Fundamentals of Electromagnetics

Time Domain vs. Frequency Domain
Part 1: The Fourier Transform

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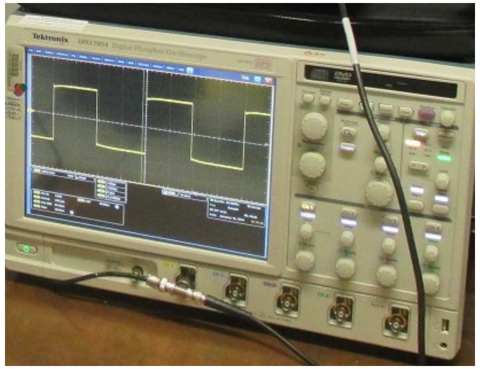
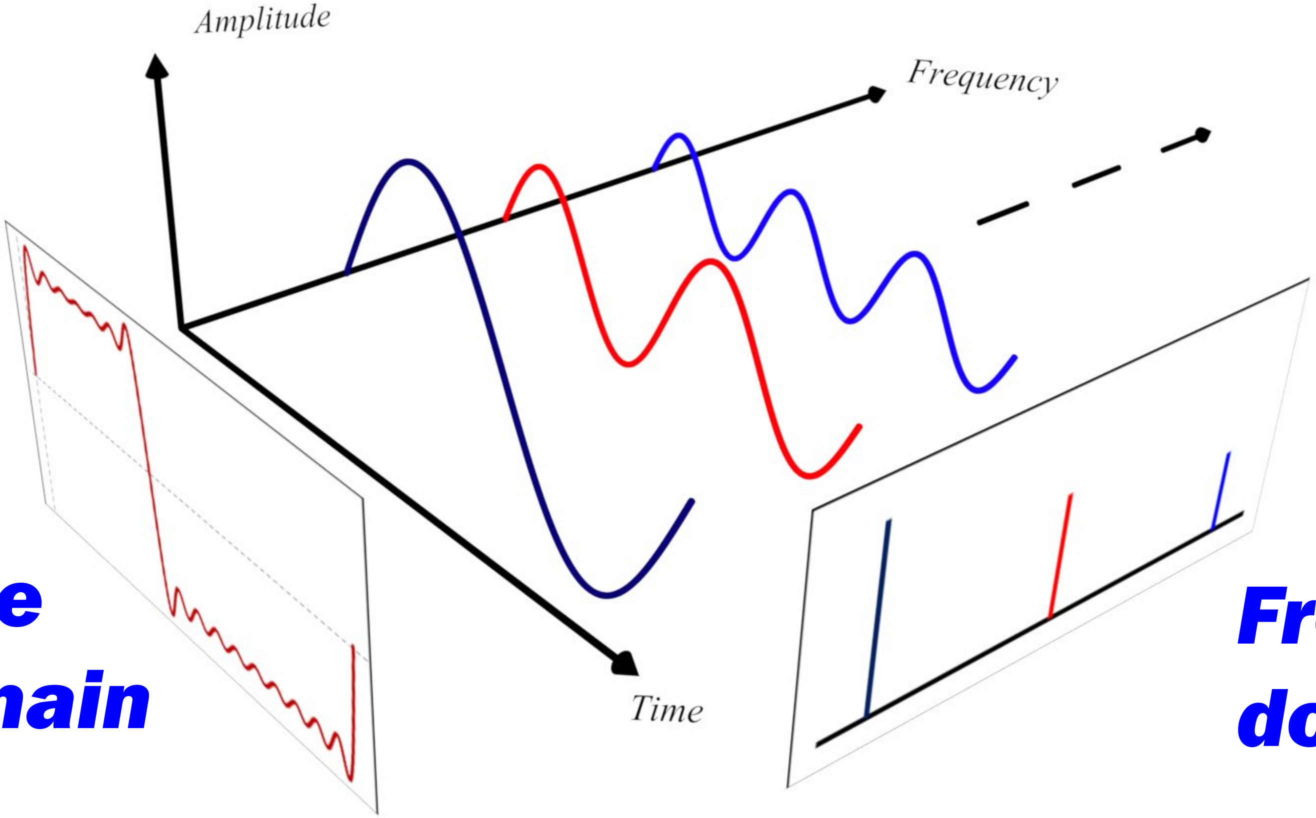
Topics

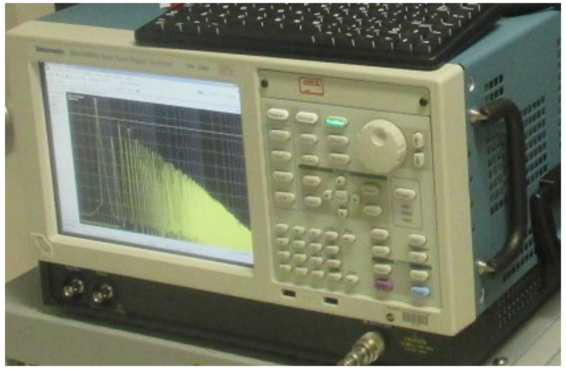
- **Time Domain vs. Frequency Domain**
- **Average Power**
- **Introduction to Fourier Series Expansions**
- **Fourier Transform**
- **Summary**

- ***Shameless plug: "Fundamentals of Electromagnetics" video series***
  - *Publicly available on YouTube; search for above title*
  - *Direct playlist link:* https://youtube.com/playlist?list=PLtrpQ-gPvnJn2r9Mw49jij7Ky0mb6RJYF&si=UxEKqVRgsR9w6nZ7

Time Domain vs. Frequency Domain







Time domain

Frequency domain

Fourier Transform

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \qquad (\omega = 2\pi f)$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

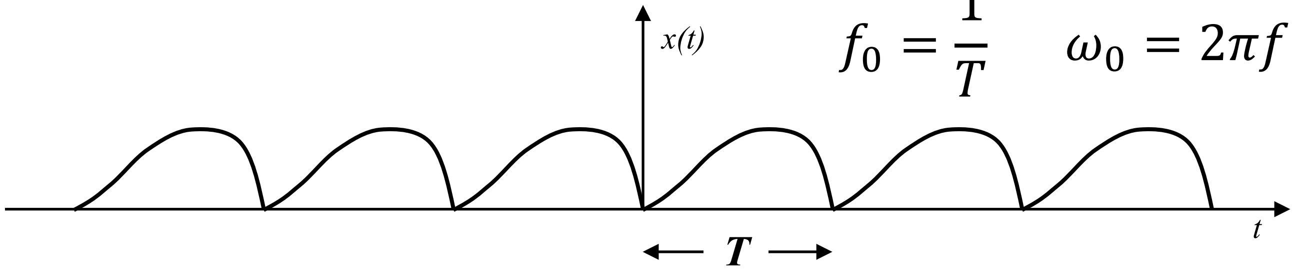
Inverse Fourier Transform

Fourier Series

- Any periodic signal of period T (fundamental frequency f_0 = 1/T) can be expressed as "Fourier Series," i.e. an infinite sum of sinusoidal components at integer multiples of fundamental frequency f_0
- *Recommend reviewing "On the Nature of Sinusoids" section of "The Math" session*

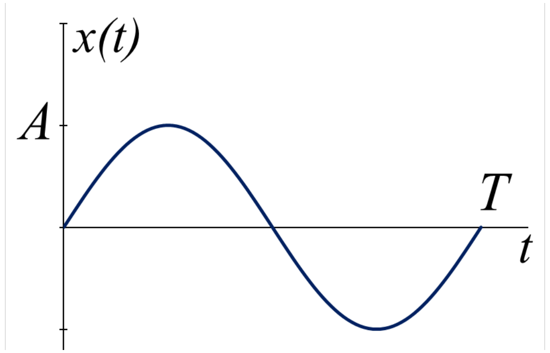
$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

$$f_0 = \frac{1}{T} \qquad \omega_0 = 2\pi f$$



Jean-Baptiste Joseph Fourier
21 March 1768
16 May 1830

Average Value of Sinusoid



$$x(t) = A\cos(\omega_0 t + \theta)$$

$$\omega_0 = 2\pi f_0 = \frac{2\pi}{T}$$

Time average integrated over period T:

$$[x(t)]_{av} = \frac{1}{T}\int_{t_1}^{t_1+T} x(t)\,dt$$

t_1 = arbitrary start time

$$= \frac{1}{T}\int_{t_1}^{t_1+T} A\cos(\omega_0 t + \theta)\,dt$$

$$= \frac{A}{\omega_0 T}\left[\sin(\omega_0 t + \theta)\right]_{t_1}^{t_1+T}$$

$$= \frac{A}{\omega_0 T}\{\sin[\omega_0(t_1 + T) + \theta] - \sin(\omega_0 t_1 + \theta)\}$$

$$= \frac{A}{\omega_0 T}\{\sin(\omega_0 t_1 + \theta + \underline{\omega_0 T}) - \sin(\omega_0 t_1 + \theta)\}$$

$$\omega_0 T = 2\pi$$

$$= \frac{A}{\omega_0 T}\{\sin(\omega_0 t_1 + \theta \underline{+ 2\pi}) - \sin(\omega_0 t_1 + \theta)\}$$

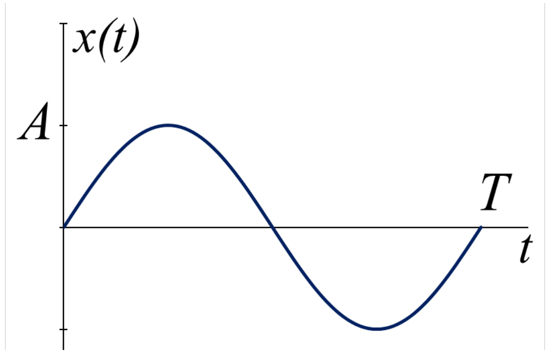
Full cycle puts you right back where you started

$$= \frac{A}{\omega_0 T}\{\underbrace{\sin(\omega_0 t_1 + \theta) - \sin(\omega_0 t_1 + \theta)}_{= 0}\}$$

$$= 0$$

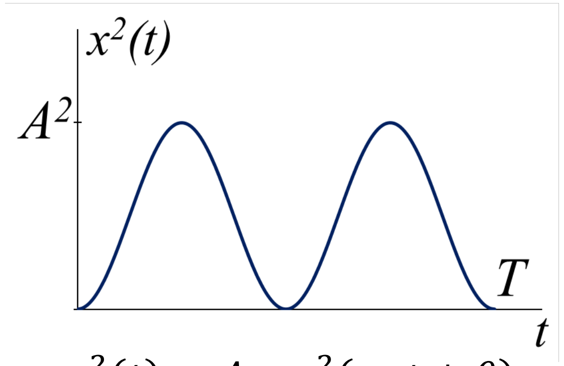
The integral of any sinusoidal function over a period equals 0

Average Power



$$x(t) = A\cos(\omega_0 t + \theta)$$

$$\omega_0 = 2\pi f_0 = \frac{2\pi}{T}$$



$$x^2(t) = A\cos^2(\omega_0 t + \theta)$$

If x(t) is a potential across a resistance R:

Instantaneous power: $$P = \frac{x^2(t)}{R}$$

Average power: $$P_{av} = \frac{1}{R}\cdot[x^2(t)]_{av}$$

$$P_{av} = \frac{1}{R}\cdot\frac{1}{T}\int_{t_1}^{t_1+T} x^2(t)\,dt$$ ***t_1 = arbitrary start time***

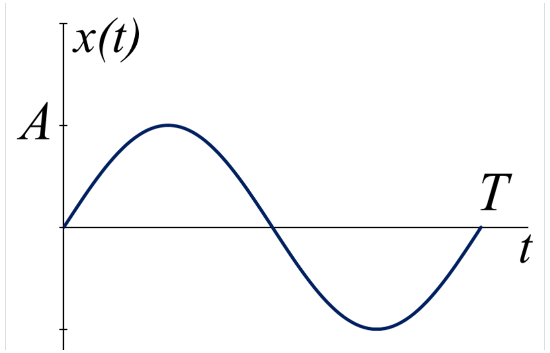
If x(t) is a current through a resistance R:

Instantaneous power: $$P = x^2(t)\cdot R$$

Average power: $$P_{av} = R\cdot[x^2(t)]_{av}$$

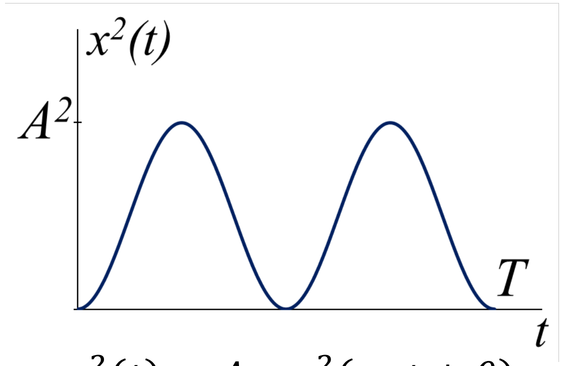
$$P_{av} = R\cdot\frac{1}{T}\int_{t_1}^{t_1+T} x^2(t)\,dt$$

Average Power and Root Mean Square (RMS)



$$x(t) = A\cos(\omega_0 t + \theta)$$

$$\omega_0 = 2\pi f_0 = \frac{2\pi}{T}$$



$$x^2(t) = A\cos^2(\omega_0 t + \theta)$$

In either case:

$$P_{av} \propto [x^2(t)]_{av} = \frac{1}{T}\int_{t_1}^{t_1+T} x^2(t)\,dt$$

"Root mean square" (rms):

$$[x(t)]_{rms} = \sqrt{\frac{1}{T}\int_{t_1}^{t_1+T} x^2(t)\,dt}$$

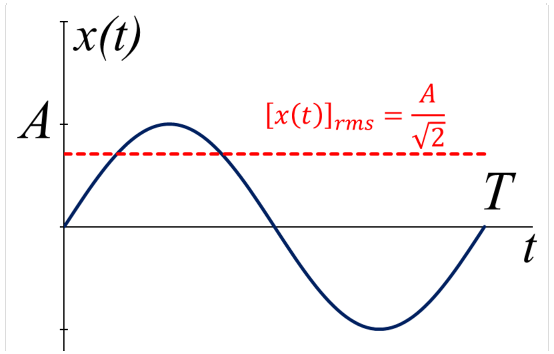
Square root of the mean of the square

The rms value of a voltage or current is the equivalent DC value that would dissipate the same average power P_{av} in a load resistance R

$$P_{av} = \frac{(V_{rms})^2}{R} \qquad P_{av} = (I_{rms})^2 R$$

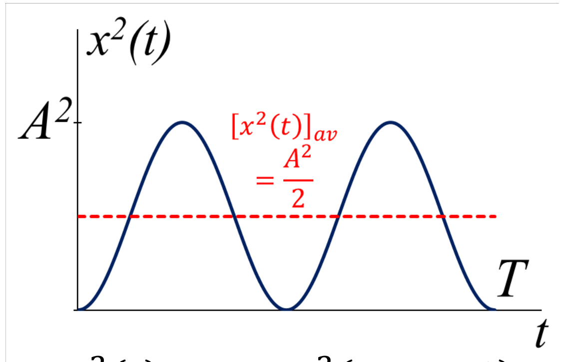
Frequency components are commonly expressed as rms values…

Sinusoid Average Power, "Traditional Method"



$$x(t) = A\cos(\omega_0 t + \theta)$$

$$\omega_0 = 2\pi f_0 = \frac{2\pi}{T}$$



$$x^2(t) = A\cos^2(\omega_0 t + \theta)$$

$$[x^2(t)]_{av} = \frac{1}{T}\int_{t_1}^{t_1+T} x^2(t)\,dt$$

$$= \frac{1}{T}\int_{t_1}^{t_1+T} A^2\cos^2(\omega_0 t + \theta)\,dt \qquad \cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

$$= \frac{A^2}{T}\int_{t_1}^{t_1+T} \frac{1}{2}[1 + \cos(2\omega_0 t + 2\theta)]\,dt$$

$$= \frac{A^2}{2T}\left[t + \frac{1}{2\omega_0}\sin(2\omega_0 t + 2\theta)\right]_{t_1}^{t_1+T}$$

$$= \frac{A^2}{2T}\left[\underline{(t_1 + T - t_1)} + \frac{1}{2\omega_0}\sin(2\omega_0 t_1 + \underline{2\omega_0 T} + 2\theta) - \frac{1}{2\omega_0}\sin(2\omega_0 t_1 + 2\theta)\right]$$

t_1 cancels and is therefore arbitrary

$\omega_0 T = 2\pi$

$$= \frac{A^2}{2T}\left[T + \frac{1}{2\omega_0}[\sin(2\omega_0 t_1 + 2\theta + \underline{4\pi}) - \sin(2\omega_0 t_1 + 2\theta)]\right]$$

2 full cycles put you right back where you started

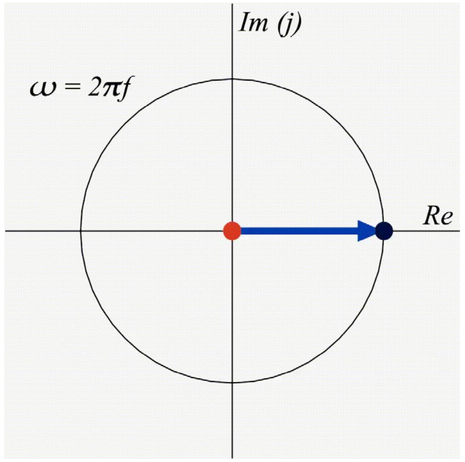
$$= \frac{A^2}{2\cancel{T}}\left[\cancel{T} + \frac{1}{2\omega_0}\underbrace{[\sin(2\omega_0 t_1 + 2\theta) - \sin(2\omega_0 t_1 + 2\theta)]}_{=\,0}\right]$$

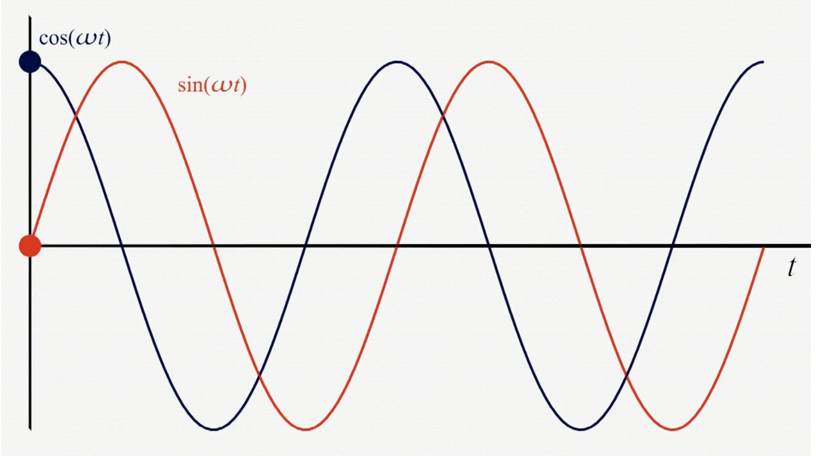
θ cancels and is therefore arbitrary

$$[x^2(t)]_{av} = \frac{A^2}{2} \qquad [x(t)]_{rms} = \frac{A}{\sqrt{2}} \approx 0.707A$$

Recall: Euler's Formula and Projections of Circular Motion

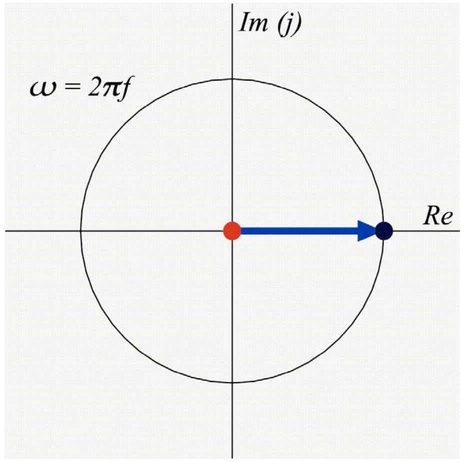
$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

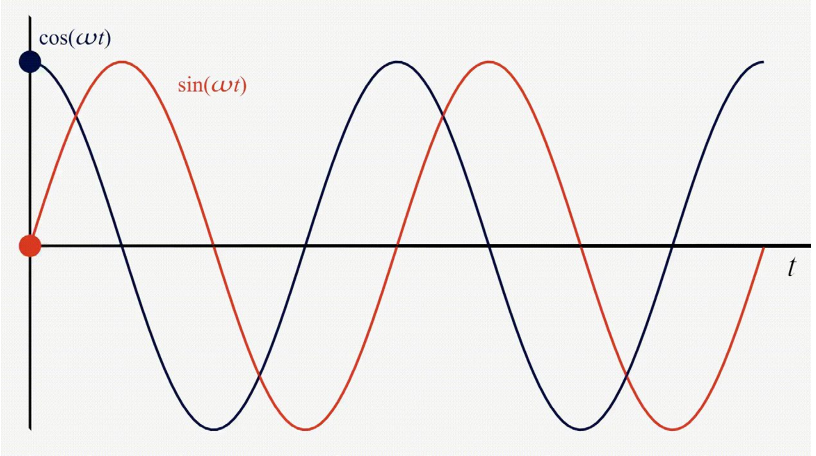




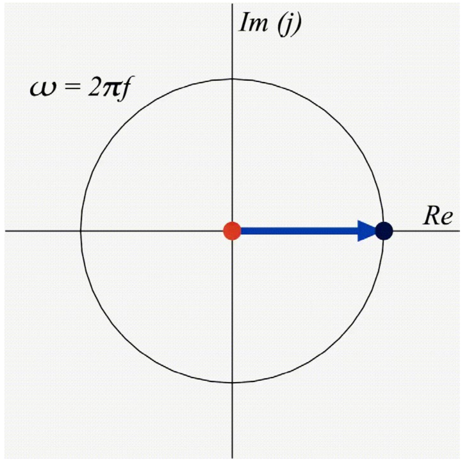
Recall: Euler's Formula and Projections of Circular Motion

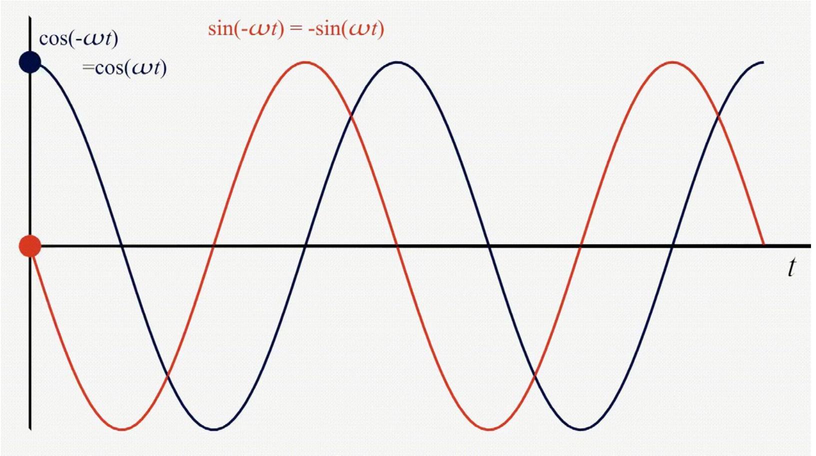
$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$





$$e^{-j\omega t} = \cos(\omega t) - j\sin(\omega t)$$



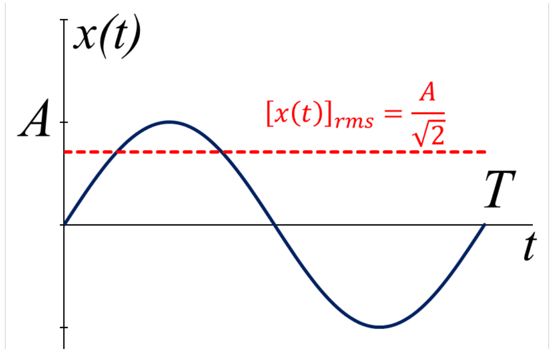


$$\cos(\omega t) = \frac{1}{2}\left(e^{j\omega t} + e^{-j\omega t}\right)$$

$$j\sin(\omega t) = \frac{1}{2}\left(e^{j\omega t} - e^{-j\omega t}\right)$$

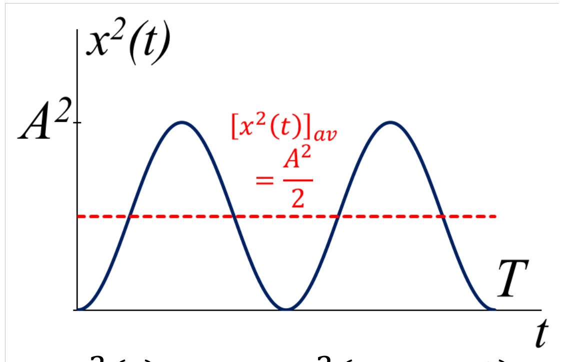
Equal contributions of +/- ω components

Sinusoid Average Power, “Exponential Method 1”



$$x(t) = A\cos(\omega_0 t + \theta)$$

$$\omega_0 = 2\pi f_0 = \frac{2\pi}{T}$$



$$x^2(t) = A\cos^2(\omega_0 t + \theta)$$

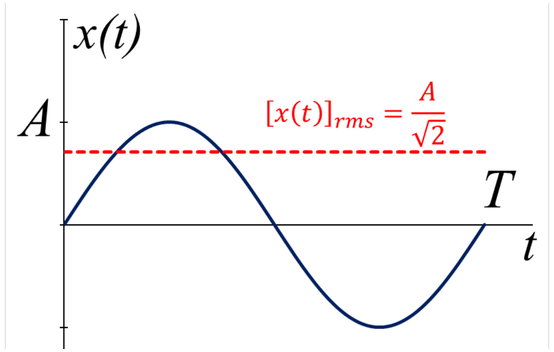
$$x(t) = \frac{A}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)$$ ***θ = 0 for simplicity***

$$x^2(t) = \left[\frac{A}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)\right]\left[\frac{A}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)\right]$$

$$= \frac{A^2}{4}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)\underbrace{e^{-j\omega_0 t}}_{\text{Conjugate of } +\omega_0 \text{ term}} + \frac{A^2}{4}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)\underbrace{e^{j\omega_0 t}}_{\text{Conjugate of } -\omega_0 \text{ term}}$$

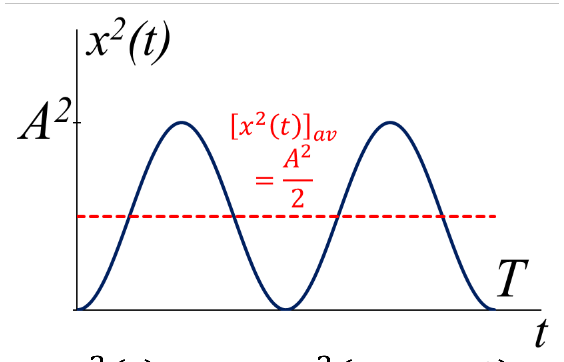
$$x^2(t) = \frac{A^2}{4}\left(e^0 + e^{-j2\omega_0 t}\right) + \frac{A^2}{4}\left(e^0 + e^{j2\omega_0 t}\right) \qquad e^0 = 1$$

Sinusoid Average Power, "Exponential Method 1"



$$x(t) = A\cos(\omega_0 t + \theta)$$

$$\omega_0 = 2\pi f_0 = \frac{2\pi}{T}$$



$$x^2(t) = A\cos^2(\omega_0 t + \theta)$$

$$x(t) = \frac{A}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)$$ **θ = 0 for simplicity**

$$x^2(t) = \left[\frac{A}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)\right]\left[\frac{A}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)\right]$$

$$= \frac{A^2}{4}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)\underbrace{e^{-j\omega_0 t}}_{\text{Conjugate of } +\omega_0 \text{ term}} + \frac{A^2}{4}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)\underbrace{e^{j\omega_0 t}}_{\text{Conjugate of } -\omega_0 \text{ term}}$$

$$x^2(t) = \frac{A^2}{4}\left(1 + e^{-j2\omega_0 t}\right) + \frac{A^2}{4}\left(1 + e^{j2\omega_0 t}\right)$$

$$[x^2(t)]_{av} = \frac{1}{T}\cdot\frac{A^2}{4}\int_0^T \left(1 + e^{-j2\omega_0 t}\right)dt + \frac{1}{T}\cdot\frac{A^2}{4}\int_0^T \left(1 + e^{j2\omega_0 t}\right)dt$$ **t_1 = 0 for simplicity**

$$= \frac{A^2}{4T}\left[t + \frac{1}{(-j2\omega_0)}\left(e^{-j2\omega_0 T}\right)\right]_0^T + \frac{A^2}{4T}\left[t + \frac{1}{(j2\omega_0)}\left(e^{j2\omega_0 T}\right)\right]_0^T$$

$$= \frac{A^2}{4T}\left[(T-0) + \frac{1}{(-j2\omega_0)}\left(e^{-j2\omega_0 T} - e^0\right)\right] + \frac{A^2}{4T}\left[(T-0) + \frac{1}{(j2\omega_0)}\left(e^{j2\omega_0 T} - e^0\right)\right]$$

$e^0 = 1$
$\omega_0 T = 2\pi$

$$= \frac{A^2}{4\cancel{T}}\left[\cancel{T} + \frac{1}{(-j2\omega_0)}\underbrace{\left(e^{-j4\pi} - 1\right)}_{=0}\right] + \frac{A^2}{4\cancel{T}}\left[\cancel{T} + \frac{1}{(j2\omega_0)}\underbrace{\left(e^{j4\pi} - 1\right)}_{=0}\right]$$

$e^{j2n\pi} = 1$

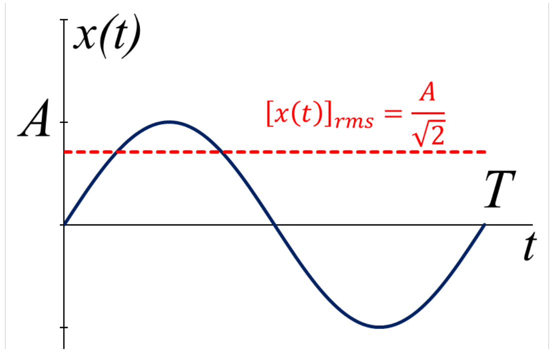
The integral of any sinusoidal or complex exponential function over an integer multiple of periods equals 0

$$= \frac{A^2}{4} + \frac{A^2}{4}$$

$$[x^2(t)]_{av} = \frac{A^2}{2} \qquad [x(t)]_{rms} = \frac{A}{\sqrt{2}}$$

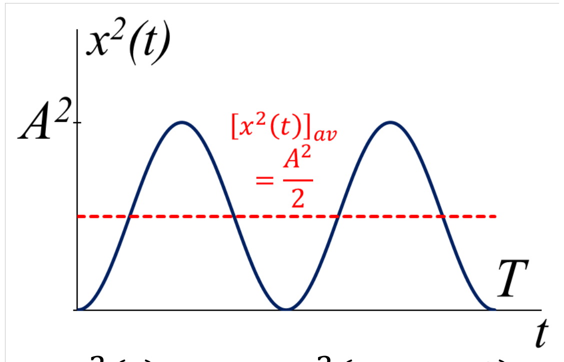
Same answers (as they must be)

Sinusoid Average Power, “Exponential Method 1” Observations



$$x(t) = A\cos(\omega_0 t + \theta)$$

$$\omega_0 = 2\pi f_0 = \frac{2\pi}{T}$$



$$x^2(t) = A\cos^2(\omega_0 t + \theta)$$

$$x(t) = \frac{A}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)$$

$$x^2(t) = \left[\frac{A}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)\right]\left[\frac{A}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)\right]$$

$$= \frac{A^2}{4}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)\underbrace{e^{-j\omega_0 t}}_{\text{Conjugate of } +\omega_0 \text{ term}} + \frac{A^2}{4}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)\underbrace{e^{j\omega_0 t}}_{\text{Conjugate of } -\omega_0 \text{ term}}$$

$$x^2(t) = \frac{A^2}{4}\left(1 + e^{-j2\omega_0 t}\right) + \frac{A^2}{4}\left(1 + e^{j2\omega_0 t}\right)$$

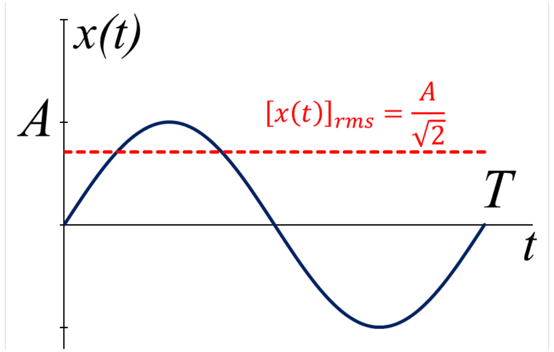
$$[x^2(t)]_{av} = \frac{1}{T}\cdot\frac{A^2}{4}\int_0^T \left(\underline{1} + \underbrace{e^{-j2\omega_0 t}}_{\text{Integrates to 0}}\right)dt + \frac{1}{T}\cdot\frac{A^2}{4}\int_0^T \left(\underline{1} + \underbrace{e^{j2\omega_0 t}}_{\text{Integrates to 0}}\right)dt$$

Multiplying any complex exponential by its own conjugate cancels the exponential component, leaving a constant that averages over a period to itself

Multiplying any other frequency component by that same conjugate leaves another complex exponential that integrates over a period to 0

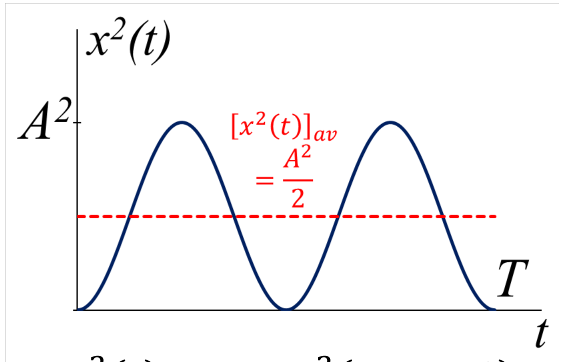
Seems rather useful…

Sinusoid Average Power, “Exponential Method 2”



$$x(t) = A\cos(\omega_0 t + \theta)$$

$$\omega_0 = 2\pi f_0 = \frac{2\pi}{T}$$



$$x^2(t) = A\cos^2(\omega_0 t + \theta)$$

$$x(t) = \frac{A}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right) = \frac{A}{2}e^{j\omega_0 t} + \frac{A}{2}e^{-j\omega_0 t}$$

$$x_+(t) = \frac{A}{2}e^{j\omega_0 t} \qquad x_-(t) = \frac{A}{2}e^{-j\omega_0 t}$$

$$x_+^*(t) = \frac{A}{2}e^{-j\omega_0 t} \qquad x_-^*(t) = \frac{A}{2}e^{j\omega_0 t}$$

$$[x_+^2(t)]_{av} = x_+(t)x_+^*(t) = \frac{A^2}{4} \qquad [x_-^2(t)]_{av} = x_-(t)x_-^*(t) = \frac{A^2}{4}$$

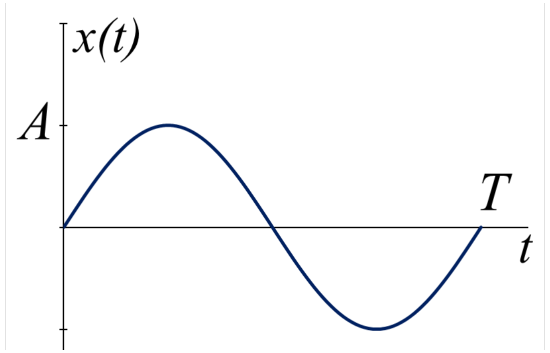
$$[x^2(t)]_{av} = [x_+^2(t)]_{av} + [x_-^2(t)]_{av} = \frac{A^2}{2}$$

Same answer (as it must be)

Power contribution from any frequency component (positive or negative) is given directly by multiplying by its complex conjugate (single step)

(…this only works due to the findings of the previous analysis…)

Sinusoid Amplitude



$$x(t) = A\cos(\omega_0 t + \theta)$$

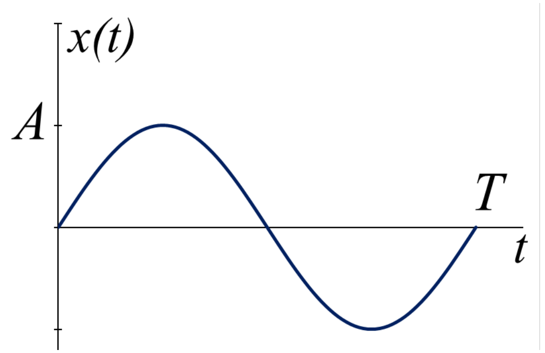
$$\omega_0 = 2\pi f_0 = \frac{2\pi}{T}$$

$$x(t) = \frac{A}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right) = \frac{A}{2}e^{j\omega_0 t} + \frac{A}{2}e^{-j\omega_0 t}$$

Preceding discussion was all about calculating average power of sinusoidal waveform with known amplitude

What if you didn't know the amplitude of a given frequency component and wanted to find it?

Sinusoid Amplitude (cont.)



$$x(t) = A\cos(\omega_0 t + \theta)$$

$$\omega_0 = 2\pi f_0 = \frac{2\pi}{T}$$

$$x(t) = \frac{A}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right) = \underline{\frac{A}{2}}e^{j\omega_0 t} + \underline{\frac{A}{2}}e^{-j\omega_0 t}$$

$$\frac{1}{T}\int_0^T x(t)\underline{e^{-j\omega_0 t}}dt = \frac{1}{T}\int_0^T \frac{A}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)e^{-j\omega_0 t}dt$$

Conjugate of $+\omega_0$ term (exponential term only)

$$= \frac{A}{2T}\int_0^T \left(1 + \underline{e^{-j2\omega_0 t}}\right)dt$$

Integrates to 0

$$= \frac{A}{2\cancel{T}}(\cancel{T})$$

$$= \frac{A}{2}$$

Amplitude of $+\omega_0$ term

$$\frac{1}{T}\int_0^T x(t)\underline{e^{j\omega_0 t}}dt = \frac{1}{T}\int_0^T \frac{A}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)e^{j\omega_0 t}dt$$

Conjugate of $-\omega_0$ term (exponential term only)

$$= \frac{A}{2T}\int_0^T \left(\underline{e^{j2\omega_0 t}} + 1\right)dt$$

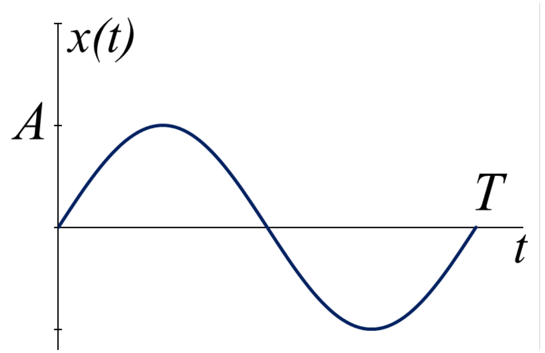
Integrates to 0

$$= \frac{A}{2\cancel{T}}(\cancel{T})$$

$$= \frac{A}{2}$$

Amplitude of $-\omega_0$ term

Sinusoid Amplitude (cont.)

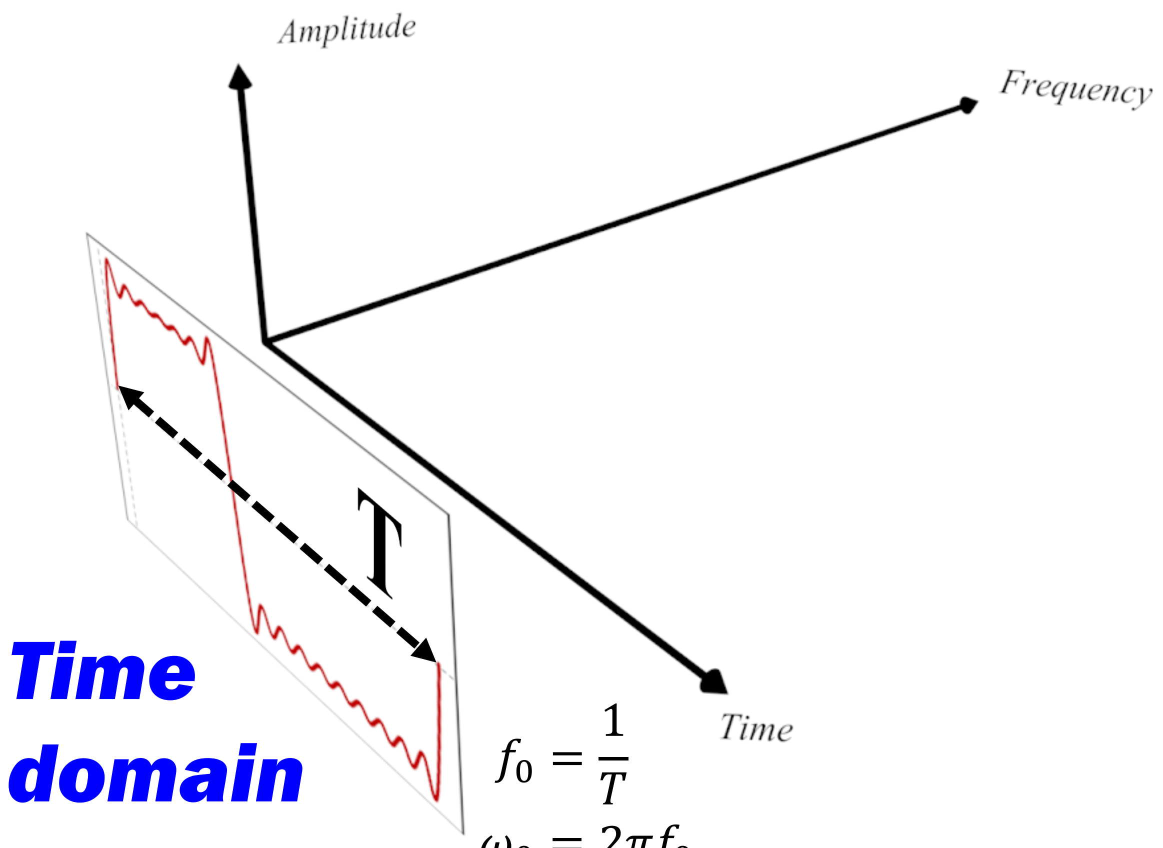


$$x(t) = A\cos(\omega_0 t + \theta)$$

$$\omega_0 = 2\pi f_0 = \frac{2\pi}{T}$$

- ***When you:***
  - *multiply by complex conjugate of desired frequency component*
  - *take time-average integral over period T*
- ***You get:***
  - *amplitude of desired frequency component only*
  - *all other frequency components integrate to 0*

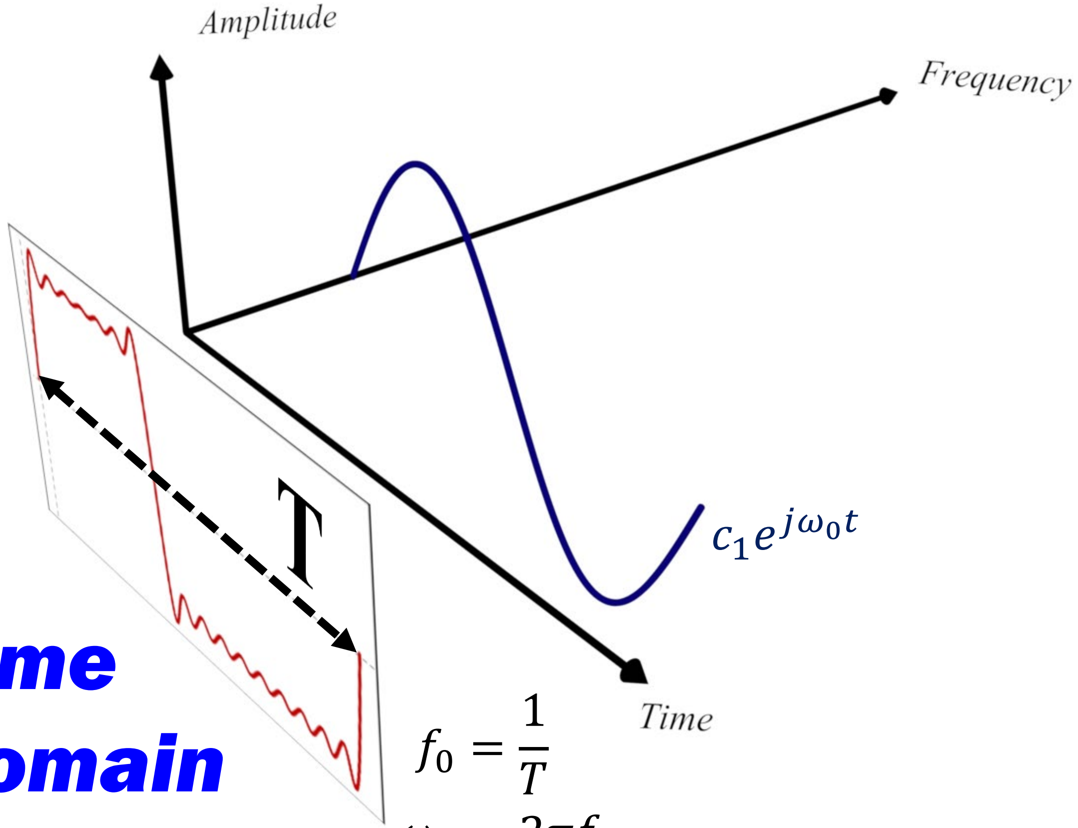
Fourier Series Expansion



$$f_0 = \frac{1}{T}$$
$$\omega_0 = 2\pi f_0$$

Any periodic signal of period T (fundamental frequency f_0 = 1/T) can be expressed as an infinite sum of sinusoidal components at integral multiples of the fundamental frequency (a.k.a "harmonics")

Fourier Series Expansion

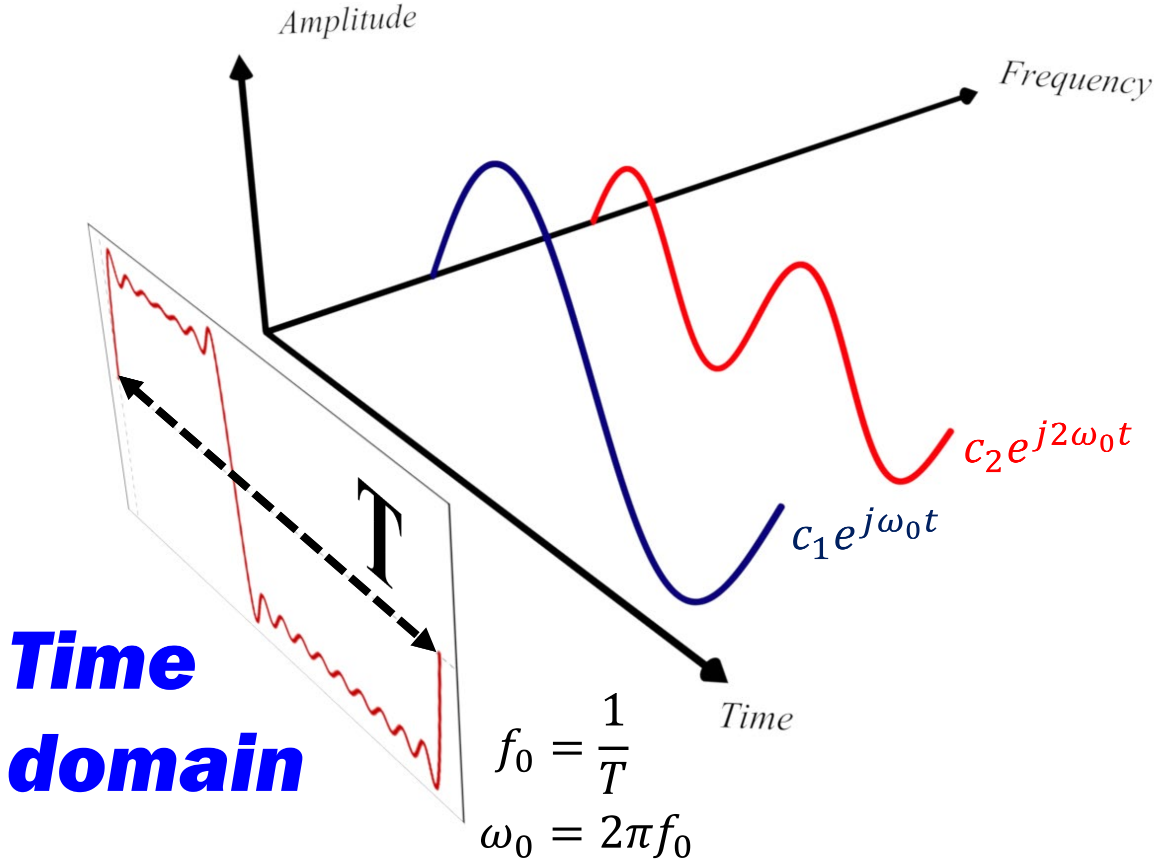


Time domain

$$f_0 = \frac{1}{T}$$
$$\omega_0 = 2\pi f_0$$

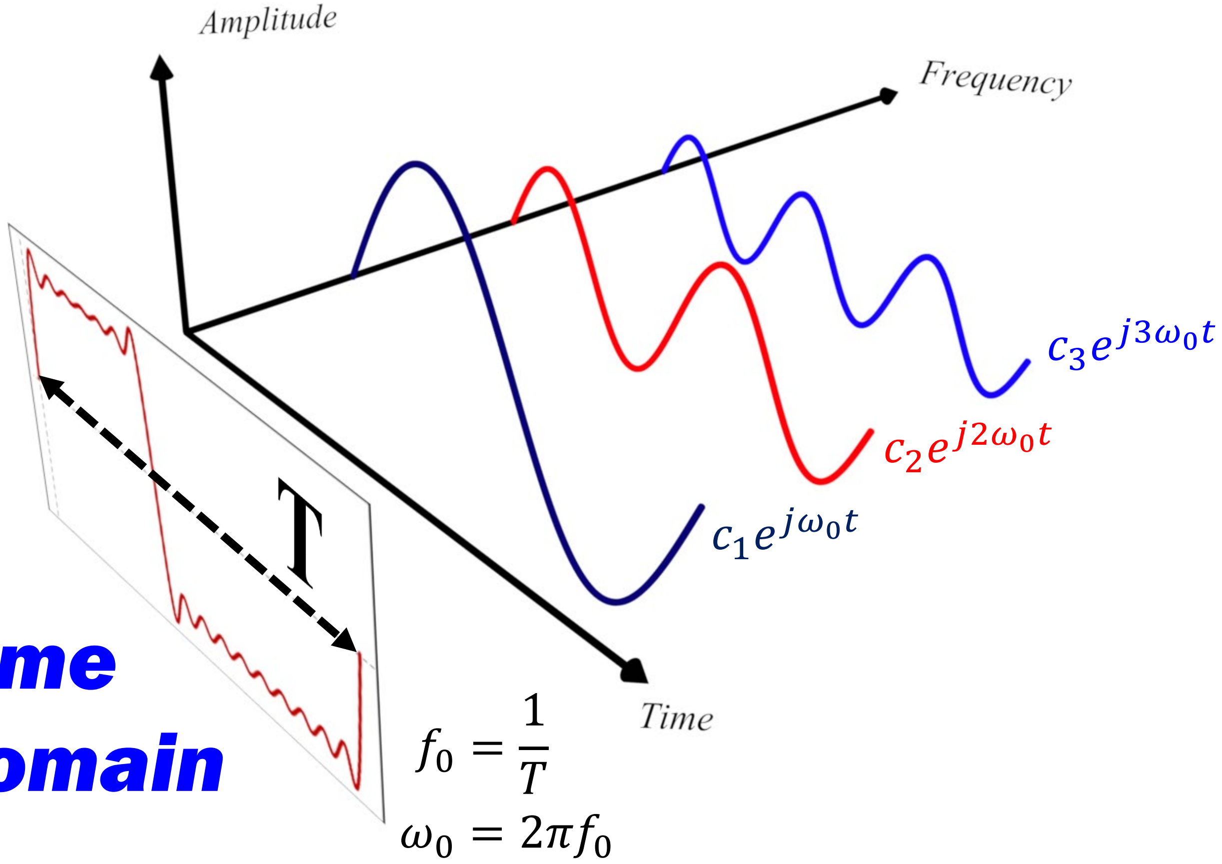
Any periodic signal of period T (fundamental frequency f_0 = 1/T) can be expressed as an infinite sum of sinusoidal components at integral multiples of the fundamental frequency (a.k.a "harmonics")

Fourier Series Expansion



Any periodic signal of period T (fundamental frequency f_0 = 1/T) can be expressed as an infinite sum of sinusoidal components at integral multiples of the fundamental frequency (a.k.a "harmonics")

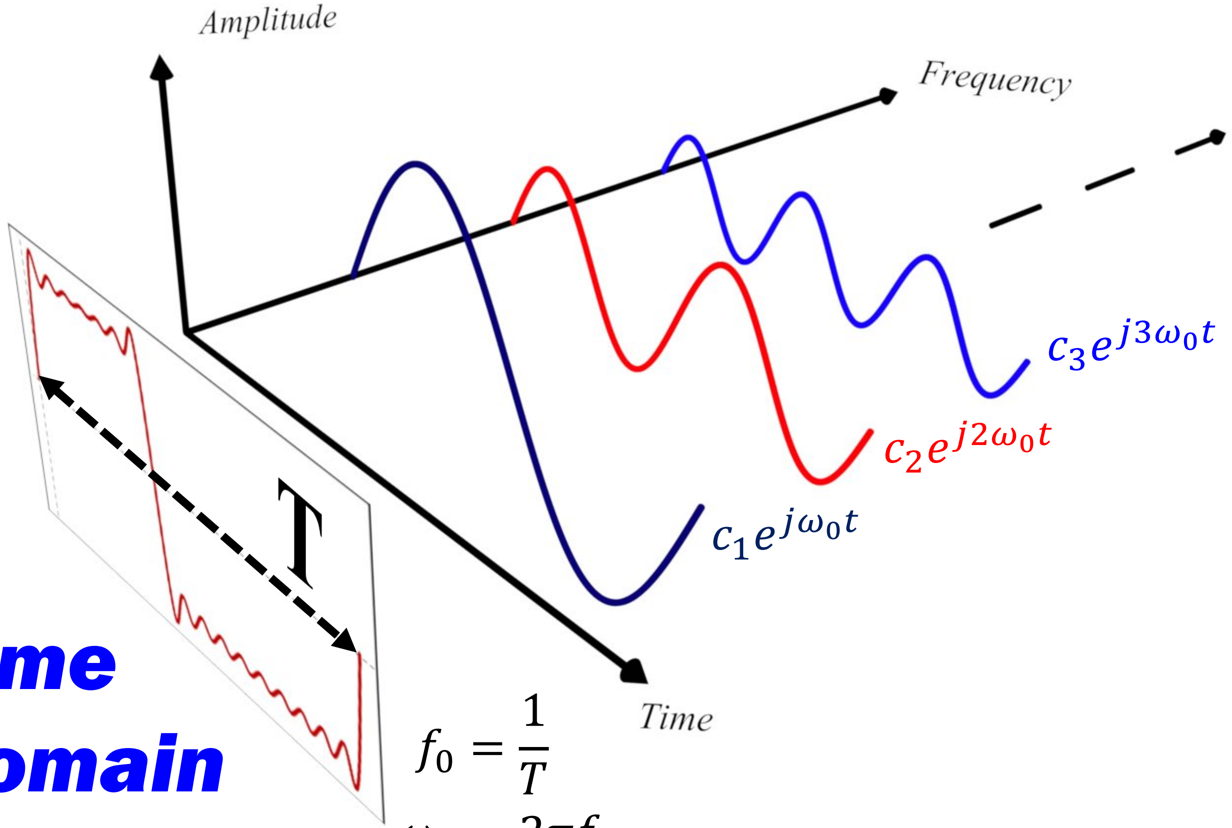
Fourier Series Expansion



Time domain

Any periodic signal of period T (fundamental frequency f_0 = 1/T) can be expressed as an infinite sum of sinusoidal components at integral multiples of the fundamental frequency (a.k.a "harmonics")

Fourier Series Expansion



Time domain

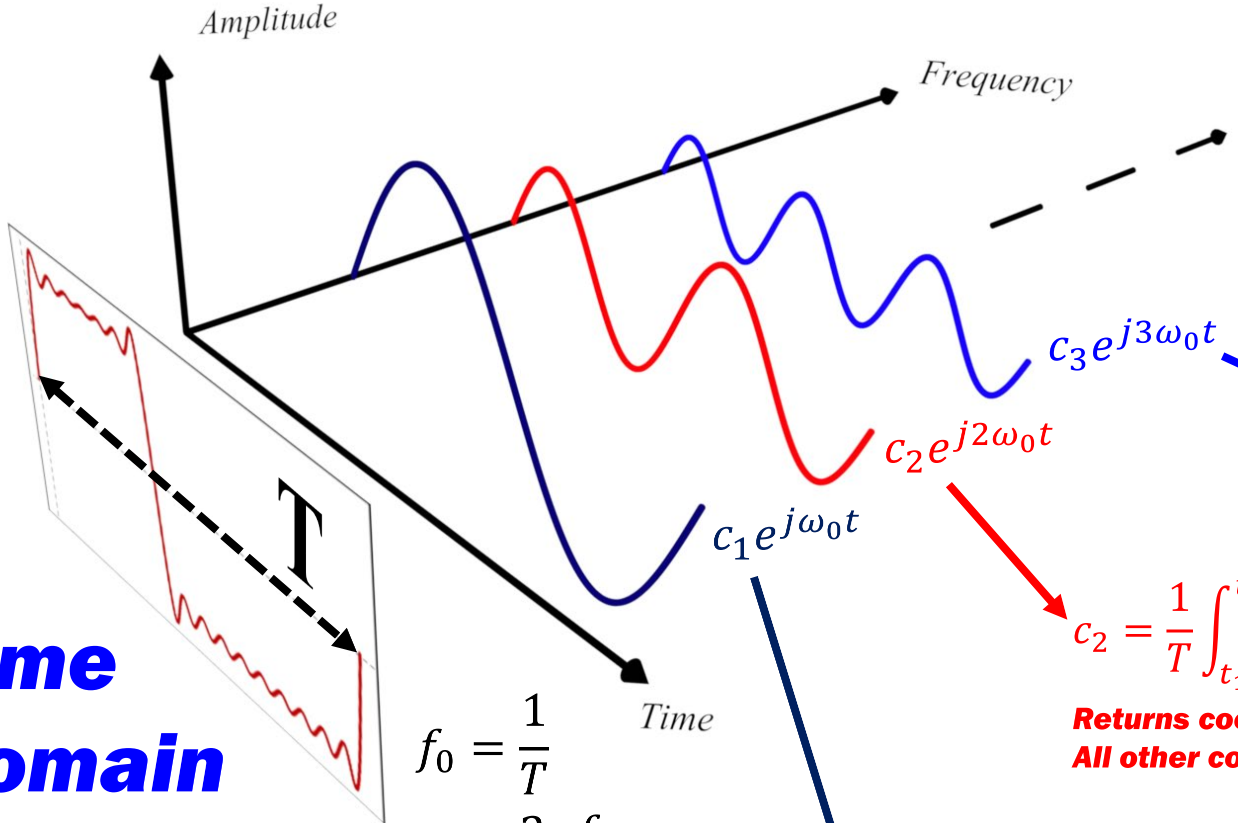
$$f_0 = \frac{1}{T}$$
$$\omega_0 = 2\pi f_0$$

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

Sum goes from -∞ to +∞ to include positive and negative frequency components (+ω and –ω)

Any periodic signal of period T (fundamental frequency f_0 = 1/T) can be expressed as an infinite sum of sinusoidal components at integer multiples of the fundamental frequency (a.k.a "harmonics")

Fourier Series Expansion



Time domain

$$f_0 = \frac{1}{T}$$

$$\omega_0 = 2\pi f_0$$

$$c_3 = \frac{1}{T}\int_{t_1}^{t_1+T} x(t)\underline{e^{-j3\omega_0 t}}\,dt$$

Returns coeffiecient of 3rd harmonic only
All other components integrate to 0

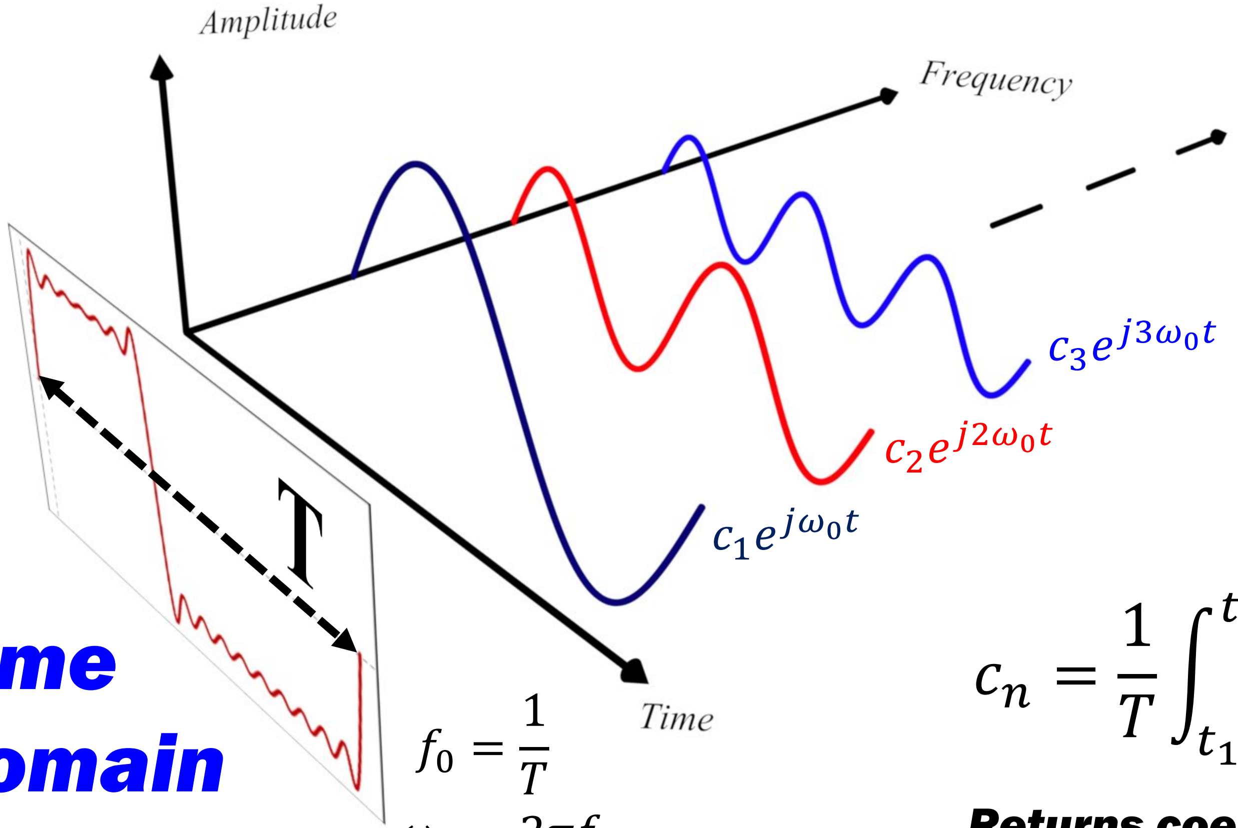
$$c_2 = \frac{1}{T}\int_{t_1}^{t_1+T} x(t)\underline{e^{-j2\omega_0 t}}\,dt$$

Returns coeffiecient of 2nd harmonic only
All other components integrate to 0

$$c_1 = \frac{1}{T}\int_{t_1}^{t_1+T} x(t)\underline{e^{-j\omega_0 t}}\,dt$$

Returns coeffiecient of
fundamental component only (n = 1)
All other components integrate to 0

Fourier Series Expansion



Time domain

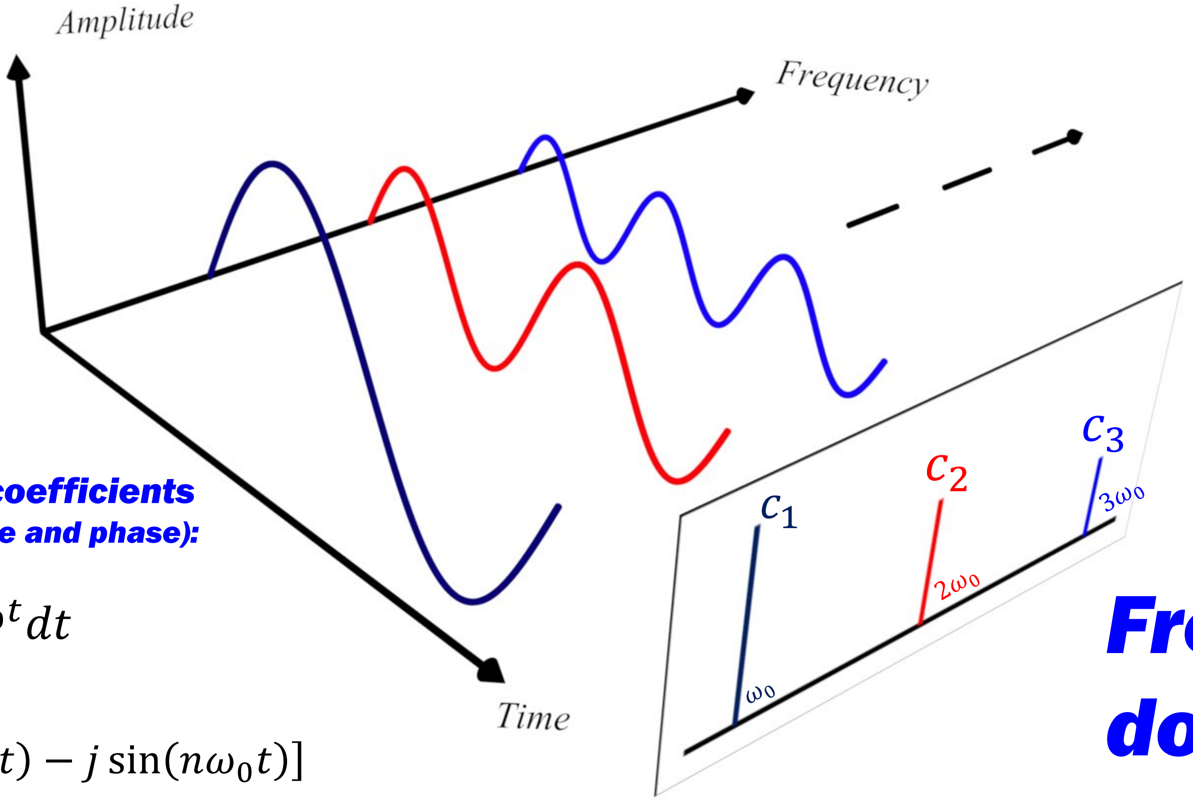
$$f_0 = \frac{1}{T}$$

$$\omega_0 = 2\pi f_0$$

$$c_n = \frac{1}{T}\int_{t_1}^{t_1+T} x(t)\underline{e^{-jn\omega_0 t}}\,dt$$

Returns coeffiecient of nth harmonic only
All other components integrate to 0

Fourier Series Expansion (cont.)



Frequency domain

General Fourier series expansion coefficients (can be complex numbers with magnitude and phase):

$$c_n = \frac{1}{T}\int_{t_1}^{t_1+T} x(t)e^{-jn\omega_0 t}dt$$

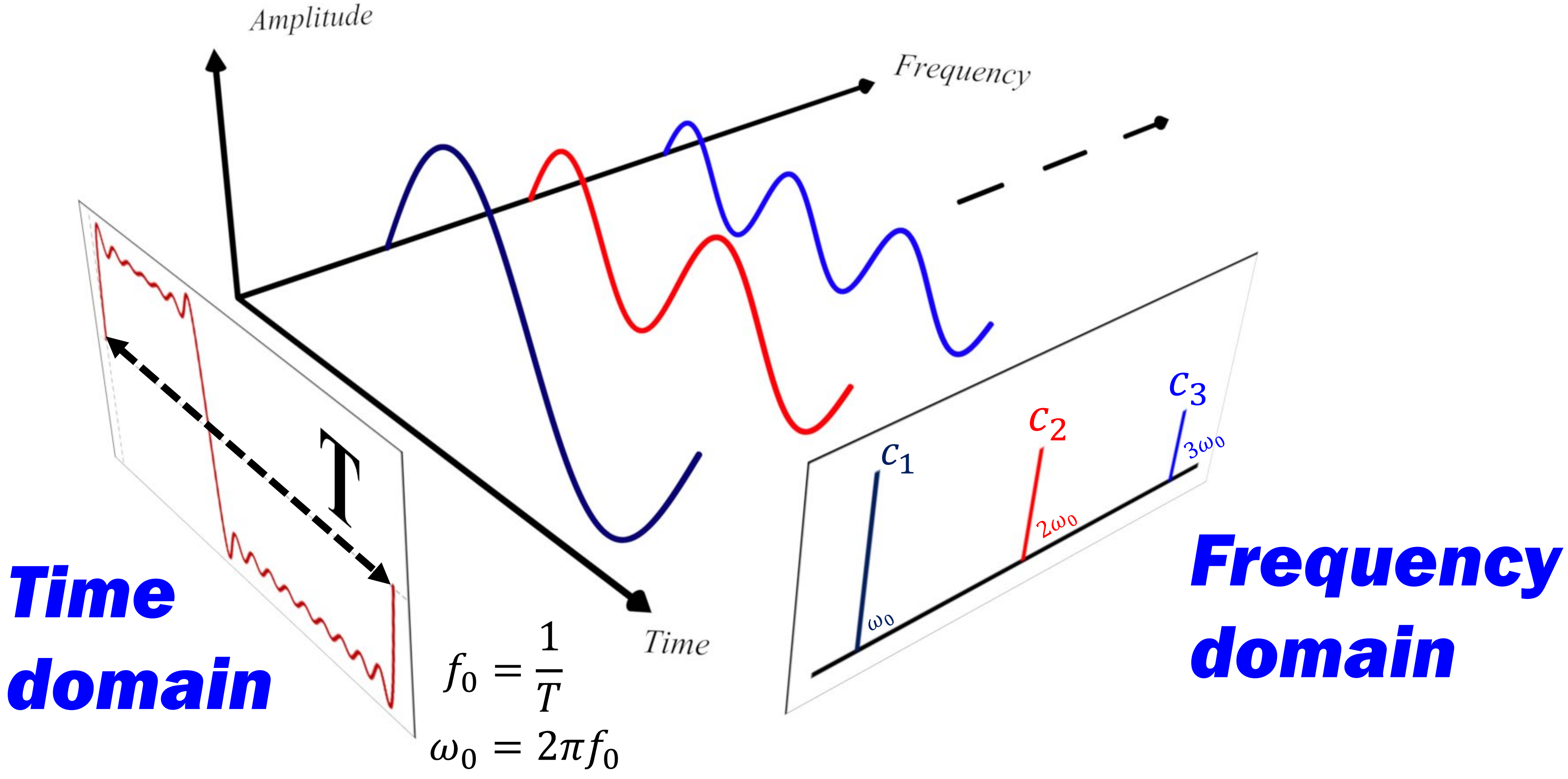
$$c_n = \frac{1}{T}\int_{t_1}^{t_1+T} x(t)[\cos(n\omega_0 t) - j\sin(n\omega_0 t)]$$

Real and imaginary parts give magnitude and phase

$$|c_n| = \sqrt{(Re_n)^2 + (Im_n)^2}$$

$$\angle c_n = \tan^{-1}\left(\frac{Im_n}{Re_n}\right)$$

Fourier Series Expansion (cont.)

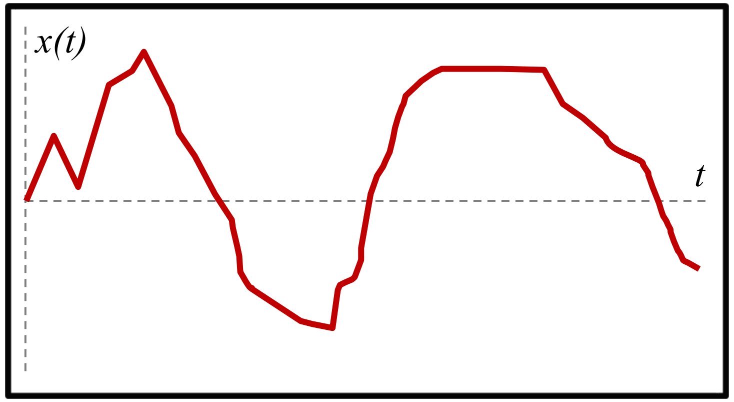


Fourier series expansion provides frequency content for periodic waveforms

Extremely useful, but…

Non-Periodic Waveform?

Time domain

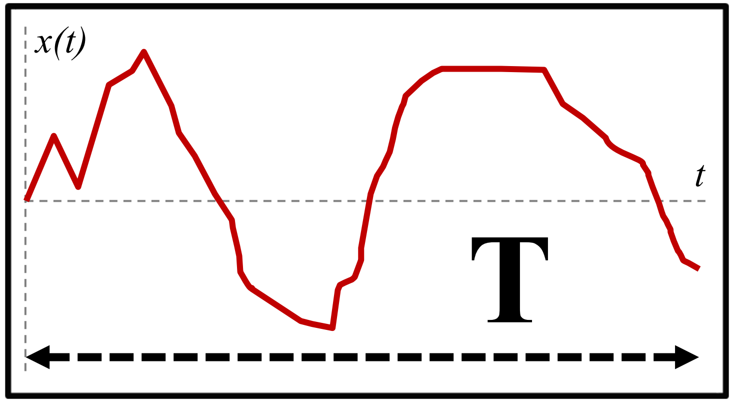


Frequency domain

?

What about a non-periodic waveform?

Time domain

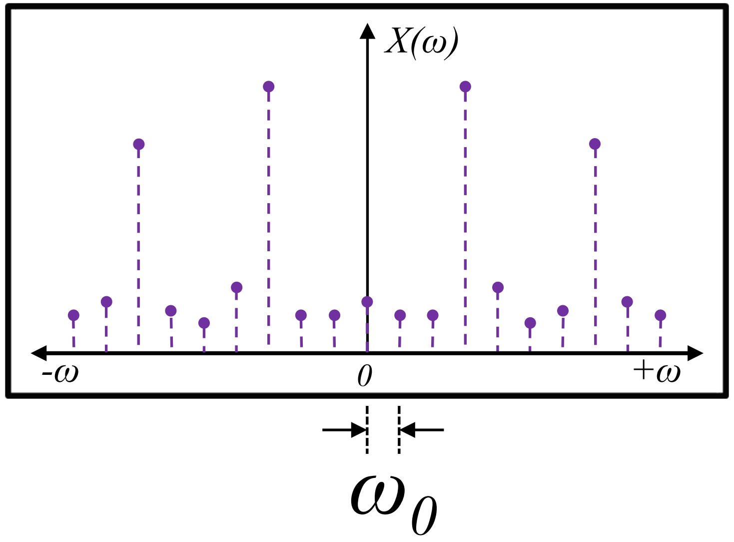


T = ***sampling interval (seconds)***

Effective base frequency:

$$f_0 = \frac{1}{T} \qquad \omega_0 = \frac{2\pi}{T}$$

$$c_n = \frac{1}{T}\int_{t_1}^{t_1+T} x(t)e^{-jn\omega_0 t}dt$$

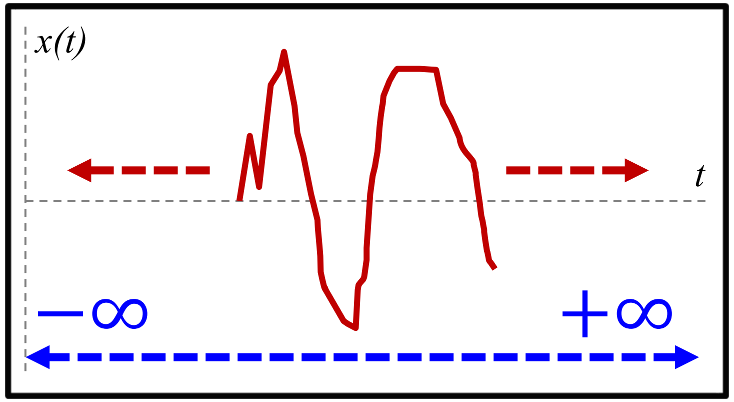


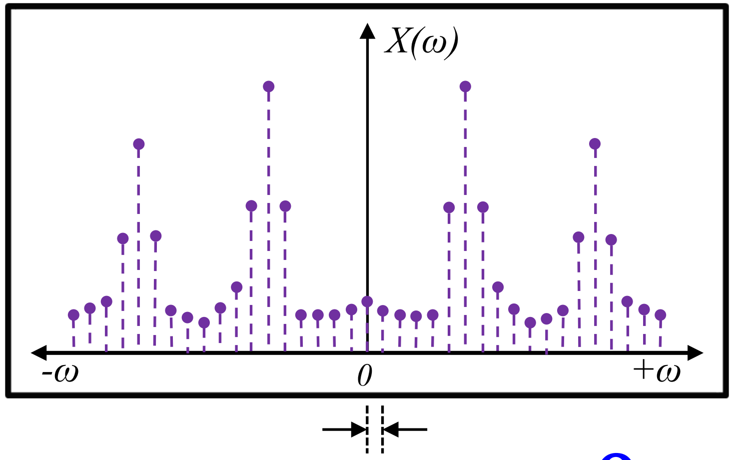
Discrete peaks at harmonics of ω_0

Fourier Series Approximation (cont.)

$$c_n = \frac{1}{T}\int_{t_1}^{t_1+T} x(t)e^{-jn\omega_0 t}\,dt$$

Time domain





T *extends to +/- ∞*

Effective base frequency:

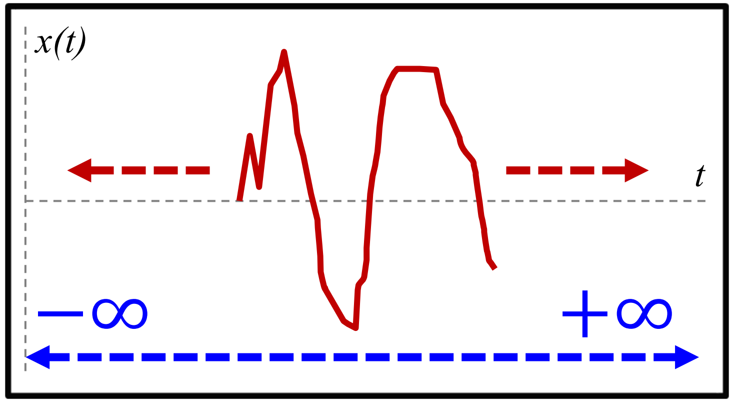
$$f_0 = \frac{1}{T} \qquad \omega_0 = \frac{2\pi}{T}$$

$$\omega_0 \rightarrow 0$$

("dω")

Fourier Transform (cont.)

Time domain



T *extends to +/- ∞*

Effective base frequency:

$$f_0 = \frac{1}{T} \qquad \omega_0 = \frac{2\pi}{T}$$

$$c_n = \frac{1}{T}\int_{t_1}^{t_1+T} x(t)e^{-jn\omega_0 t}dt$$

$$Tc_n = \int_{t_1}^{t_1+T} x(t)e^{-jn\omega_0 t}dt$$

$$Tc_n = \int_{\underline{-\infty}}^{\underline{\infty}} x(t)e^{-jn\omega_0 t}dt = X(n\omega_0)$$

$T \to \infty$
$\omega_0 \to 0$

n^{th} frequency "component"

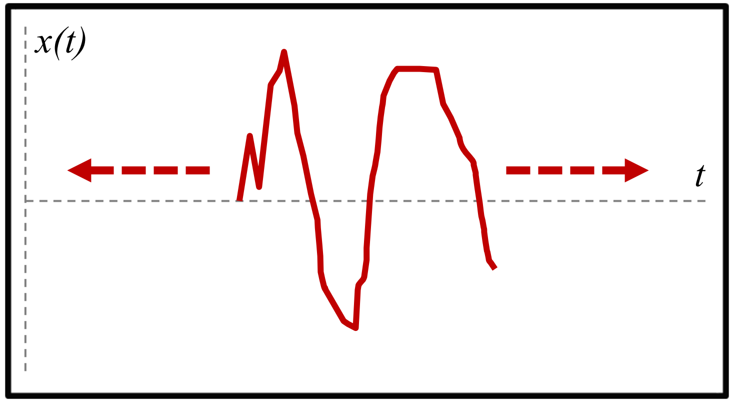
We'll come back to this…

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

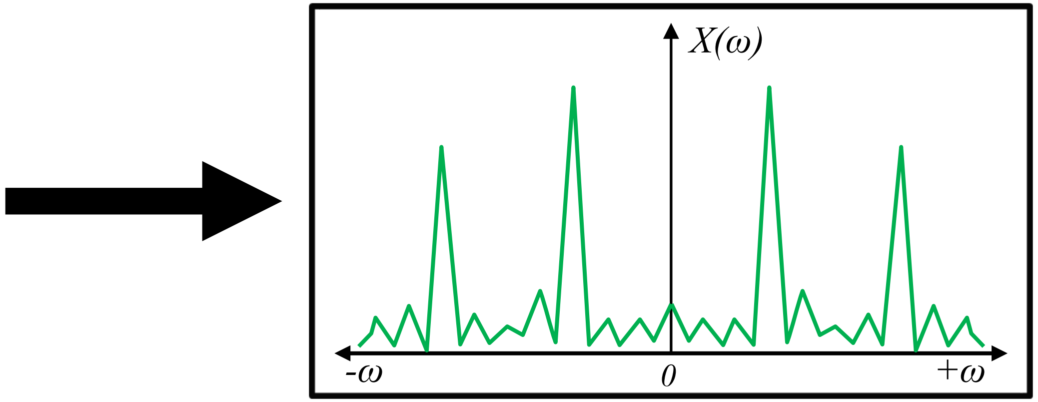
Fourier Transform
("Forward," $t \to \omega$)

Fourier Transform (cont.)

Time domain



Frequency domain



$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

Fourier Transform
(“Forward,” $t \rightarrow \omega$)

Inverse Fourier Transform

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$ $\omega_0 = \frac{2\pi}{T} \rightarrow \frac{T\omega_0}{2\pi} = 1$

$$= \frac{T\omega_0}{2\pi} \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

$$= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} T c_n e^{jn\omega_0 t} \omega_0$$

$Tc_n = X(n\omega_0)$
from previous slide

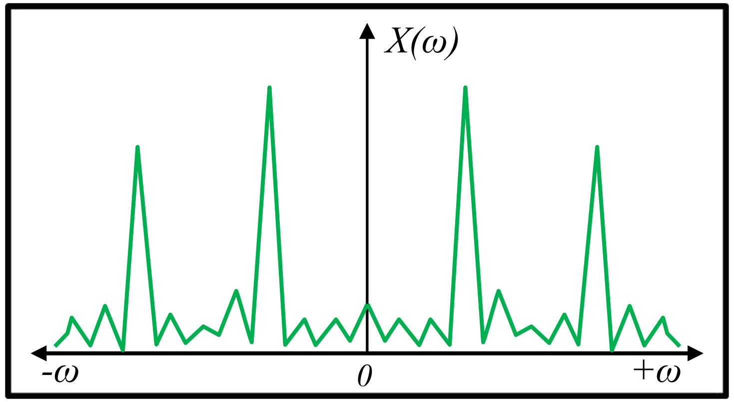
$$= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} X(n\omega_0) e^{jn\omega_0 t} \omega_0$$

$T \rightarrow \infty$
$\omega_0 \rightarrow 0$ ("$d\omega$")

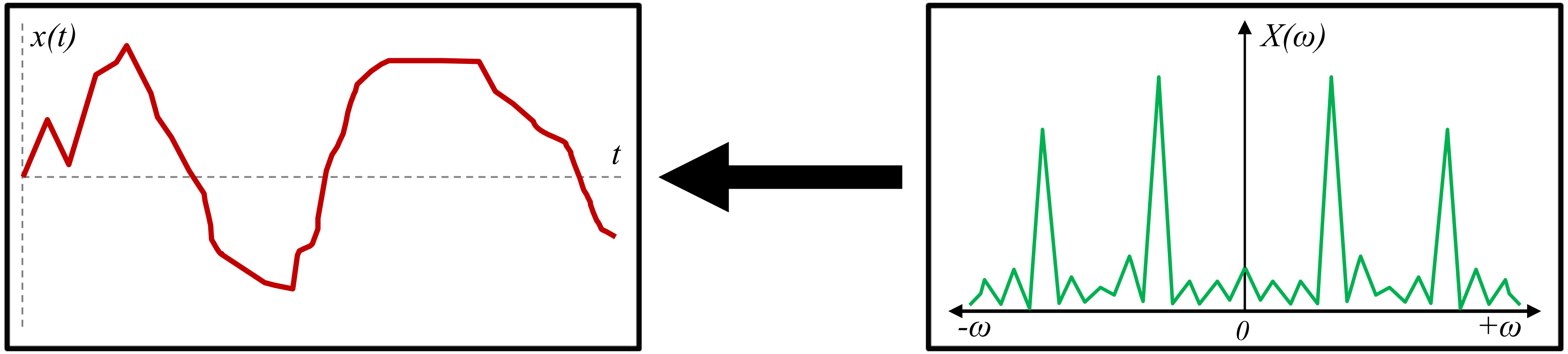
$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

Inverse Fourier Transform
($\omega \rightarrow t$)

Frequency domain



Time domain **Frequency domain**



$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$

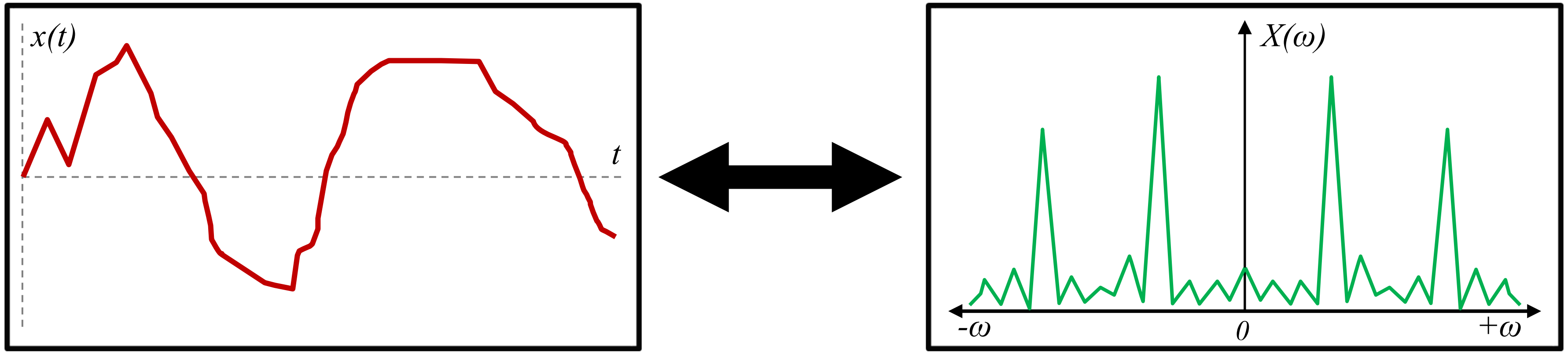
Inverse Fourier Transform

(ω → t)

Fourier Transform Pair

$$x(t) \leftrightarrow X(\omega)$$

"Fourier Transform pair"



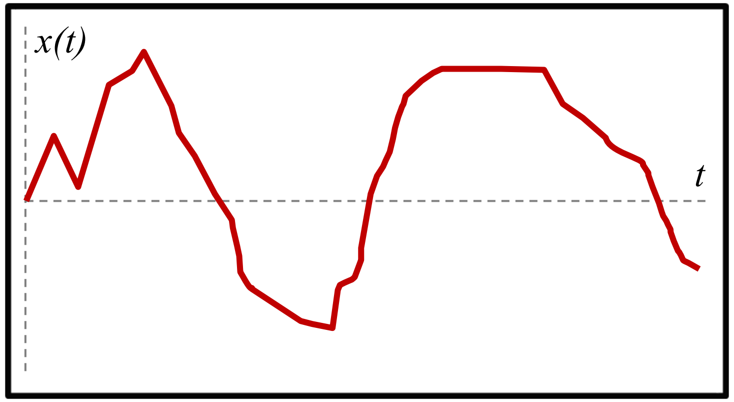
$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$

Inverse Fourier Transform
($\omega \rightarrow t$)

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

Fourier Transform
("Forward," $t \rightarrow \omega$)

Ambitious Integration Limits

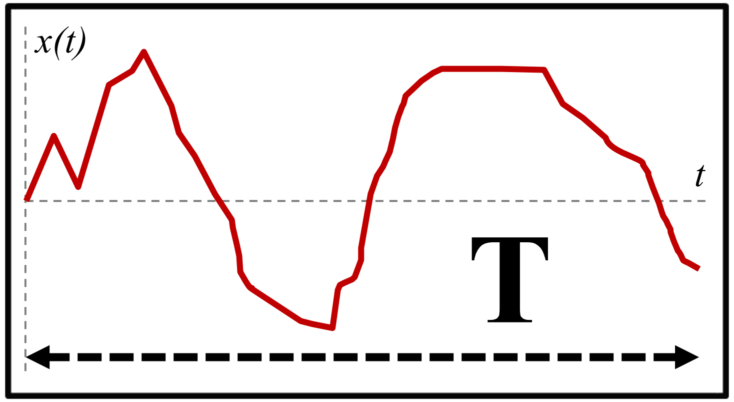
-∞  ∞

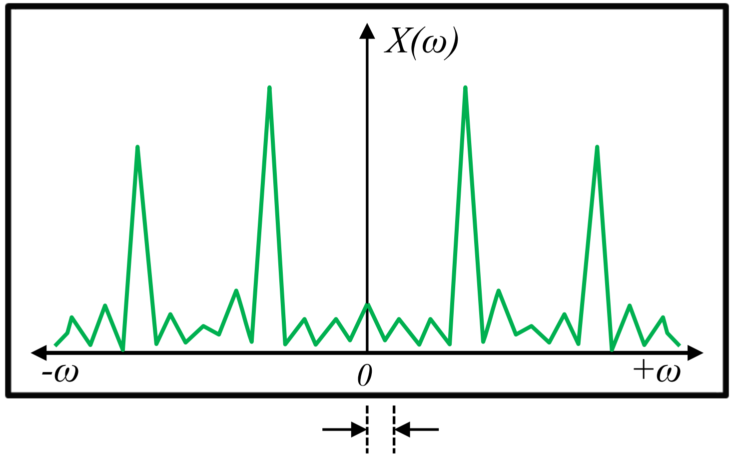
Still not enough…

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

Finite Sampling Interval and Resolution Bandwidth (RBW)

$$\mathrm{T} < \infty$$





T = ***sampling interval (seconds)***

Effective base frequency:

$$f_0 = \frac{1}{T} \quad \omega_0 = \frac{2\pi}{T}$$

$$\Delta\omega = \omega_0$$

Minimum resolvable frequency (a.k.a. "resolution bandwidth")

$$\Delta f = \frac{\Delta\omega}{2\pi} = \frac{1}{T}$$

Correlation to Fourier Series Coefficients

Recall:

$$Tc_n = X(n\omega_0)$$

n^{th} frequency "component" of Fourier transform does not directly equal c_n

$$c_n = \frac{X(n\omega_0)}{T}$$

Need to divide by sampling interval T to retrieve c_n

$$c_n = X(n\omega_0)\Delta f$$

Equivalent to multiplying by resolution bandwidth

$$c_n = \frac{1}{2\pi}X(n\omega_0)\Delta\omega$$

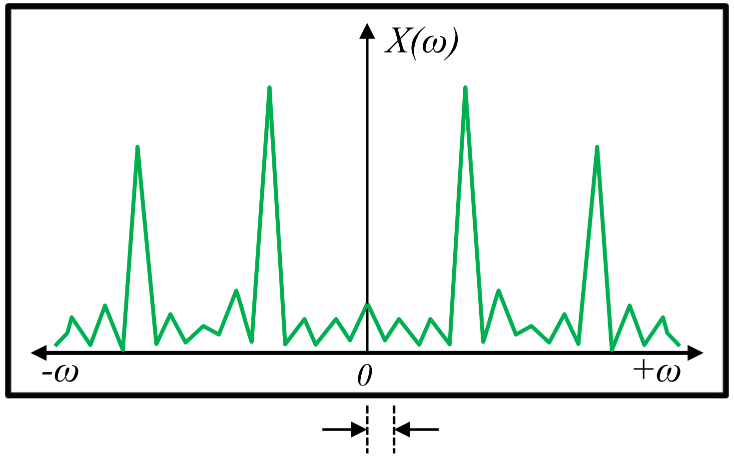
Need $1/2\pi$ factor when in terms of ω

$\Delta\omega \to 0$ ("$d\omega$")

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$

Direct Fourier Transform magnitudes must be scaled by resolution bandwidth in order to retrieve accurate magnitudes of frequency components

Spectrum analyzer performs calculation for you, but you must be aware of this for hand calculations

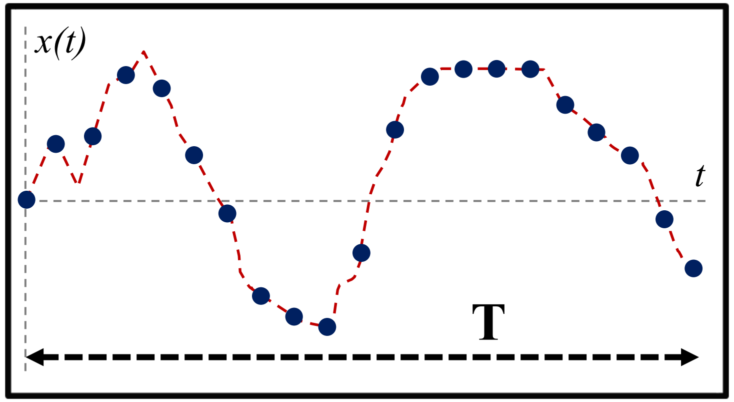


$$\Delta\omega = \omega_0$$

$$\Delta f = \frac{\Delta\omega}{2\pi} = \frac{1}{T}$$

Discrete Fourier Transform (DFT)

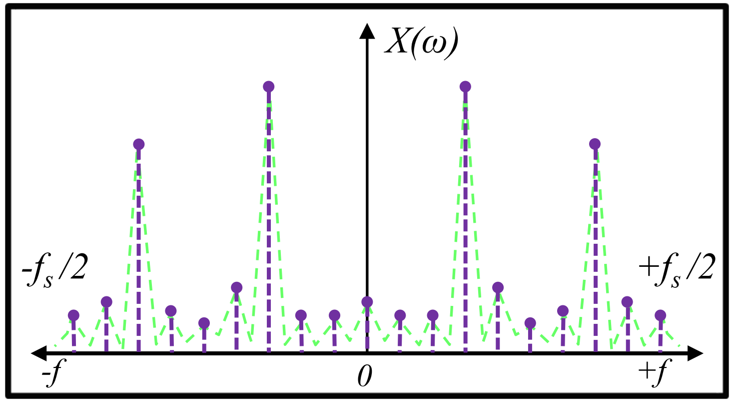
$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$



$T \rightarrow N$

$t \rightarrow n/N$

$\omega \rightarrow 2\pi k$



f_s = *sampling rate (samples/second)*

N = *number of time samples* = $f_s T$

$n = n^{th}$ *time sample*

Effective base frequency:

$$f_0 = \frac{1}{T} \qquad \omega_0 = \frac{2\pi}{T}$$

Maximum resolvable frequency = +/- $f_s/2$ *(Nyquist)*

= +/- $N/2T$

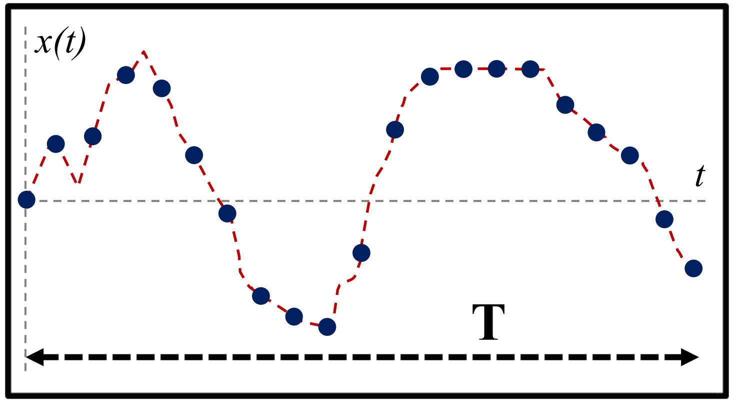
$$\textit{Number of frequency steps} = \frac{f_s}{f_0} = \frac{N/T}{1/T} = N$$

(-f and +f)

$k = k^{th}$ *frequency component*

Discrete Fourier Transform (DFT)

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$



f_s = sampling rate (samples/second)

N = number of time samples = $f_s T$

n = n^{th} time sample

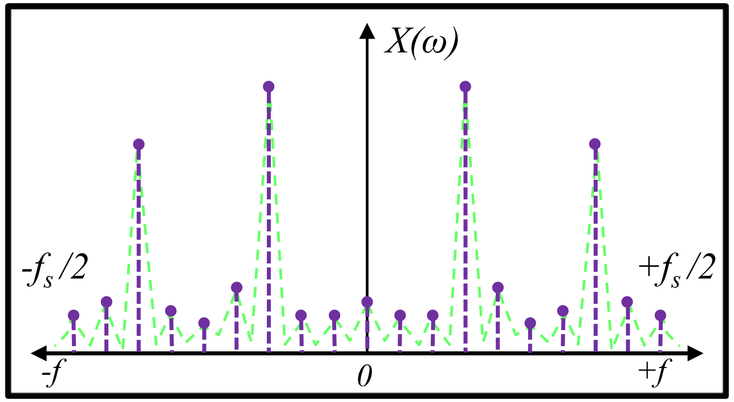
Effective base frequency:

$$f_0 = \frac{1}{T} \qquad \omega_0 = \frac{2\pi}{T}$$

$T \rightarrow N$

$t \rightarrow n/N$

$\omega \rightarrow 2\pi k$



k = k^{th} frequency component

$$X(k) = \sum_{k=0}^{N-1} x(n)e^{-j2\pi kn/N}$$

Discrete Fourier Transform (DFT)

Need to divide by N to retrieve true magnitudes (equivalent to dividing by T for continuous transform)

Fast Fourier Transform (FFT)

- **Specific implementation of DFT to facilitate more efficient calculations**
- **Developed by James Cooley and John Tukey while serving on President Kennedy's Science Advisory Committee (published in 1965 during Johnson administration)**
- **Fast DFT algorithms can be traced to Carl Friedrich Gauss's unpublished 1805 work on orbits of asteroids Pallas and Juno**
- **Excellent Veratisium video addressing history as well as technical aspects (link below)**
- **Recommended viewing:**
  - "Introduction to the Fourier Transform (Part 1)," Brian Douglas (Mathworks)
    - https://www.youtube.com/watch?v=1JnayXHhjlg
  - "Understanding the Discrete Fourier Transform and the FFT," Brian Douglas (Mathworks)
    - https://www.youtube.com/watch?v=QmgJmh2I3Fw
  - "The Remarkable Story Behind The Most Important Algorithm Of All Time," Veritasium
    - https://www.youtube.com/watch?v=nmgFG7PUHfo&t=0s

Fourier Transform: Symmetry

$$x(t) \leftrightarrow X(\omega)$$

$$\begin{cases} X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \\ x(t) = \dfrac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega \end{cases}$$

$$2\pi x(-t) = \int_{-\infty}^{\infty} X(\omega)e^{-j\omega t}d\omega$$

$$2\pi x(-\omega) = \underbrace{\int_{-\infty}^{\infty} X(t)e^{-j\omega t}dt}_{\text{Fourier Transform of } X(t)}$$

Exchange t and ω

$$X(t) \leftrightarrow 2\pi x(-\omega)$$

Fourier Transform: Derivative and Integral

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$

Derivative:

$$\frac{dx(t)}{dt} \leftrightarrow j\omega \cdot X(\omega)$$

$$\frac{d^{(k)}x(t)}{dt^{(k)}} \leftrightarrow (j\omega)^k \cdot X(\omega)$$

Integral:

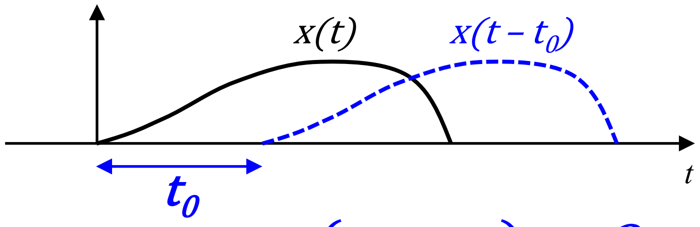
$$\int x(t) \leftrightarrow \frac{1}{j\omega} \cdot X(\omega)$$

$$\ldots\iiint x(t) \leftrightarrow \frac{1}{(j\omega)^k} \cdot X(\omega)$$

k^{th} order integral

Fourier Transform: Time/Phase Shift

$$x(t) \leftrightarrow X(\omega)$$



$$x(t - t_0) \leftrightarrow ?$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$\int_{-\infty}^{\infty} x(t - t_0) e^{-j\omega (t - t_0)} dt$$

Substitute:
$a = t - t_0$

$$= \int_{-\infty}^{\infty} x(a) e^{-j\omega (t_0 + a)} da$$

$$= \int_{-\infty}^{\infty} x(a) e^{-j\omega a} e^{-j\omega t_0} da$$

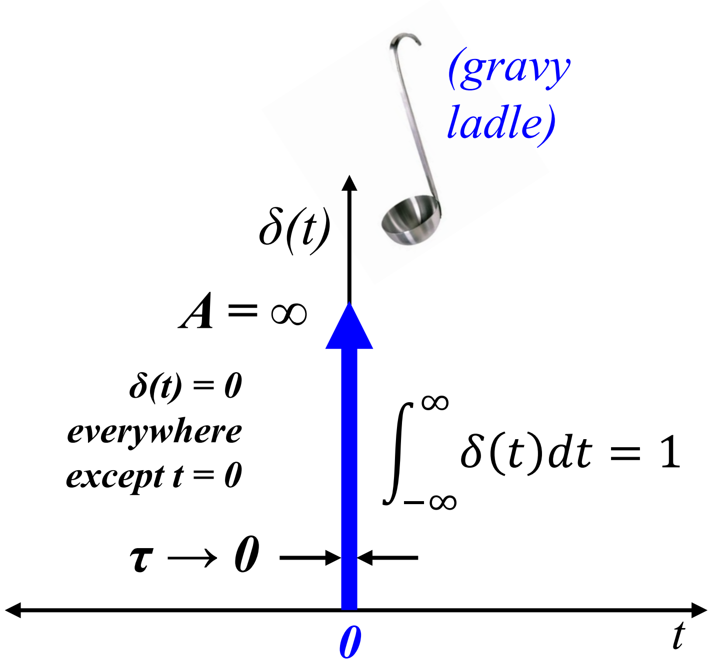
$$= e^{-j\omega t_0} \int_{-\infty}^{\infty} x(a) e^{-j\omega a}\, da$$

$$= e^{-j\omega t_0} X(\omega)$$

$$x(t - t_0) \leftrightarrow e^{-j\omega t_0} X(\omega)$$

Time shift produces phase shift in frequency domain

Fourier Transform: Ideal Impulse



"Dirac Delta" function

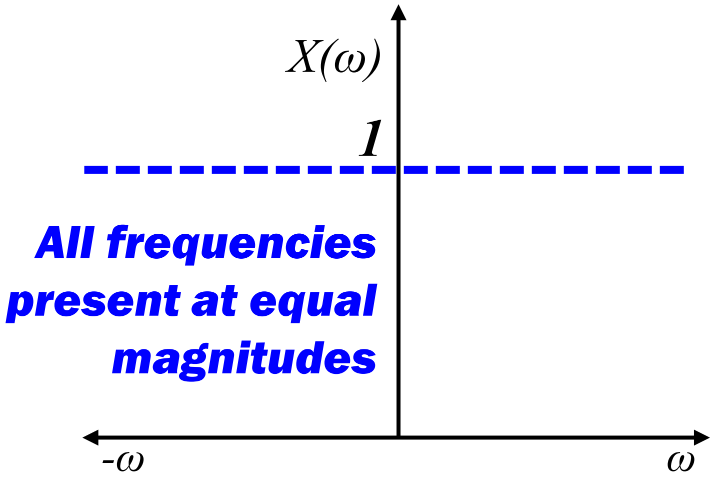
$$X(\omega) = \int_{-\infty}^{\infty} \underbrace{\delta(t)e^{-j\omega t}}_{\neq 0 \text{ only at } t=0} dt$$

$$X(\omega) = \int_{-\infty}^{\infty} \delta(t)e^{0}dt$$

$$X(\omega) = \int_{-\infty}^{\infty} \delta(t)dt$$

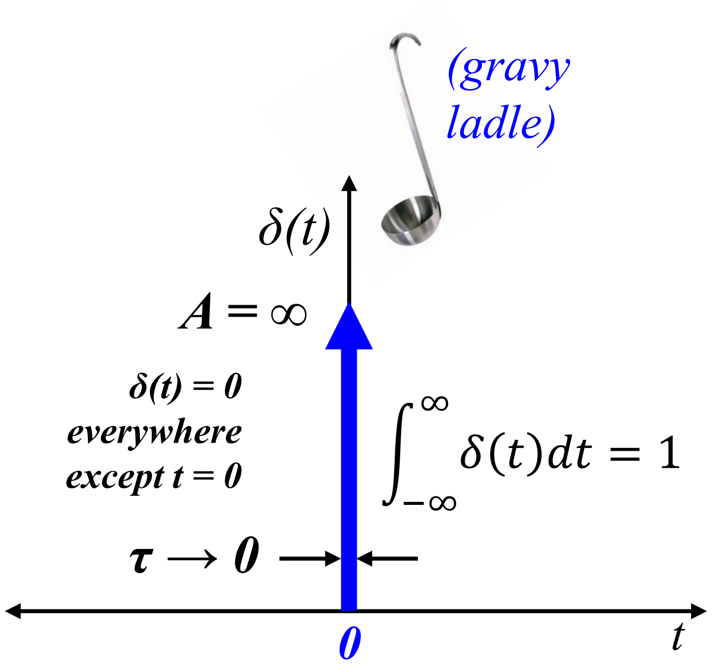
$$X(\omega) = 1$$

Independent of frequency



$$\delta(t) \leftrightarrow 1$$

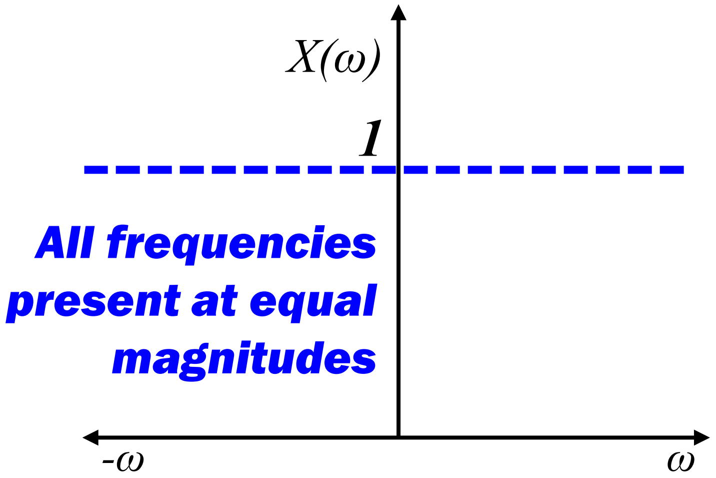
Fourier Transform: Ideal Impulse (cont.)



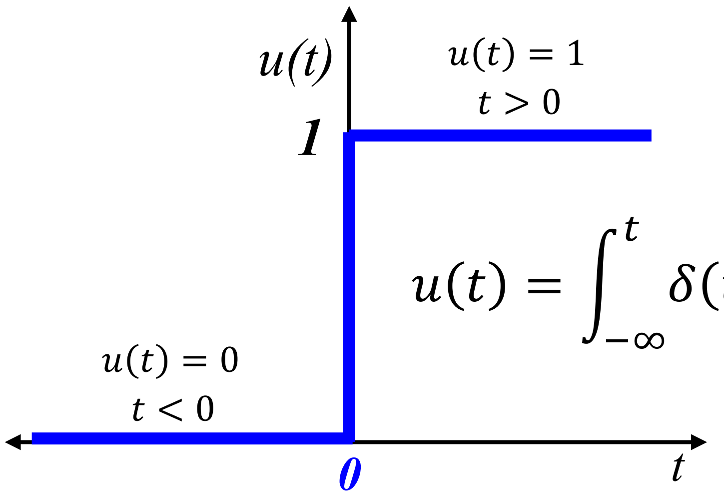
"Dirac Delta" function

Fourier transform of impulse response in time domain gives frequency response of linear system

(Extremely useful)



Fourier Transform: Unit Step Function

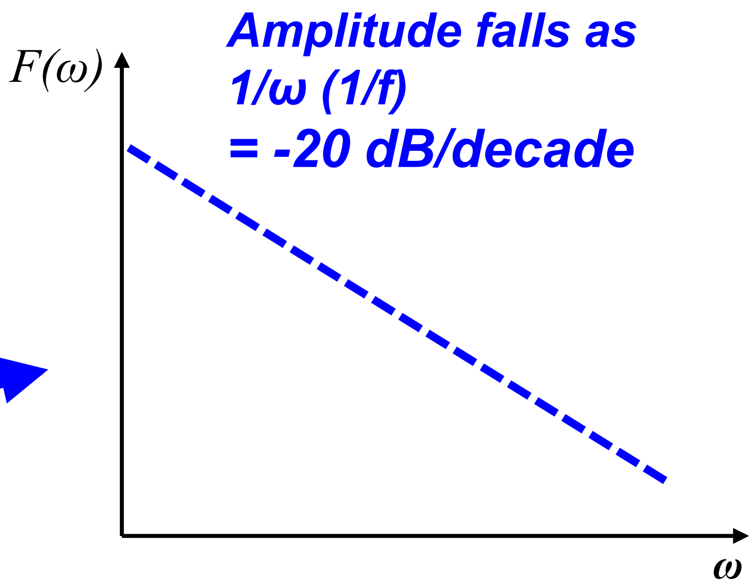


$$u(t) = \int_{-\infty}^{t} \delta(t)dt$$

$$F[u(t)] = F\left[\int_{-\infty}^{t} \delta(t)dt\right]$$

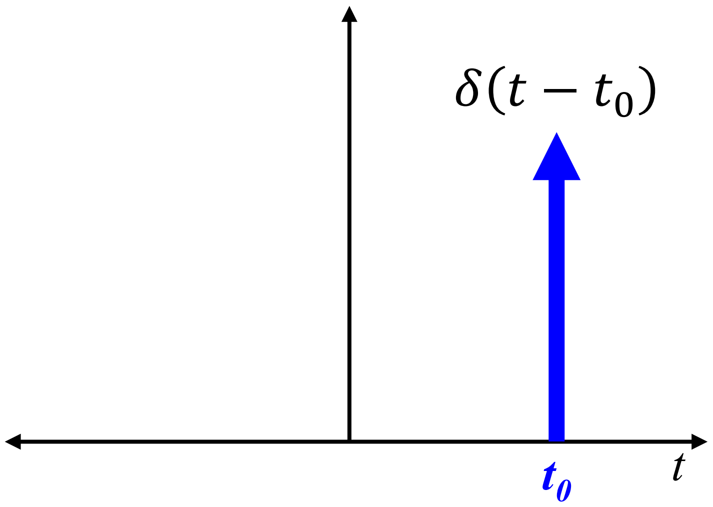
$$F[u(t)] = \frac{1}{j\omega} F[\delta(t)]$$

$$F[u(t)] = \frac{1}{j\omega}$$



$$u(t) \leftrightarrow \frac{1}{j\omega}$$

Fourier Transform: Ideal Impulse, Time-Shifted



$$\delta(t) \leftrightarrow 1$$

From time/phase shift property:

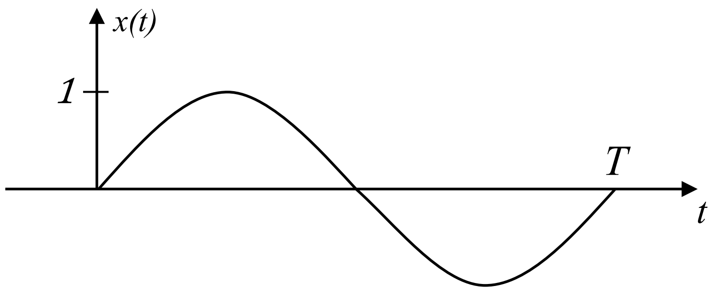
$$\delta(t - t_0) \leftrightarrow e^{-j\omega t_0}$$

From symmetry property:

$$X(t) \leftrightarrow 2\pi x(-\omega)$$

$$e^{j\omega_0 t} \leftrightarrow 2\pi\delta(\omega - \omega_0)$$

Fourier Transform: Sinusoid



$$x(t) = \cos(\omega_0 t + \theta)$$

$$\omega_0 = 2\pi f_0 = \frac{2\pi}{T}$$

$$x(t) = \cos(\omega_0 t + \theta) = \frac{1}{2}\left[e^{j(\omega_0 t+\theta)} + e^{-j(\omega_0 t+\theta)}\right]$$

$$= \frac{1}{2}e^{j\theta}e^{j\omega_0 t} + \frac{1}{2}e^{-j\theta}e^{-j\omega_0 t}$$

$$e^{j\omega_0 t} \leftrightarrow 2\pi\delta(\omega - \omega_0)$$

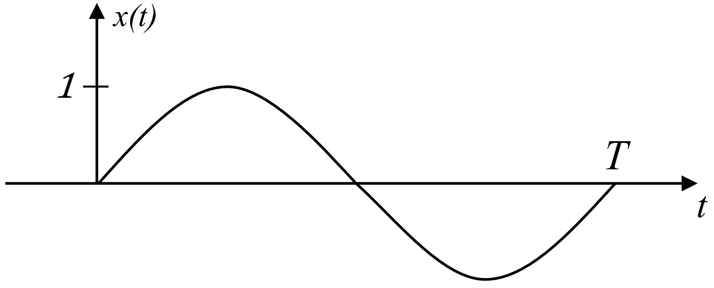
$$e^{-j\omega_0 t} \leftrightarrow 2\pi\delta(\omega + \omega_0)$$

$$\cos(\omega_0 t + \theta) \leftrightarrow \pi\delta(\omega - \omega_0)e^{j\theta} + \pi\delta(\omega + \omega_0)e^{-j\theta}$$

Phase terms

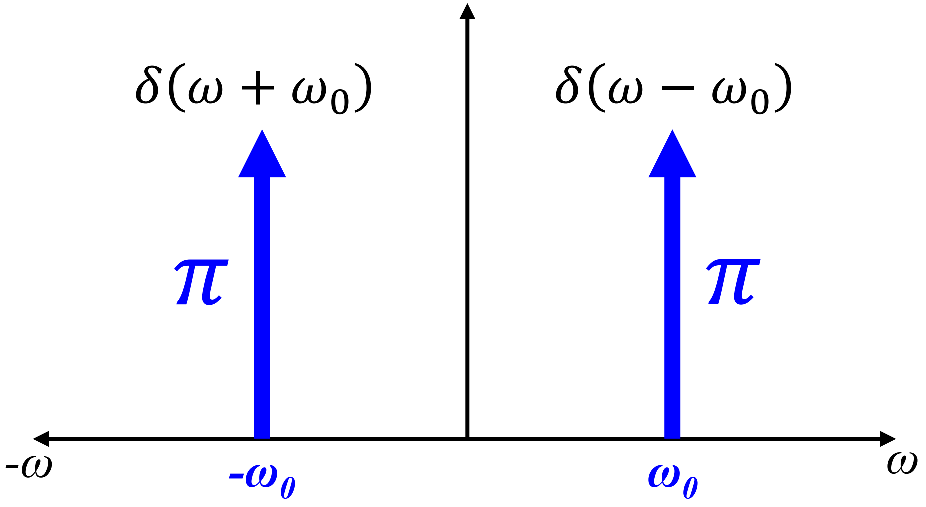
θ = 0 for simplicity: $\cos(\omega_0 t) \leftrightarrow \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$

Fourier Transform: Sinusoid (cont.)



$$x(t) = \cos(\omega_0 t + \theta)$$

$$\cos(\omega_0 t) \leftrightarrow$$
$$\pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$

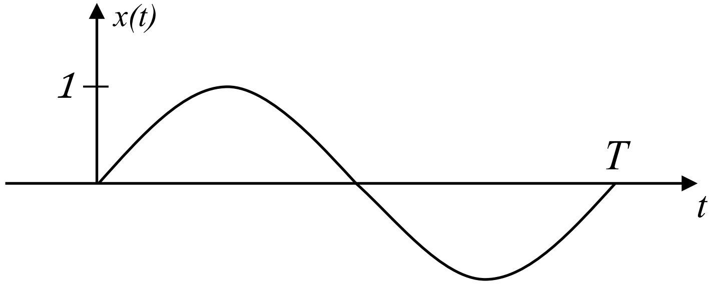


Sinusoid of frequency ω_0 in time domain maps to pair of delta functions (single peaks) in frequency domain at $+\omega_0$ and $-\omega_0$

As expected, but about those magnitudes…

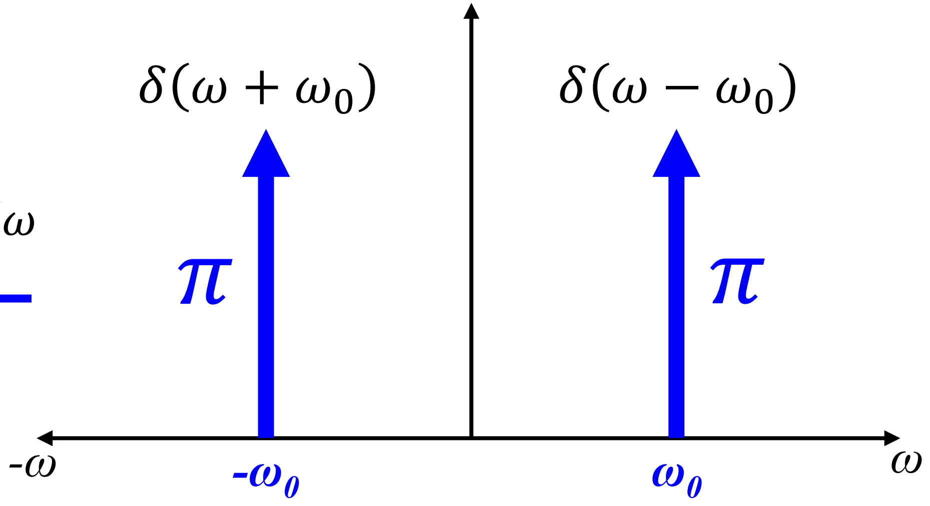
Fourier Transform: Sinusoid (cont.)

$$X(\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$



$$x(t) = \cos(\omega_0 t + \theta)$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$



Fourier transform magnitudes facilitate conversion back to time domain waveform

Always remember factor of $1/2\pi$ for "hand" calculations

Spectrum analyzer does conversion for you (but you need to be aware of it)

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}[\pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)]e^{j\omega t}d\omega$$

$$= \frac{1}{2}\left[\int_{-\infty}^{\infty}\underbrace{\delta(\omega - \omega_0)e^{j\omega t}d\omega}_{\neq 0 \text{ only at } \omega = \omega_0} + \int_{-\infty}^{\infty}\underbrace{\delta(\omega + \omega_0)e^{j\omega t}d\omega}_{\neq 0 \text{ only at } \omega = -\omega_0}\right]$$

$$= \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)$$

$$= \cos(\omega_0 t)$$

Summary

- **Fourier transforms (forward and inverse) are powerful tools for converting signals between time and frequency domains**
- **Signal is unchanged; all that changes is how we look at it**
- **Care must be taken when interpreting frequency domain magnitudes produced by forward transform**
  - Spectrum analyzer does conversion for you
  - You still need to be aware of this for any hand calculations
- **Next session: Fourier Series expansions**

References

- Papoulis, Athanasios: The Fourier Integral and Its Applications
- Paul, Clayton: Introduction to Electromagnetic Compatibility