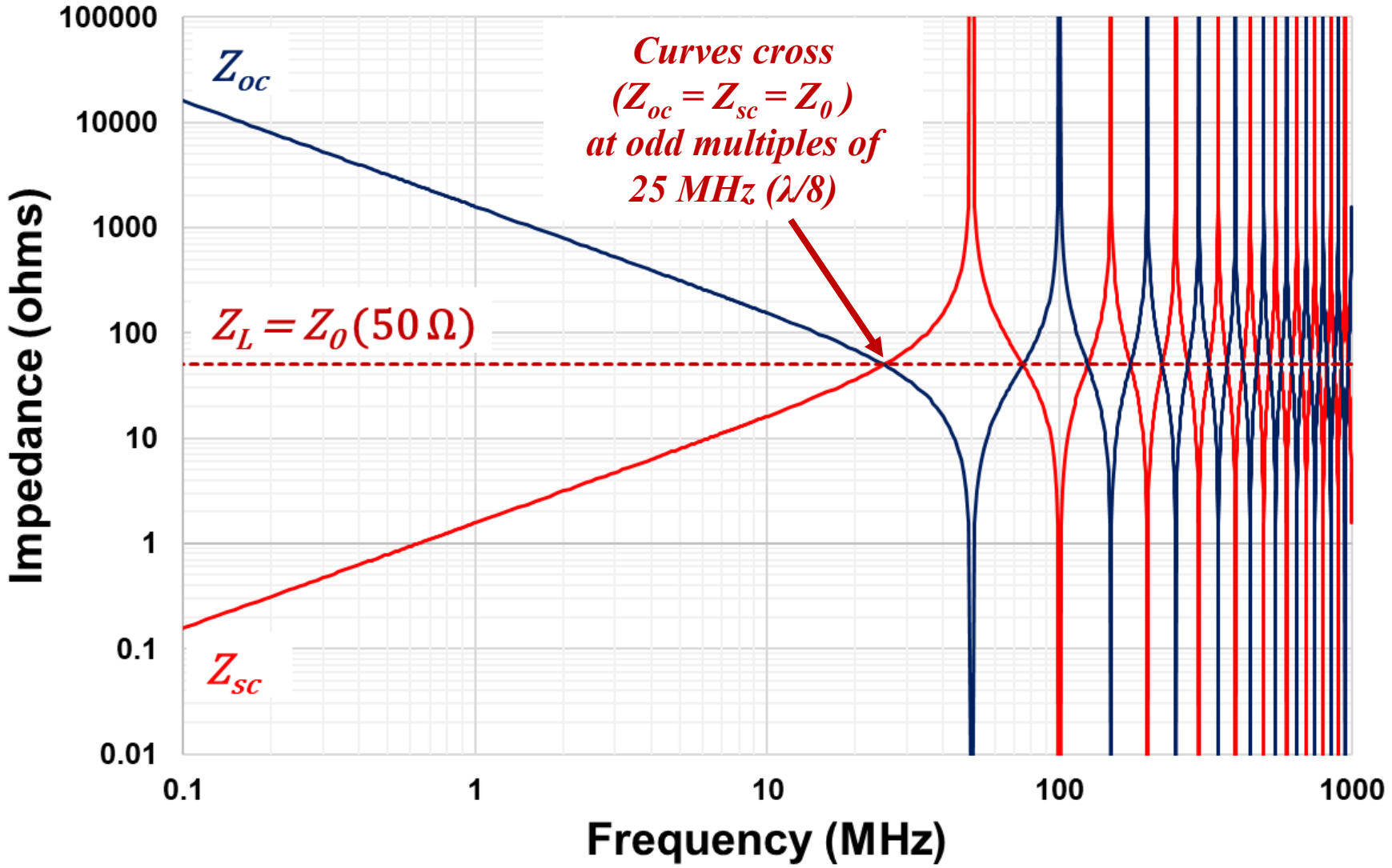
Example:

$l = 1$ *meter*

$Z_0 = 50\ \Omega$

$$\lambda \approx \frac{200}{f_{MHz}}\ (m)$$

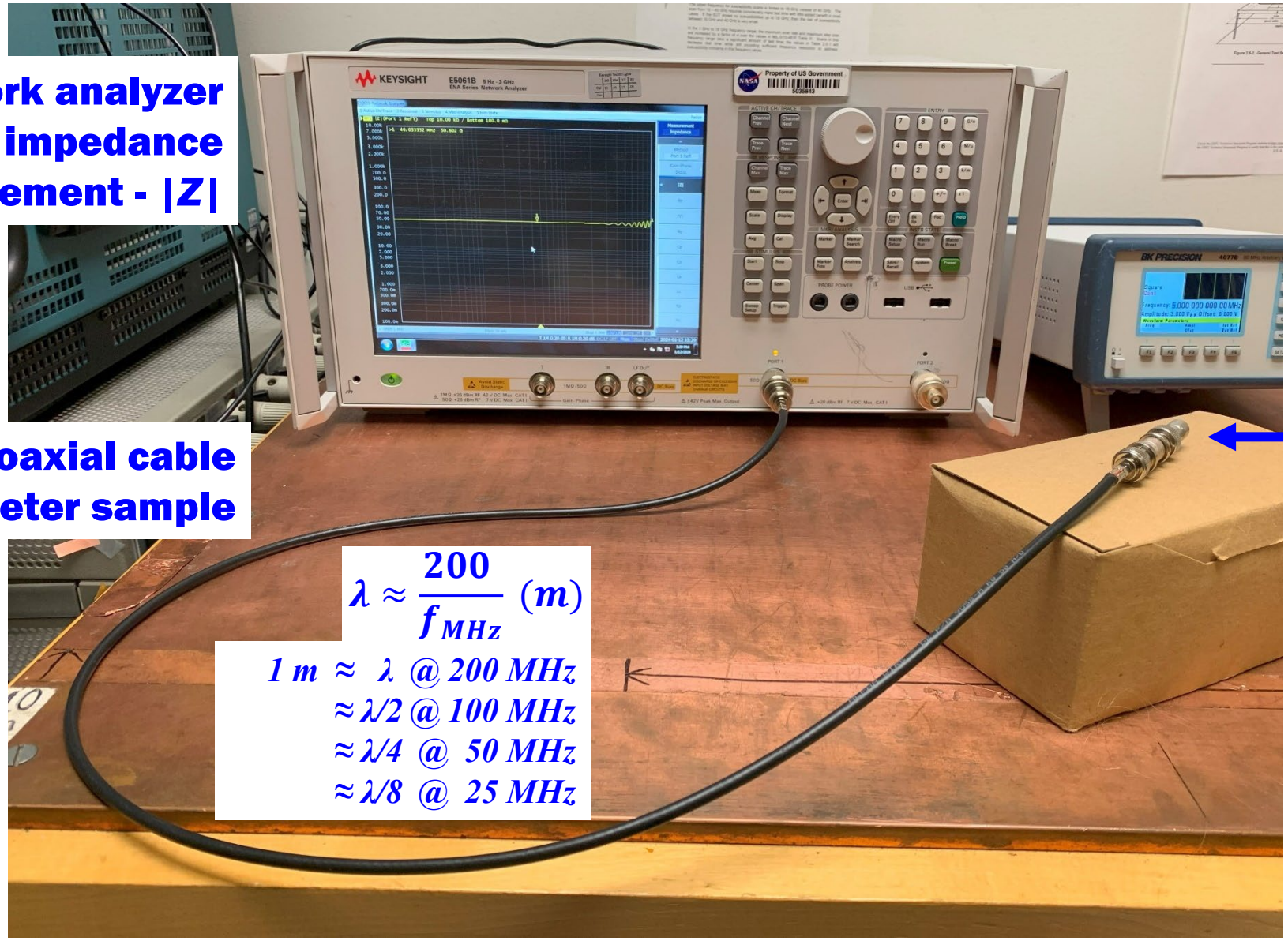
1 m ≈ λ @ 200 MHz
≈ λ/2 @ 100 MHz
≈ λ/4 @ 50 MHz
≈ λ/8 @ 25 MHz



Virtual Demonstration #2: Input Impedance

Network analyzer set up for impedance measurement - |Z|

RG-58 coaxial cable 1 meter sample



$$\lambda \approx \frac{200}{f_{MHz}} \ (m)$$

$1\ m \approx \lambda\ @\ 200\ MHz$
$\approx \lambda/2\ @\ 100\ MHz$
$\approx \lambda/4\ @\ 50\ MHz$
$\approx \lambda/8\ @\ 25\ MHz$

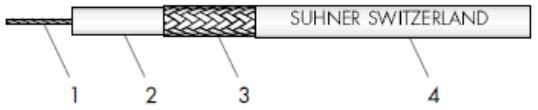
Various terminations:
50 Ω
Short
open

RG-58 C/U Coaxial Cable

SUHNER® COAXIAL CABLE
TYPE: RG 58 C/U

Item: 22510015

Cable design



1	Inner conductor	Stranded tinned copper	19x 0.187 mm	ø 0.935 mm
2	Dielectric	Solid polyethylene (PE)		ø 2.95 mm
3	Outer conductor	Tinned copper braid	96% coverage	ø 3.60 mm
4	Jacket	Non-migratory PVC	bk (RAL 9005)	ø 4.95 mm
	Print on jacket	SUHNER SWITZERLAND RG 58 C/U 50 Ohm		

Electrical data

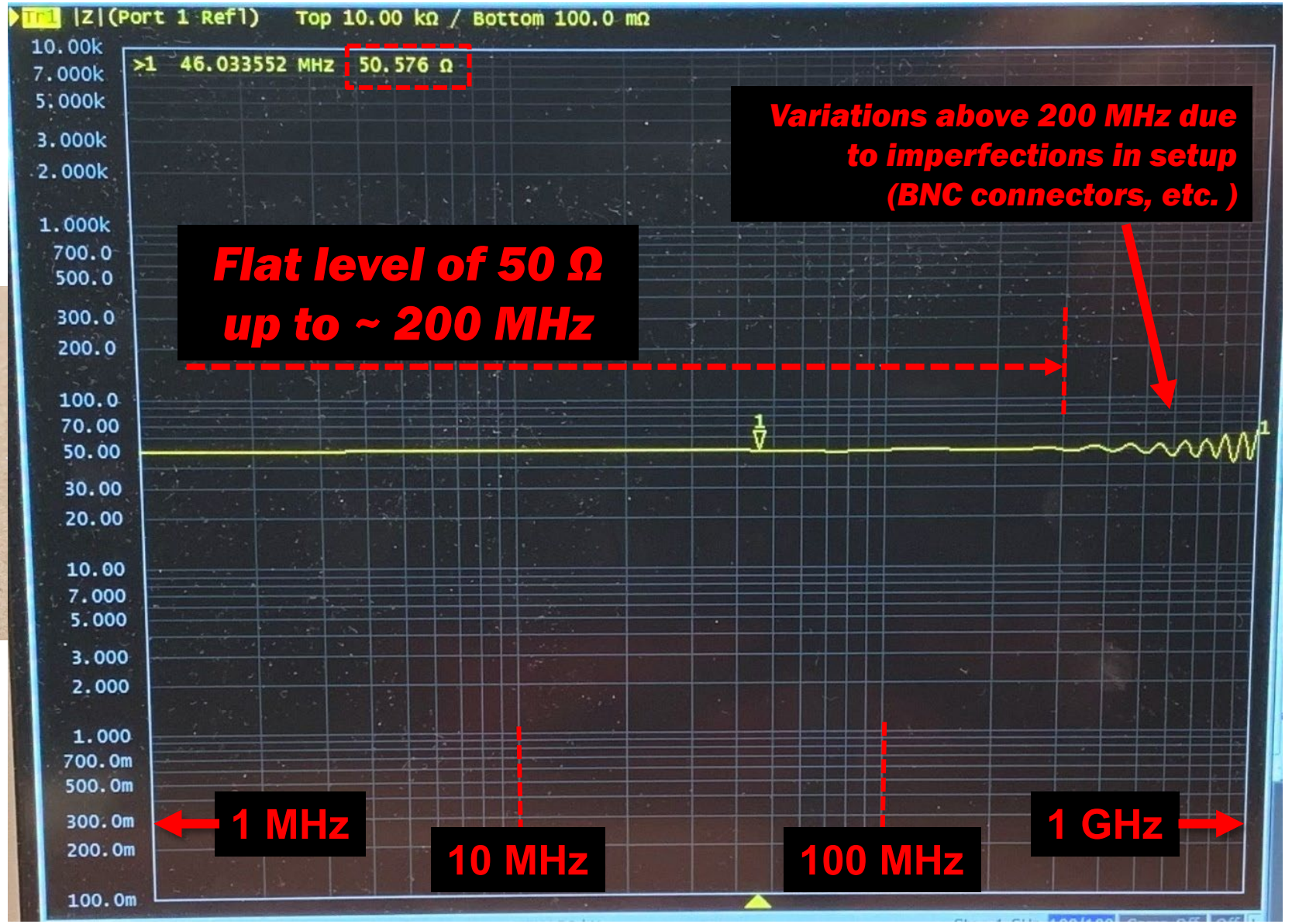
Typ. operating frequency		(GHz)	≤ 1
Impedance		(Ω)	50 ± 2
Capacitance		(pF/m)	100.7
Relative signal propagation		(%)	66.3
Signal delay		(ns/m)	5.03
Phase stability	vs temperature	(°/GHz/m)	-
	vs bending	(°/GHz)	-
Insulation resistance		(MΩm)	> 10^8
Test voltage	50 Hz / 1 min.	(kV_{rms})	5
Max. operating voltage	at sea level	(kV_{rms})	2.5
Typ. DC resistance	inner conductor	(Ω/km)	34.5
	outer conductor	(Ω/km)	13.8
Typ. screening effectiveness	1 ... 300 MHz	(dB)	> 35

$$Z_0 = \sqrt{\frac{L}{C}} \qquad L \approx 250\ nH/m$$

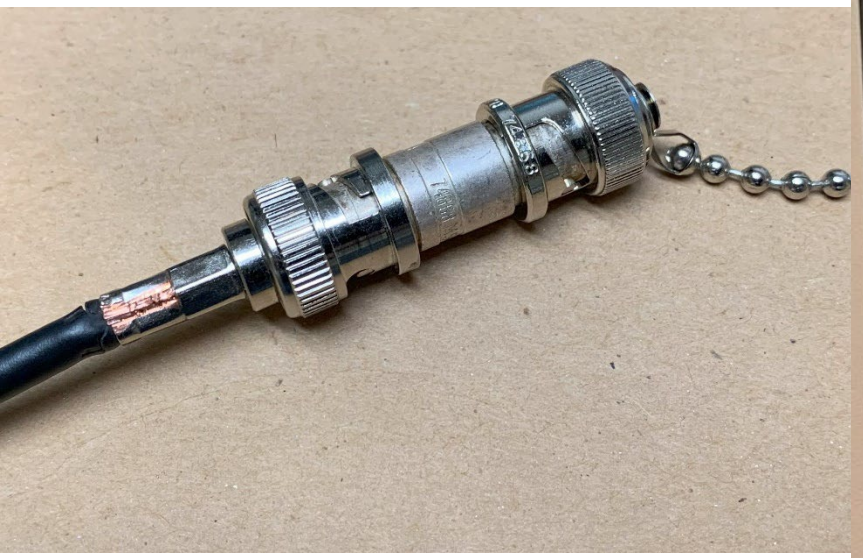
$$\frac{u}{c} = \frac{1}{\sqrt{\varepsilon_r}} \qquad \varepsilon_r \approx 2.3$$

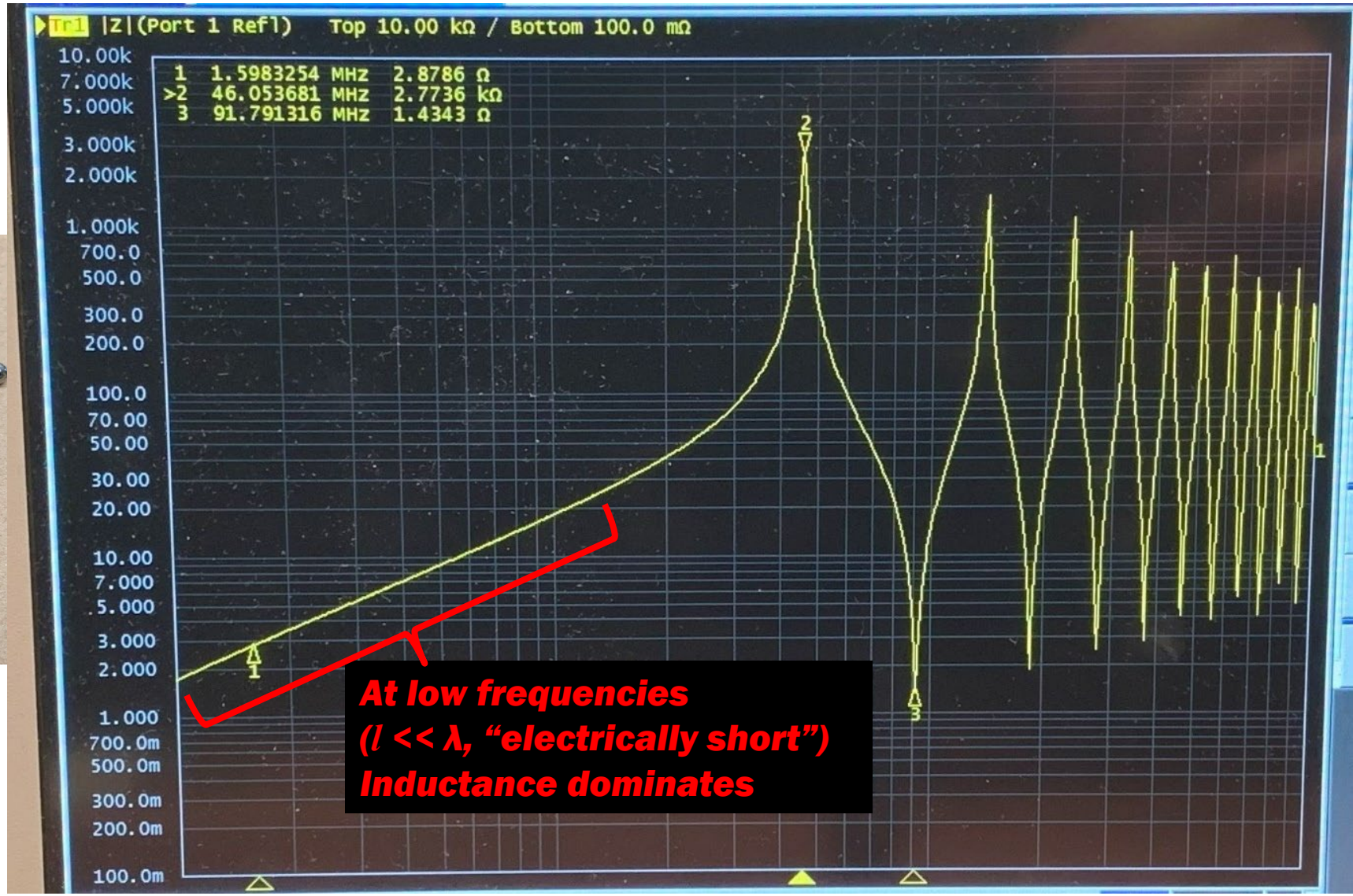
50 Ω Termination



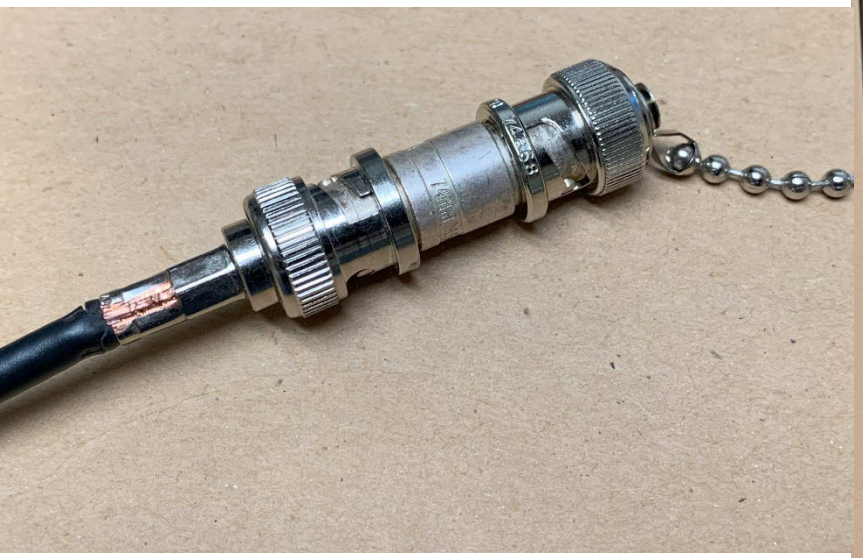


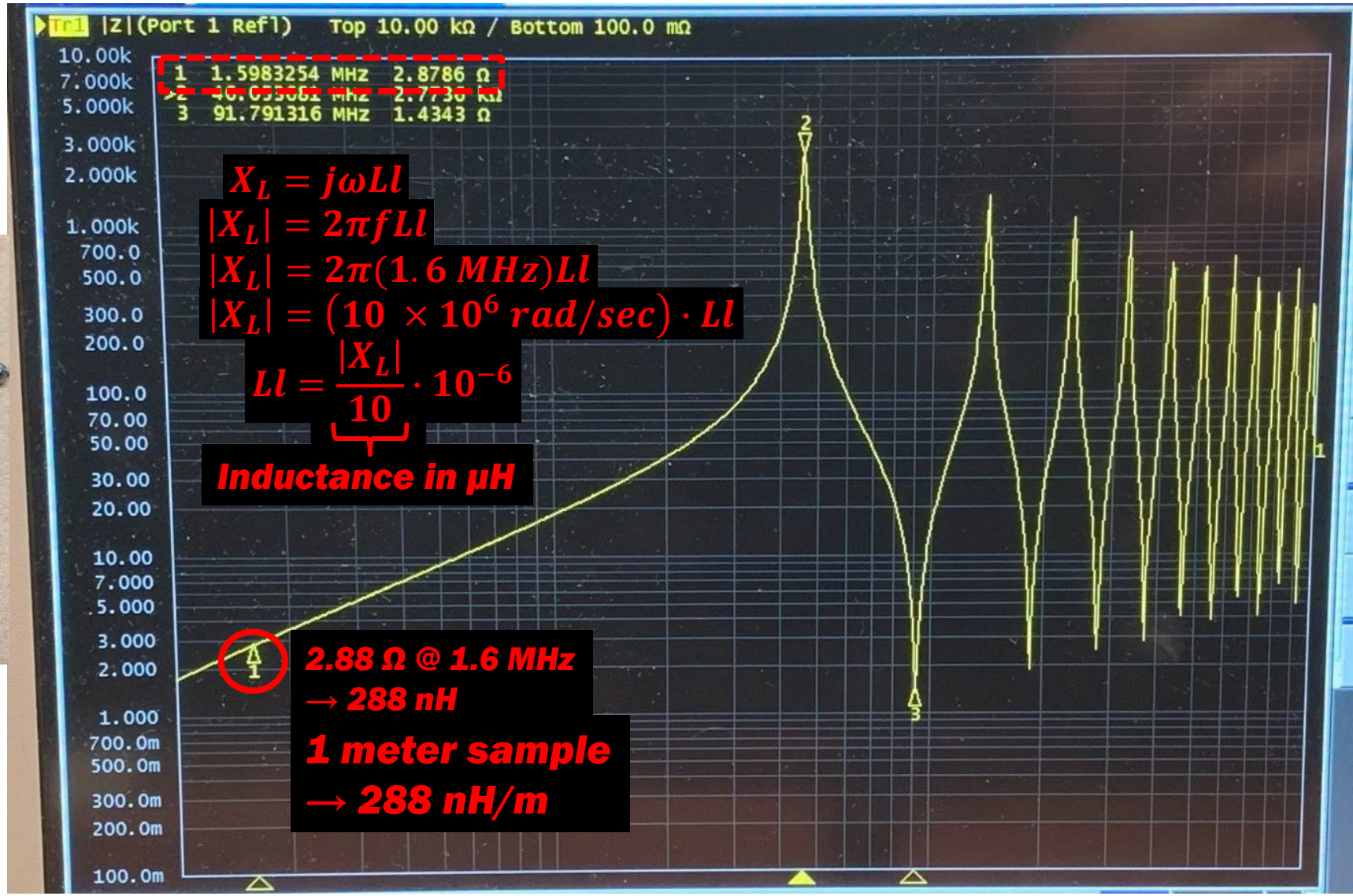
Shorted Termination



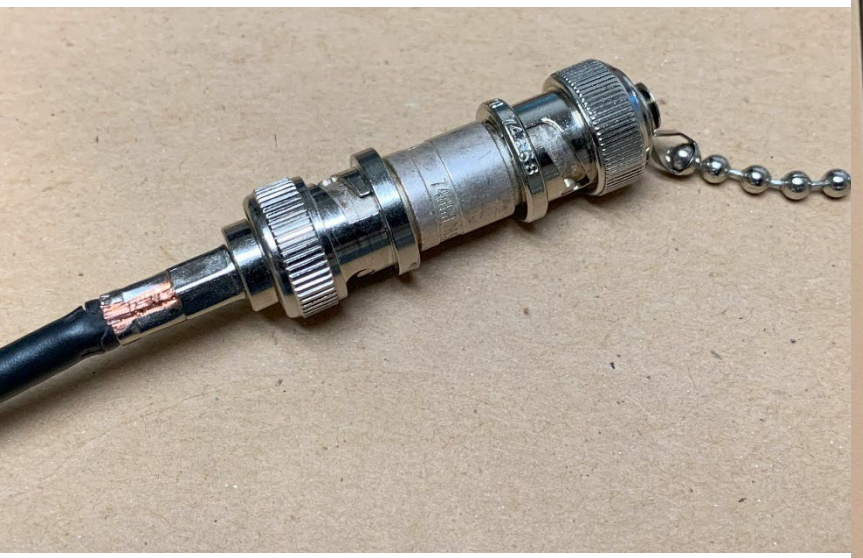


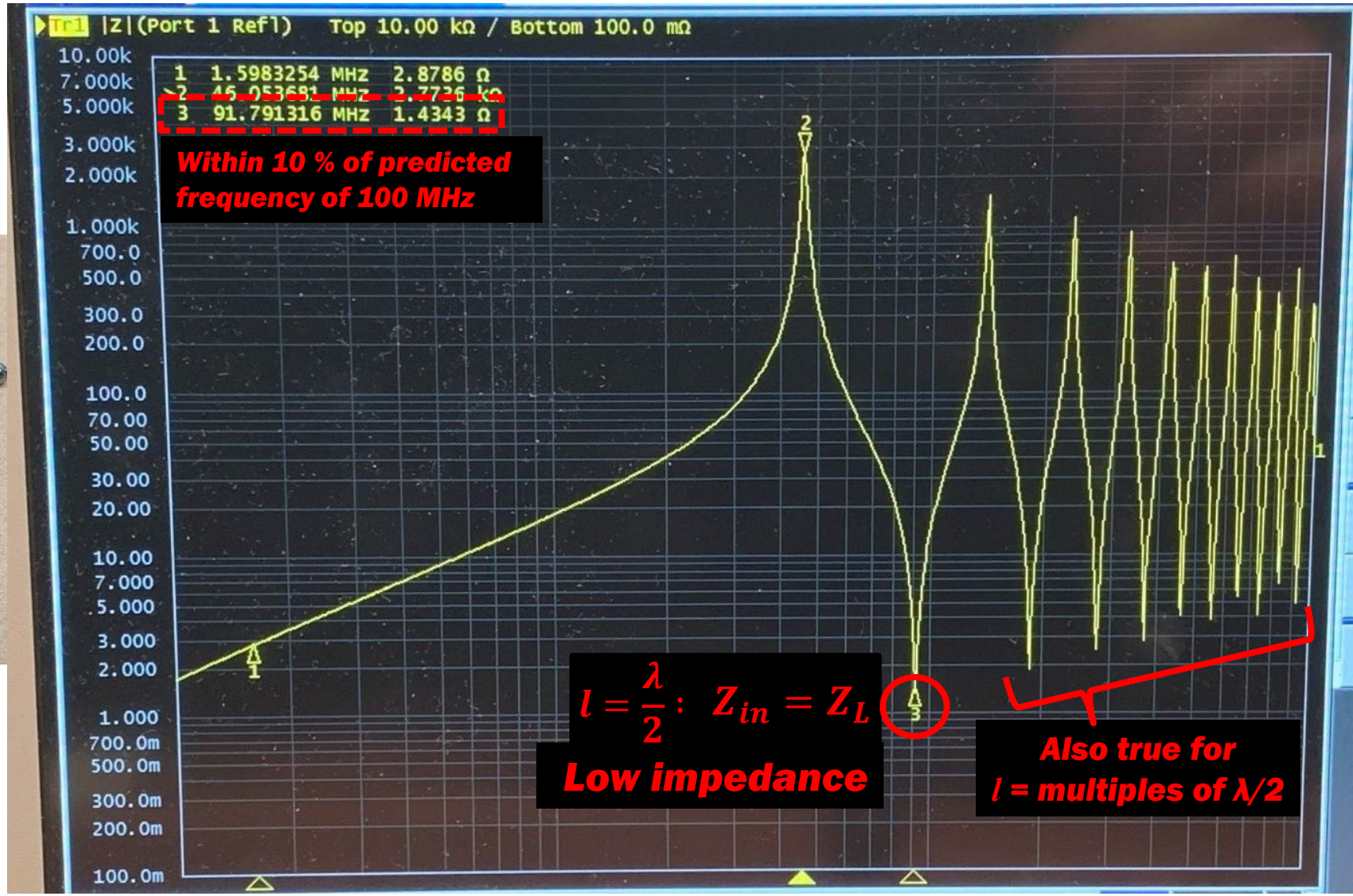
Shorted Termination: Inductance



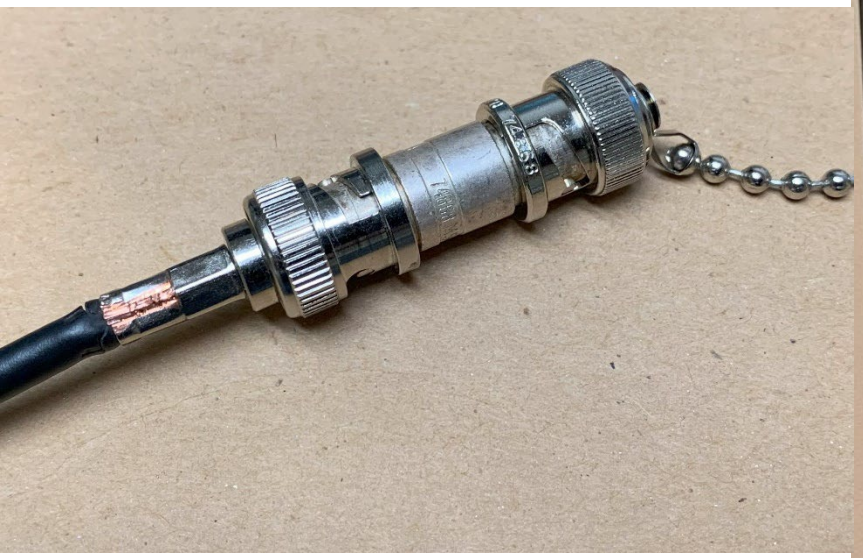


Shorted Termination: l = λ/2 (and multiples)





Shorted Termination: l = λ/4 (and odd multiples)





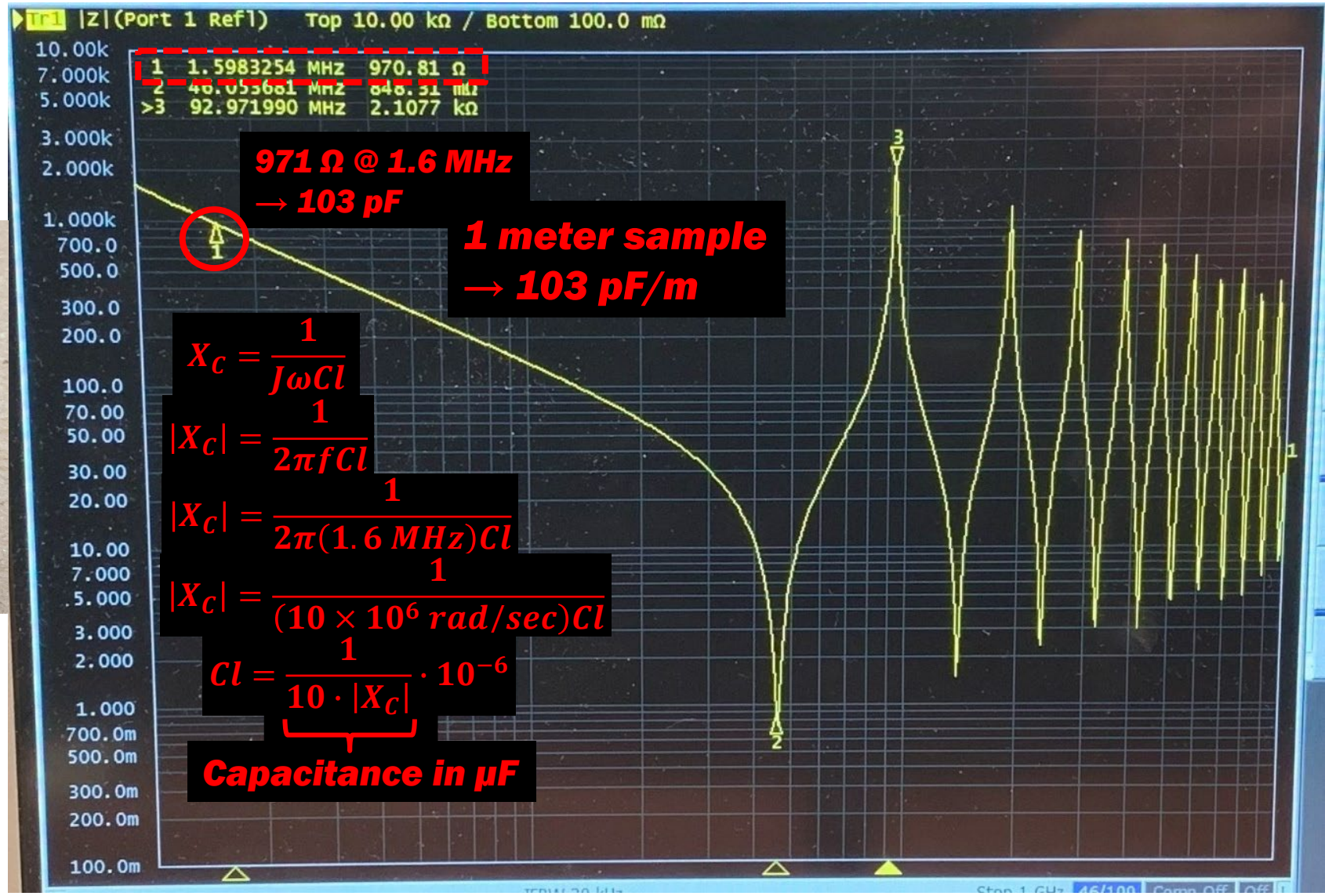
Open Termination





Open Termination: Capacitance





Open Termination: $l = \lambda/2$ (and multiples)





Open Termination: $l = \lambda/4$ (and odd multiples)





Short and Open Terminations: $l = \lambda/8$ (and odd multiples)

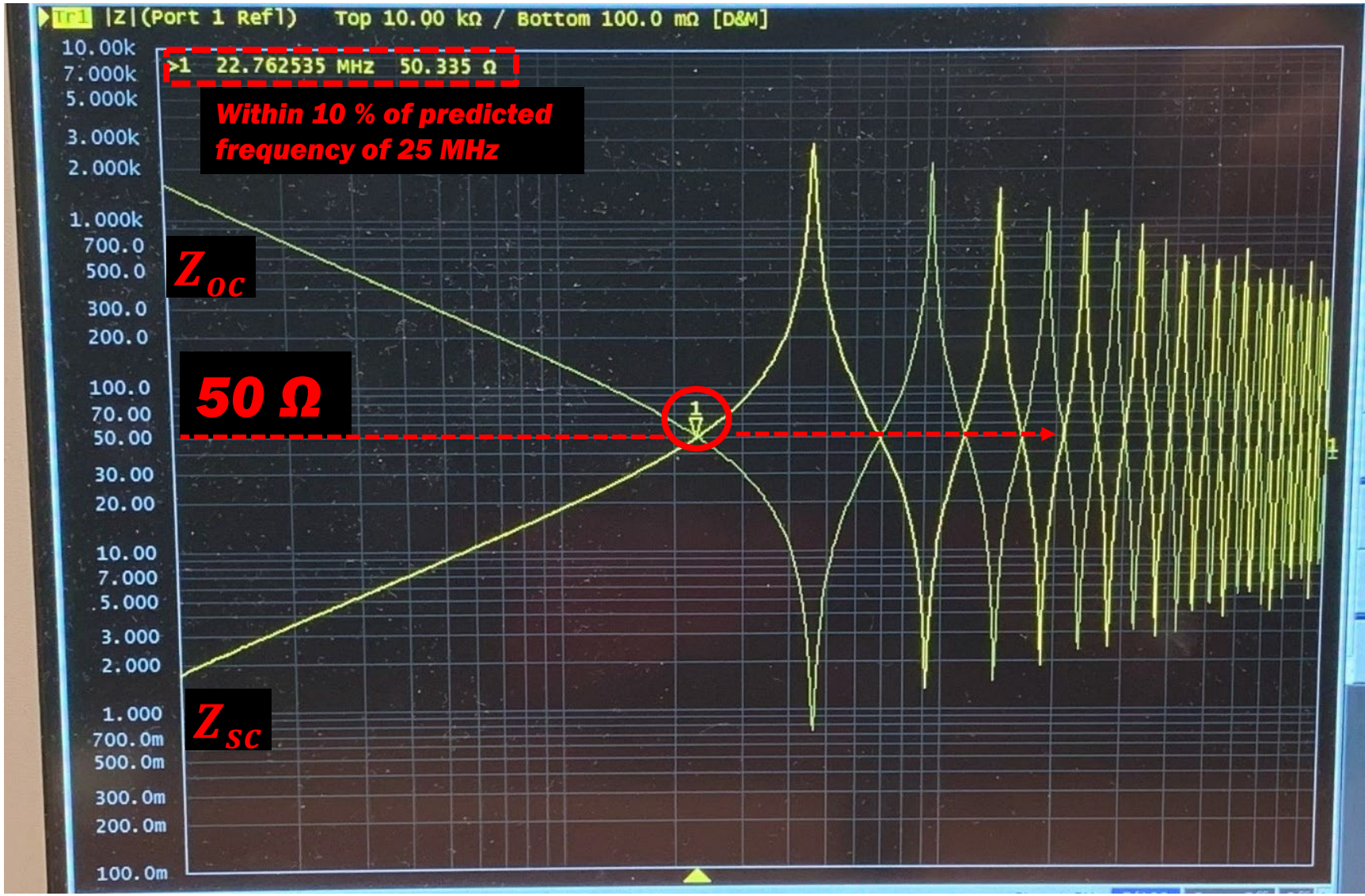
At any frequency:

$$\sqrt{Z_{sc}Z_{oc}} = Z_0$$

Where curves meet:

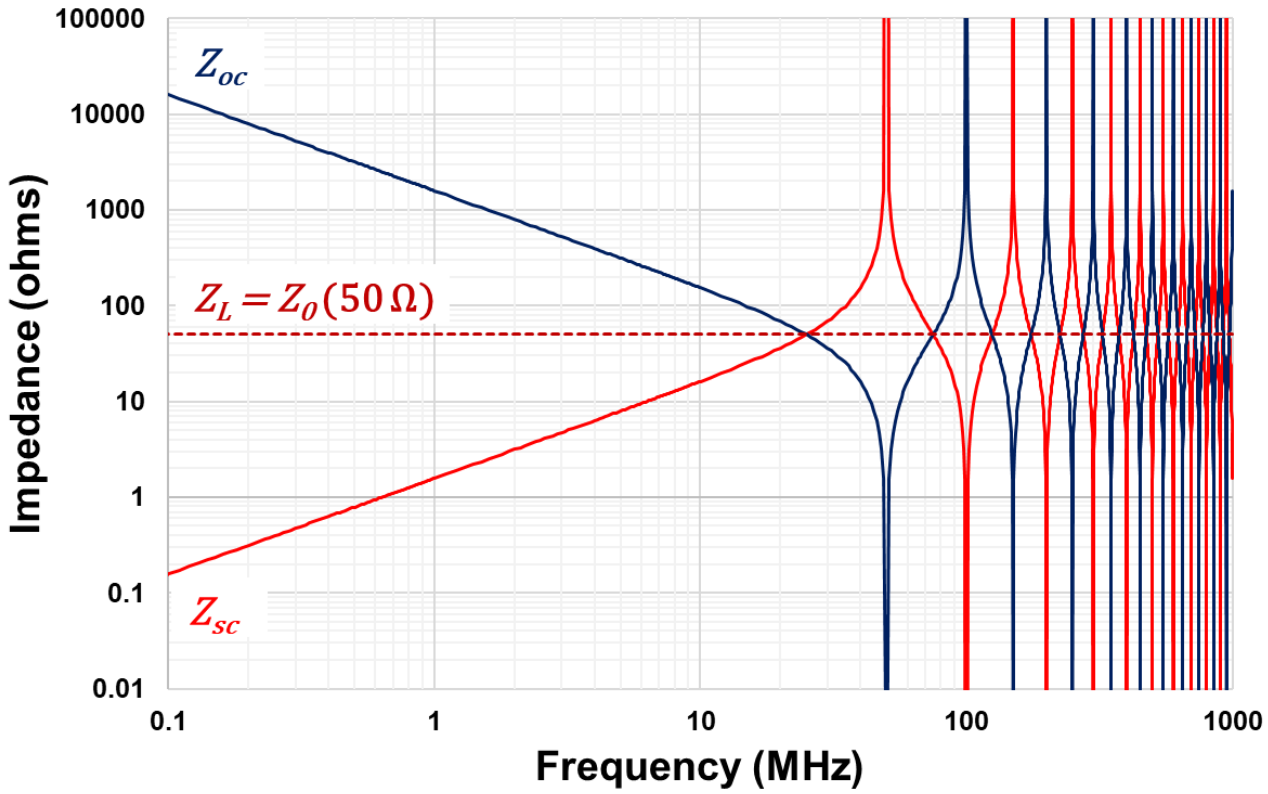
$$Z_{sc} = Z_{oc} = Z_0$$

Occurs for $l = \lambda/8$ and odd multiples

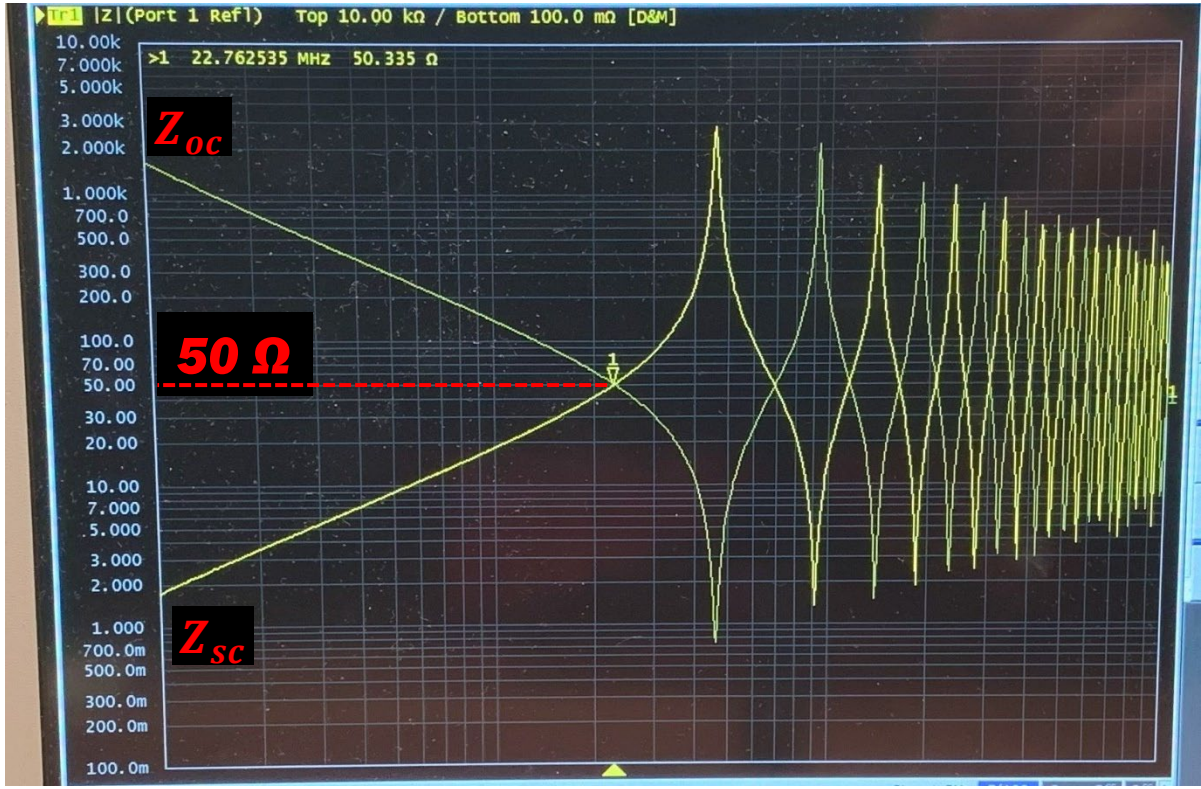


Always Two Sides of the Same Coin...

THEORY



EXPERIMENT



Transmission Lines Summary

- **Interconnections matter (!)**
- **For interconnections whose length is significant fraction of wavelength ($l > 0.1\lambda$), connections must be treated as transmission lines**
  - Use controlled characteristic impedance Z_0
  - Terminate in matched impedance
- **Velocity of propagation**
  - Air dielectric: same as speed of light in free space:
    - $c \approx 3 \times 10^8$ *m/sec* $\approx$ *1 foot/nsec*
    - $\lambda \approx 300/f_{MHz}$
  - Rules of thumb for most non-air dielectrics:
    - $u \approx (2/3)c$
    - $\lambda \approx 200/f_{MHz}$



Transmission Lines Summary (cont.)

- **Mismatched terminations:**
  - Can cause reflections and distortions in desired signal
  - Can compromise system operation
  - In digital systems, can completely freeze up state machines
  - Can look like very different impedances at some frequencies (l = odd multiples of λ/4), producing uncontrolled and undesired effects (e.g. currents flowing in undesired places, etc.)
- **Knowledge of these principles will help you to prevent problems and to recognize and diagnose such problems when you see them**



References

- Cheng, David: Field and Wave Electromagnetics
- Paul, Clayton: Introduction to Electromagnetic Compatibility
- Ott, Henry: Electromagnetic Compatibility Engineering