

Fundamentals of Electromagnetics

Time Domain vs. Frequency Domain
Part 2: Fourier Series Expansion

John McCloskey
Chief EMC Engineer
NASA/Goddard Space Flight Center
Greenbelt, MD
John.C.McCloskey@nasa.gov

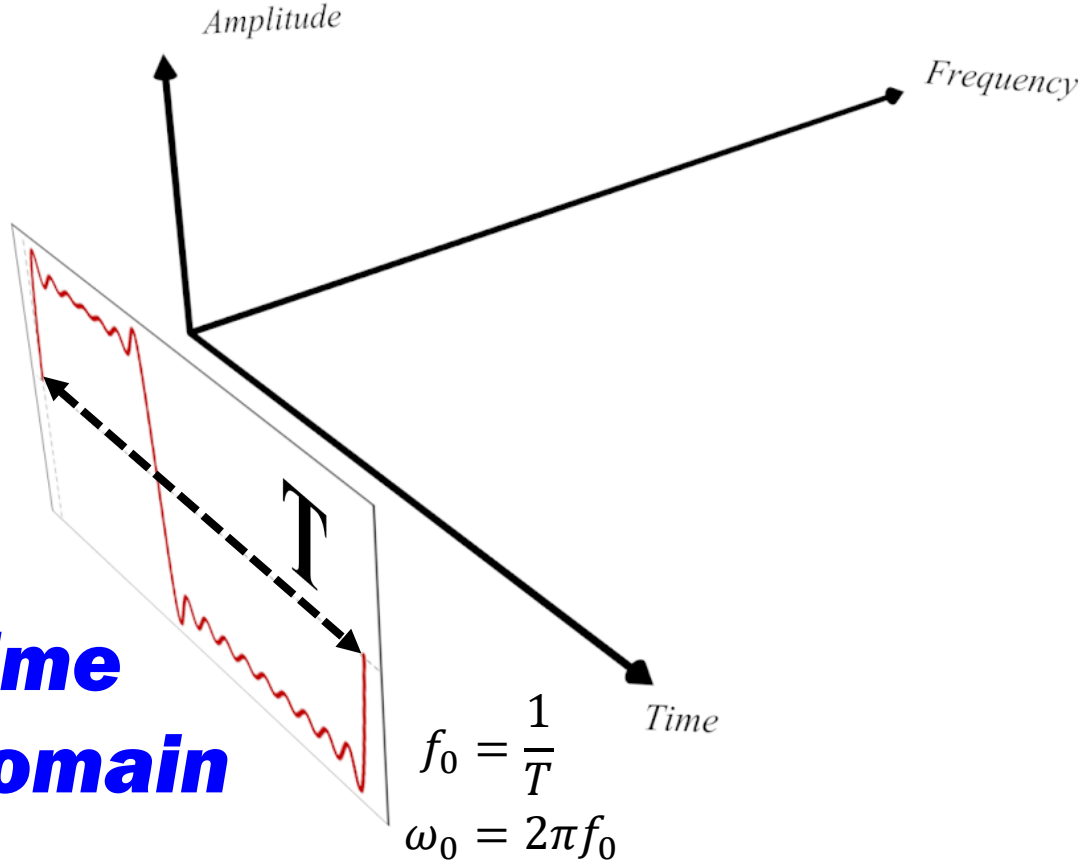


Topics

- **Fourier Series Expansion**
- **Examples**
  - Square Wave
  - Rectangular Pulse Train
  - Trapezoidal Pulse Train
- **Virtual Demonstrations: Measured Spectra and Comparison to Theory**
- **Summary**

- ***Shameless plug: "Fundamentals of Electromagnetics" video series***
  - *Publicly available on YouTube; search for above title*
  - *Direct playlist link:* https://youtube.com/playlist?list=PLtrpQ-gPvnJn2r9Mw49jij7Ky0mb6RJYF&si=UxEKqVRgsR9w6nZ7

Fourier Series Expansion



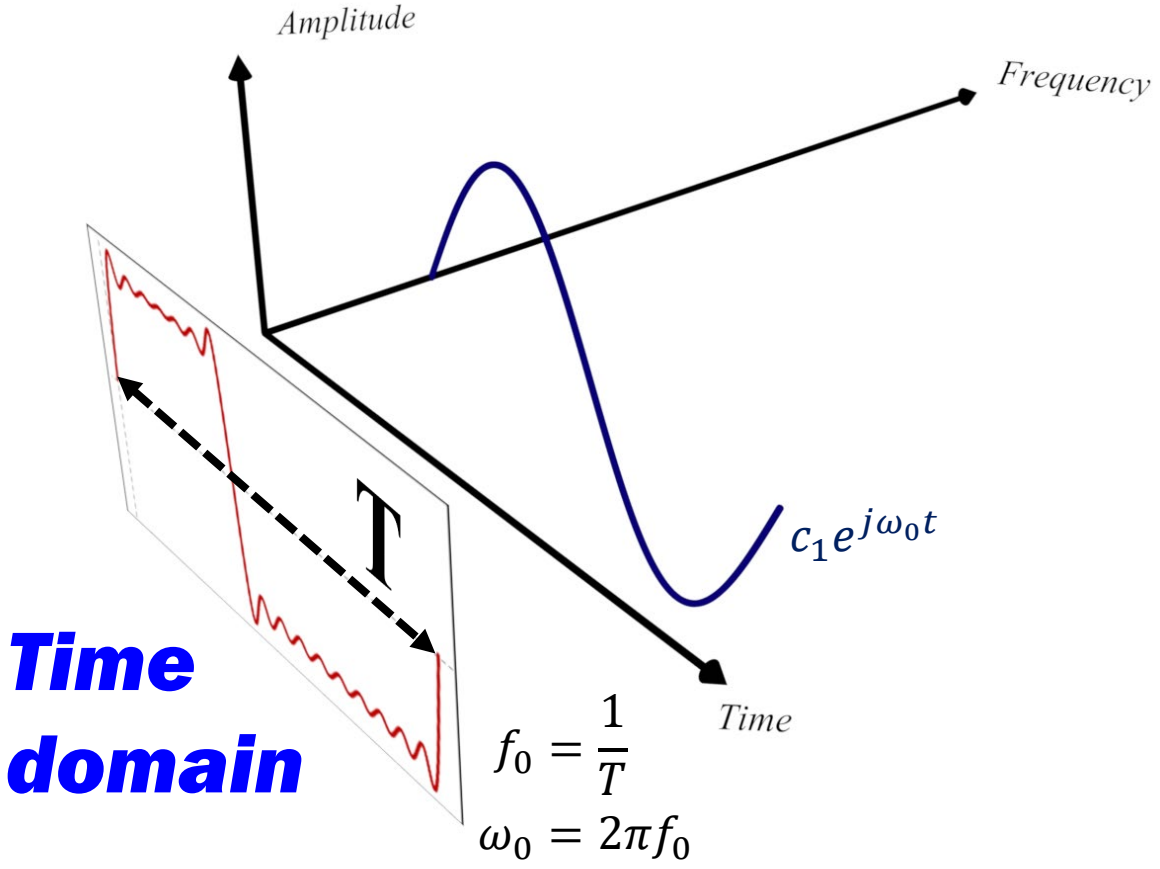
Time domain

$$f_0 = \frac{1}{T}$$

$$\omega_0 = 2\pi f_0$$

Any periodic signal of period T (fundamental frequency f_0 = 1/T) can be expressed as an infinite sum of sinusoidal components at integer multiples of the fundamental frequency (a.k.a "harmonics")

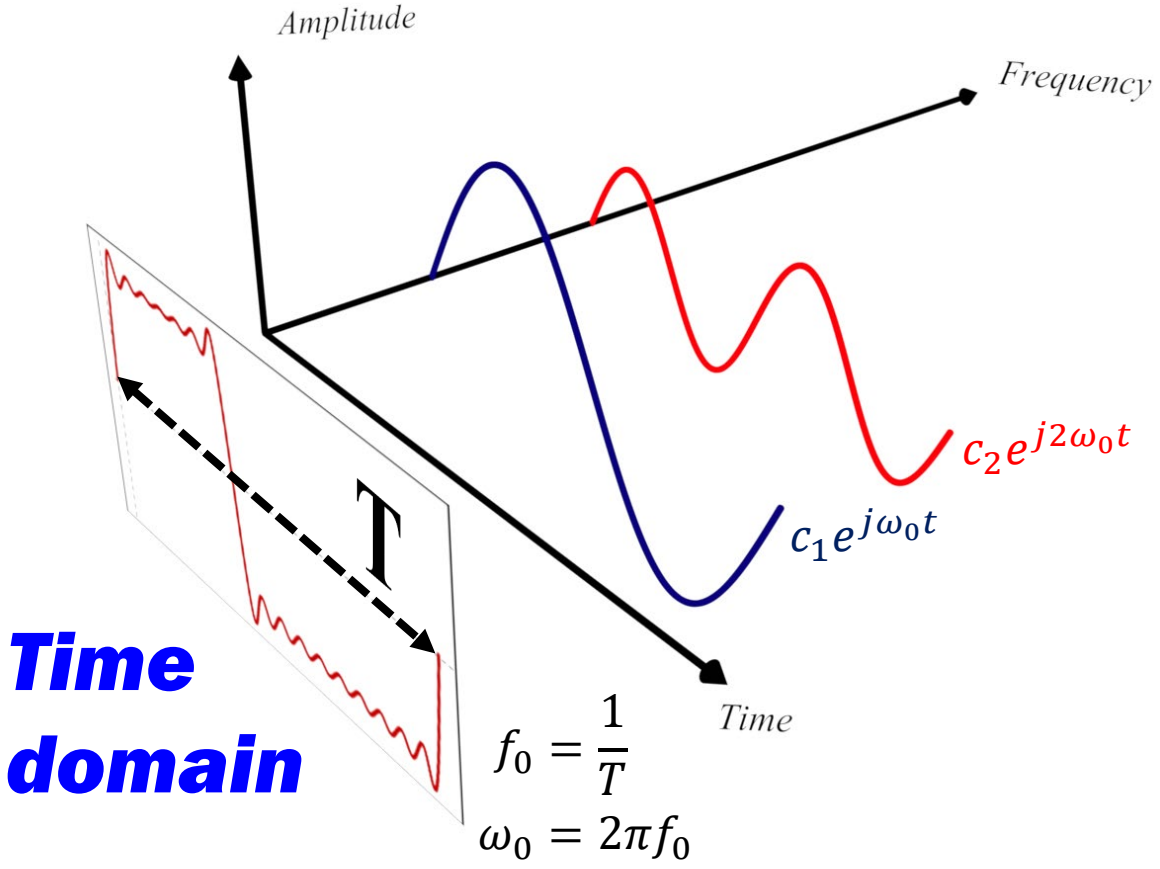
Fourier Series Expansion



Time domain

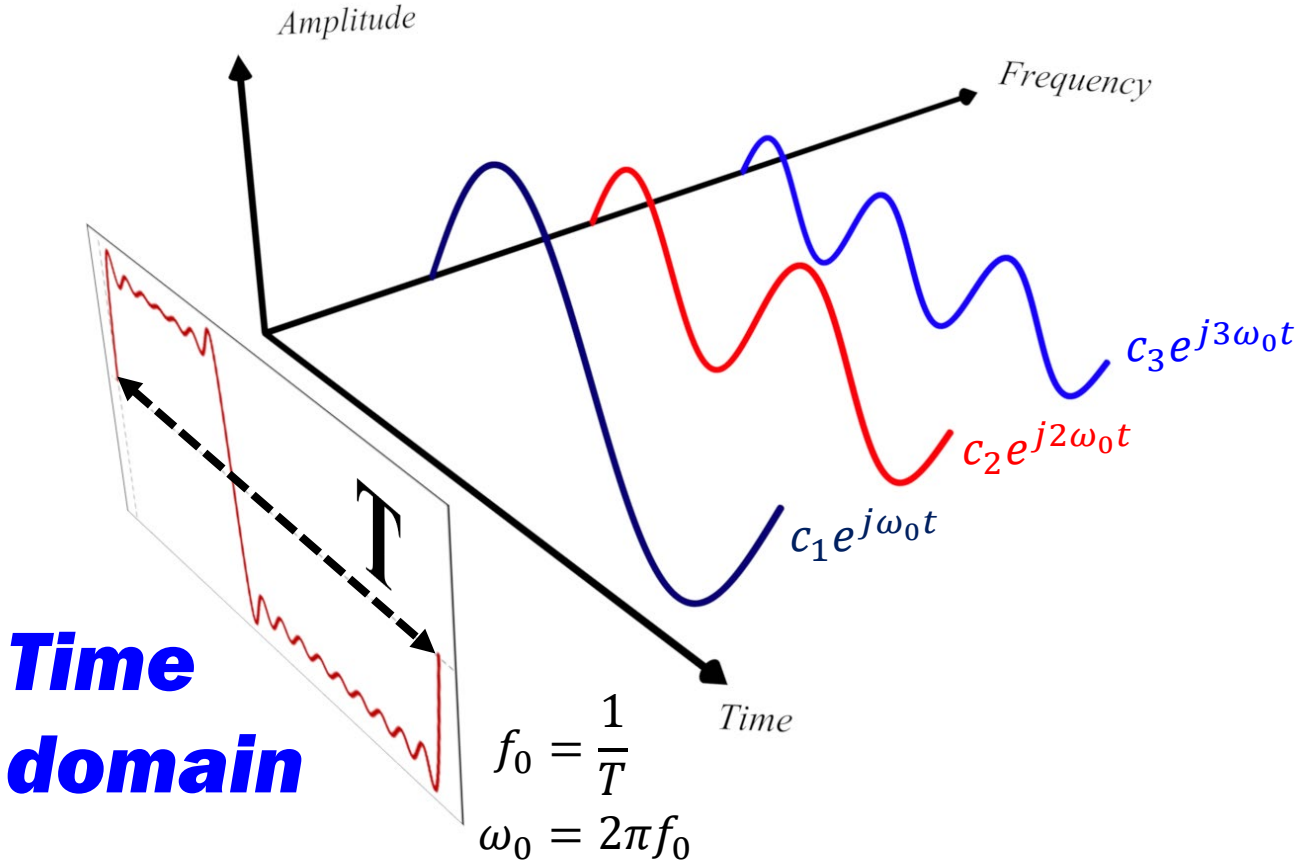
Any periodic signal of period T (fundamental frequency f_0 = 1/T) can be expressed as an infinite sum of sinusoidal components at integer multiples of the fundamental frequency (a.k.a "harmonics")

Fourier Series Expansion



Any periodic signal of period T (fundamental frequency f_0 = 1/T) can be expressed as an infinite sum of sinusoidal components at integer multiples of the fundamental frequency (a.k.a "harmonics")

Fourier Series Expansion



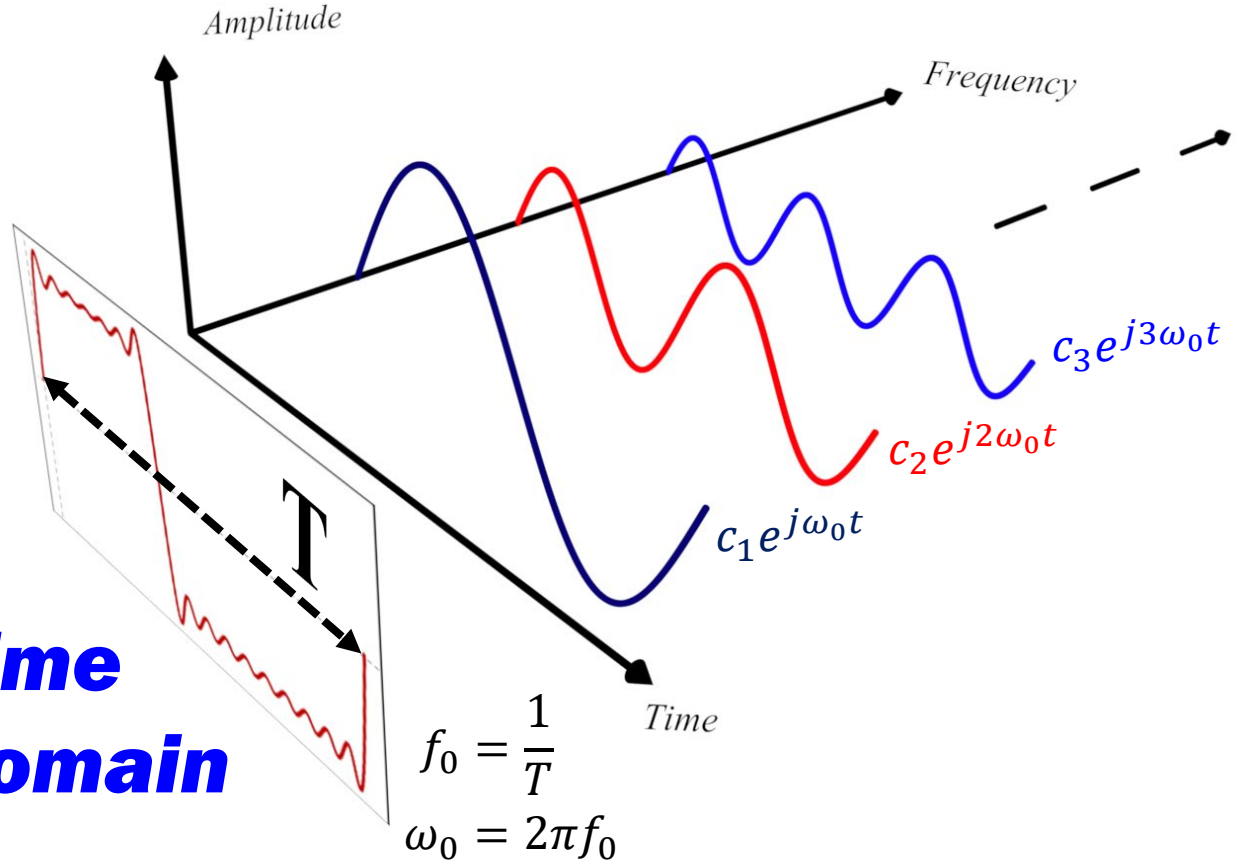
Time domain

$$f_0 = \frac{1}{T}$$

$$\omega_0 = 2\pi f_0$$

Any periodic signal of period T (fundamental frequency f_0 = 1/T) can be expressed as an infinite sum of sinusoidal components at integer multiples of the fundamental frequency (a.k.a "harmonics")

Fourier Series Expansion



Time domain

$$f_0 = \frac{1}{T}$$

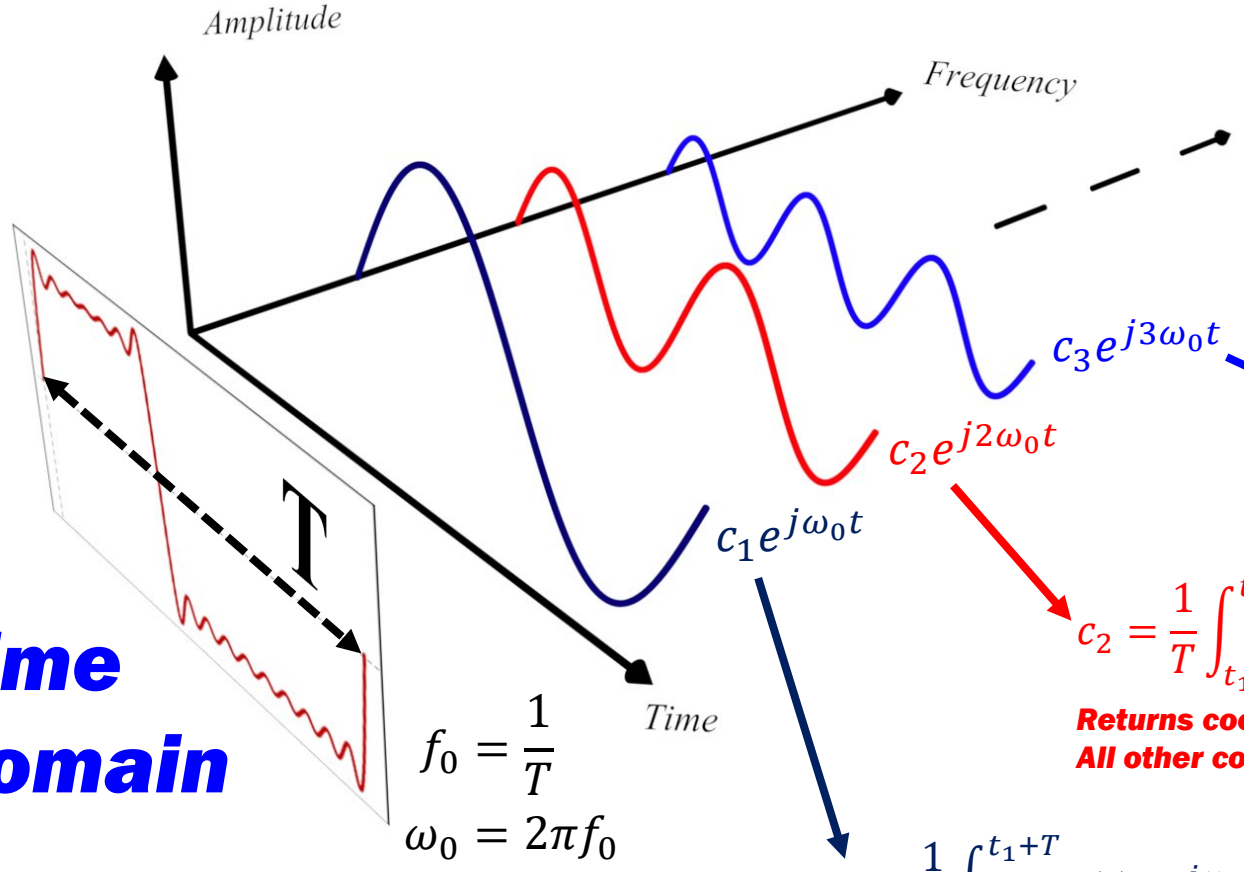
$$\omega_0 = 2\pi f_0$$

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

Sum goes from -∞ to +∞ to include positive and negative frequency components (+ω and –ω)

Any periodic signal of period T (fundamental frequency f_0 = 1/T) can be expressed as an infinite sum of sinusoidal components at integer multiples of the fundamental frequency (a.k.a "harmonics")

Fourier Series Expansion



Time domain

$$f_0 = \frac{1}{T}$$

$$\omega_0 = 2\pi f_0$$

$$c_3 = \frac{1}{T}\int_{t_1}^{t_1+T} x(t)e^{-j3\omega_0 t}dt$$

Returns coeffiecient of 3rd harmonic only
All other components integrate to 0

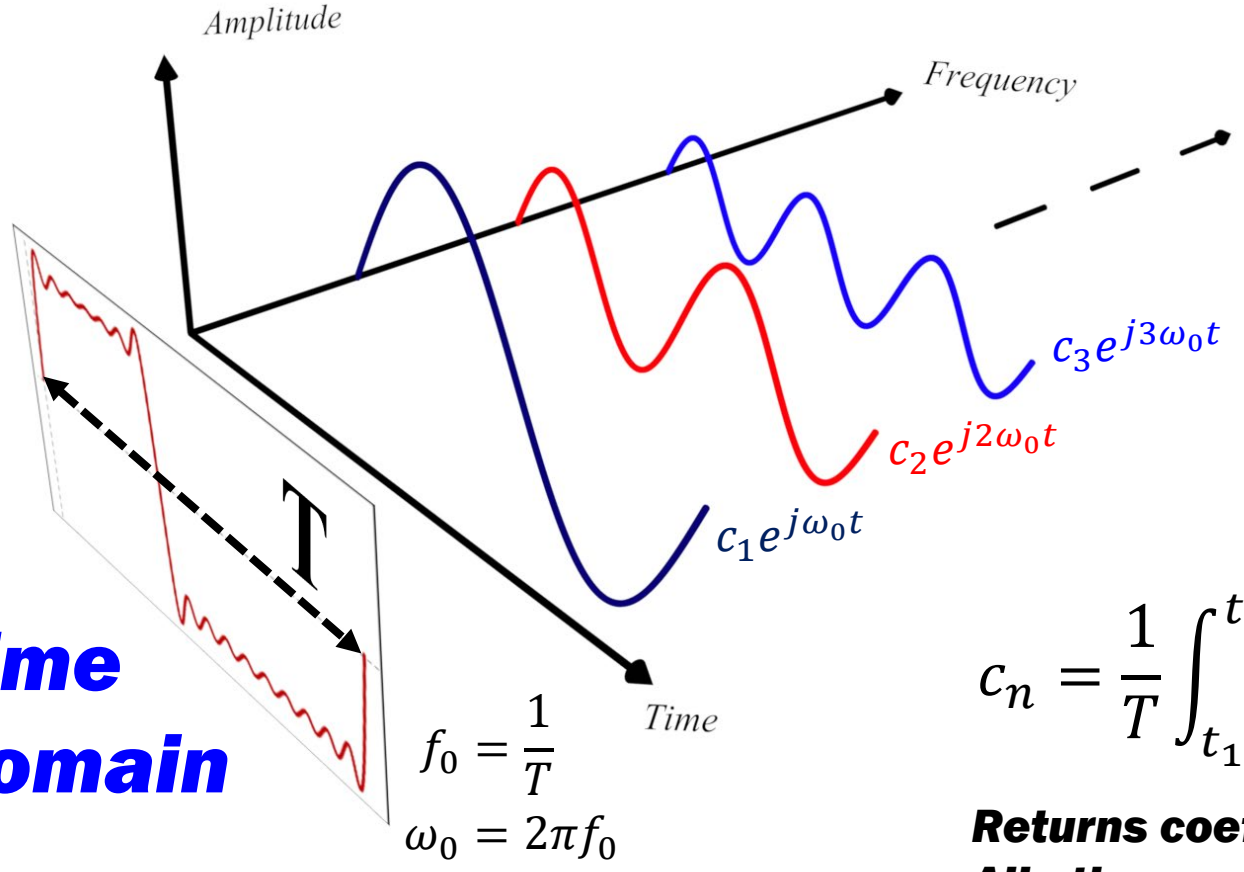
$$c_2 = \frac{1}{T}\int_{t_1}^{t_1+T} x(t)e^{-j2\omega_0 t}dt$$

Returns coeffiecient of 2nd harmonic only
All other components integrate to 0

$$c_1 = \frac{1}{T}\int_{t_1}^{t_1+T} x(t)e^{-j\omega_0 t}dt$$

Returns coeffiecient of fundamental component only (n = 1)
All other components integrate to 0

Fourier Series Expansion



Time domain

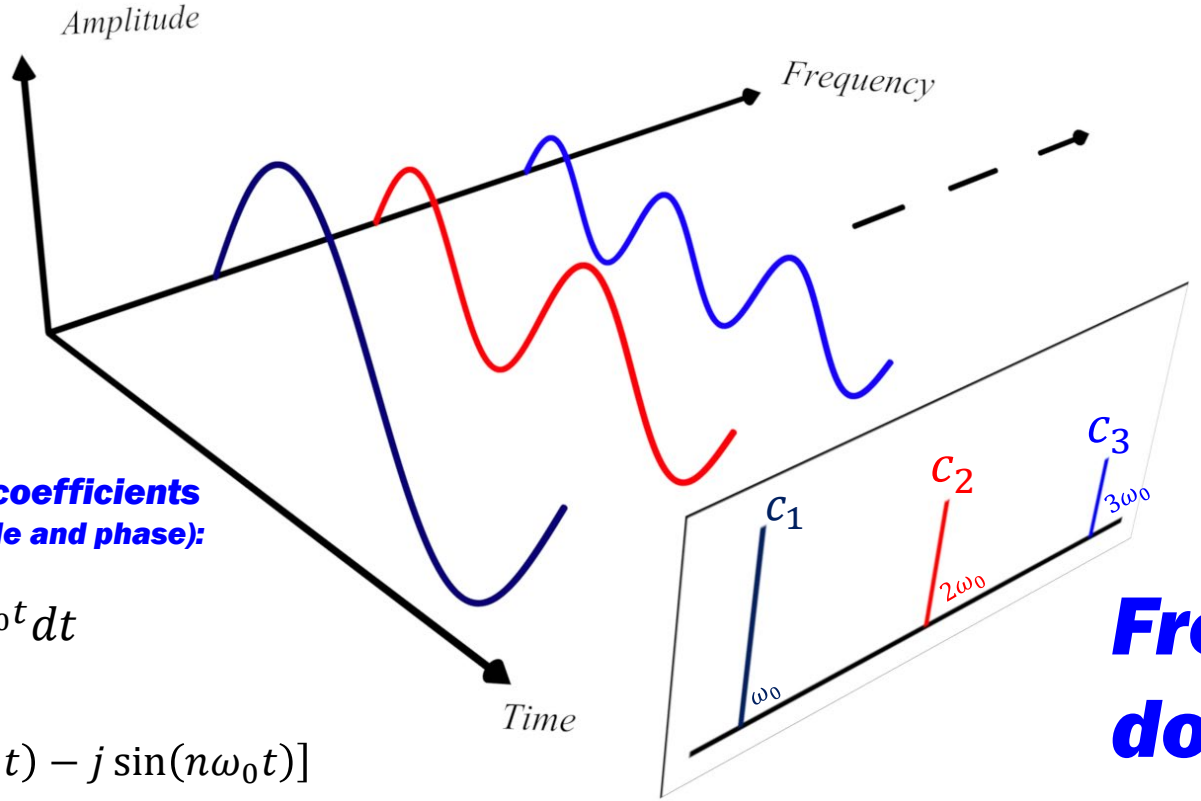
$$f_0 = \frac{1}{T}$$

$$\omega_0 = 2\pi f_0$$

$$c_n = \frac{1}{T}\int_{t_1}^{t_1+T} x(t)e^{-jn\omega_0 t}dt$$

Returns coeffiecient of n^{th} harmonic only
All other components integrate to 0

Fourier Series Expansion (cont.)



Frequency domain

General Fourier series expansion coefficients (can be complex numbers with magnitude and phase):

$$c_n = \frac{1}{T}\int_{t_1}^{t_1+T} x(t)e^{-jn\omega_0 t}dt$$

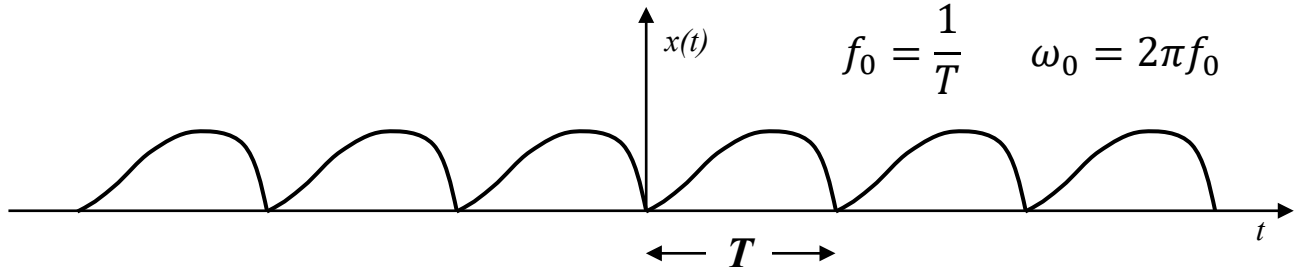
$$c_n = \frac{1}{T}\int_{t_1}^{t_1+T} x(t)[\cos(n\omega_0 t) - j\sin(n\omega_0 t)]$$

Real and imaginary parts give magnitude and phase

$$|c_n| = \sqrt{(Re_n)^2 + (Im_n)^2}$$

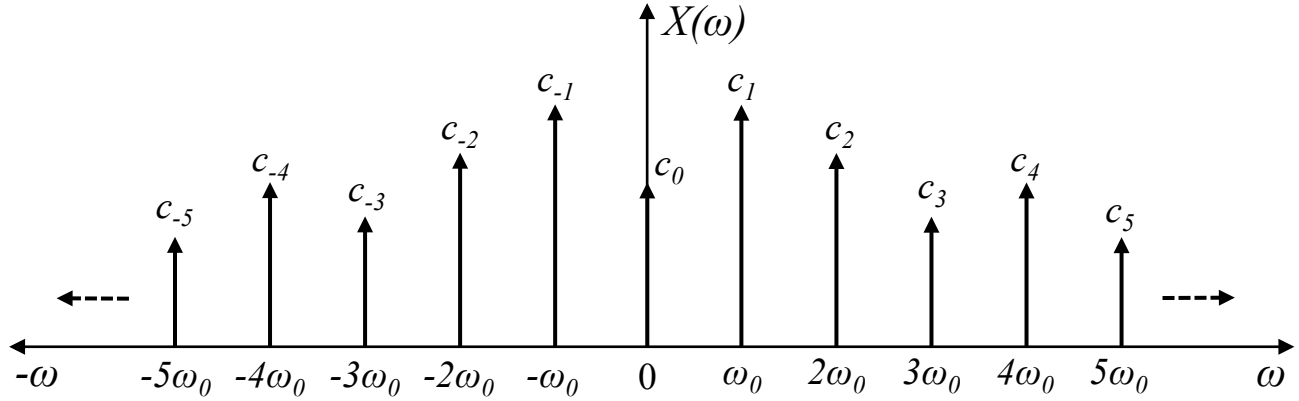
$$\angle c_n = \tan^{-1}\left(\frac{Im_n}{Re_n}\right)$$

Fourier Series (cont.)

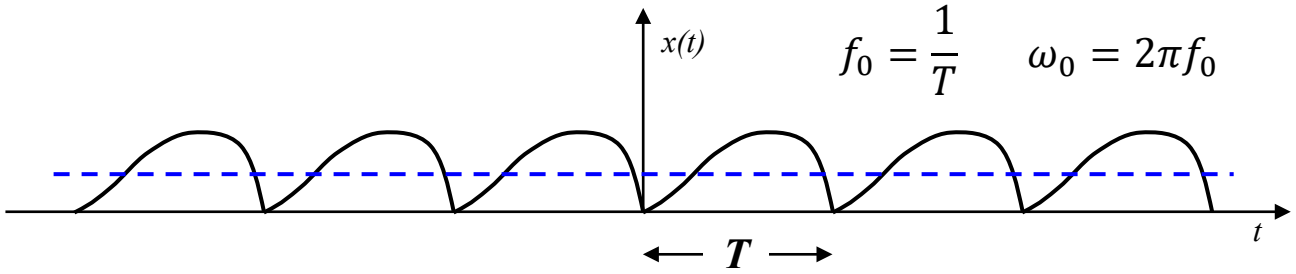


Fourier series expansion with complex exponentials produces "double-sided" frequency spectrum (-ω and +ω components)

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

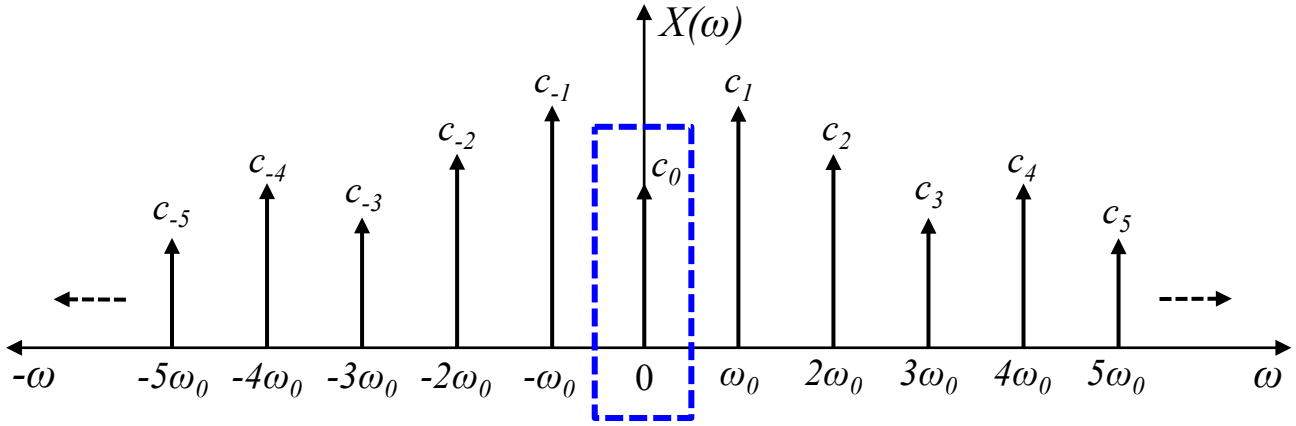


Fourier Series (cont.)



DC component
(average value integrated over a period)

$$c_0 = \frac{1}{T}\int_0^T x(t)dt$$



Fourier Series (cont.)

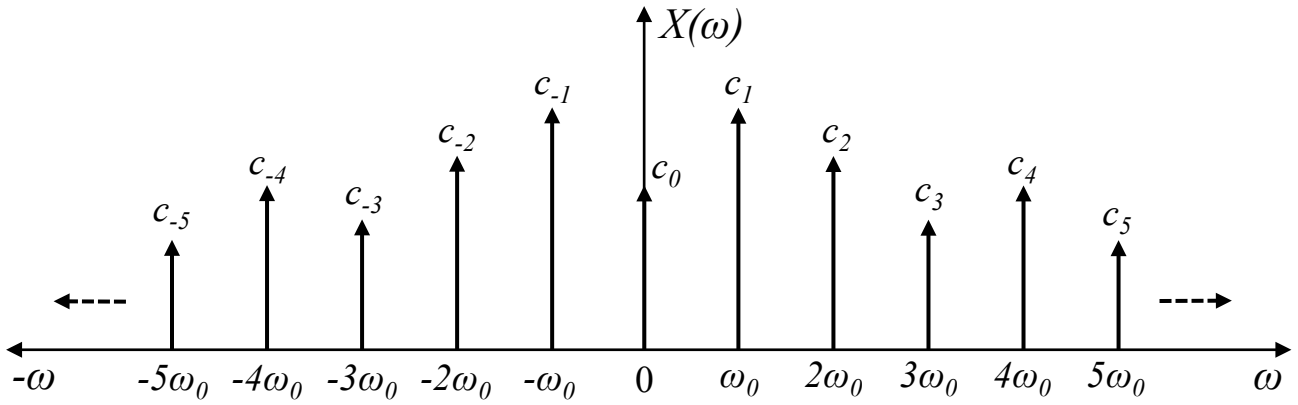
$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} = c_0 + \sum_{n=1}^{\infty} c_n e^{jn\omega_0 t} + \sum_{n=-1}^{-\infty} c_n e^{jn\omega_0 t} = c_0 + \sum_{n=1}^{\infty} c_n e^{jn\omega_0 t} + \sum_{n=1}^{\infty} c_{-n} e^{-jn\omega_0 t}$$

DC $+\omega$ **components** $-\omega$ **components** **Flip index sign**

$$c_n = \frac{1}{T}\int_{t_1}^{t_1+T} x(t)e^{-jn\omega_0 t}dt \qquad c_{-n} = \frac{1}{T}\int_{t_1}^{t_1+T} x(t)e^{jn\omega_0 t}dt$$

$$|c_{-n}| = |c_n| \qquad \angle c_{-n} = -\angle c_n$$

Complex conjugates



Fourier Series (cont.)

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} = c_0 + \sum_{n=1}^{\infty} c_n e^{jn\omega_0 t} + \sum_{n=-1}^{-\infty} c_n e^{jn\omega_0 t} = c_0 + \sum_{n=1}^{\infty} c_n e^{jn\omega_0 t} + \sum_{n=1}^{\infty} c_{-n} e^{-jn\omega_0 t}$$

$$x(t) = c_0 + \sum_{n=1}^{\infty} |c_n| e^{j\angle c_n} e^{jn\omega_0 t} + \sum_{n=1}^{\infty} |c_{-n}| e^{j\angle c_{-n}} e^{-jn\omega_0 t}$$

$|c_{-n}| = |c_n|$

$\angle c_{-n} = -\angle c_n$

$$x(t) = c_0 + \sum_{n=1}^{\infty} |c_n| e^{j\angle c_n} e^{jn\omega_0 t} + \sum_{n=1}^{\infty} |c_n| e^{-j\angle c_n} e^{-jn\omega_0 t}$$

$$x(t) = c_0 + \sum_{n=1}^{\infty} |c_n| e^{j(n\omega_0 t + \angle c_n)} + \sum_{n=1}^{\infty} |c_n| e^{-j(n\omega_0 t + \angle c_n)}$$

$$x(t) = c_0 + \sum_{n=1}^{\infty} |c_n| \left[e^{j(n\omega_0 t + \angle c_n)} + e^{-j(n\omega_0 t + \angle c_n)} \right]$$

$$x(t) = c_0 + \sum_{n=1}^{\infty} 2|c_n| \cos(n\omega_0 t + \angle c_n)$$

Recall:

$$\cos\theta = \frac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right)$$

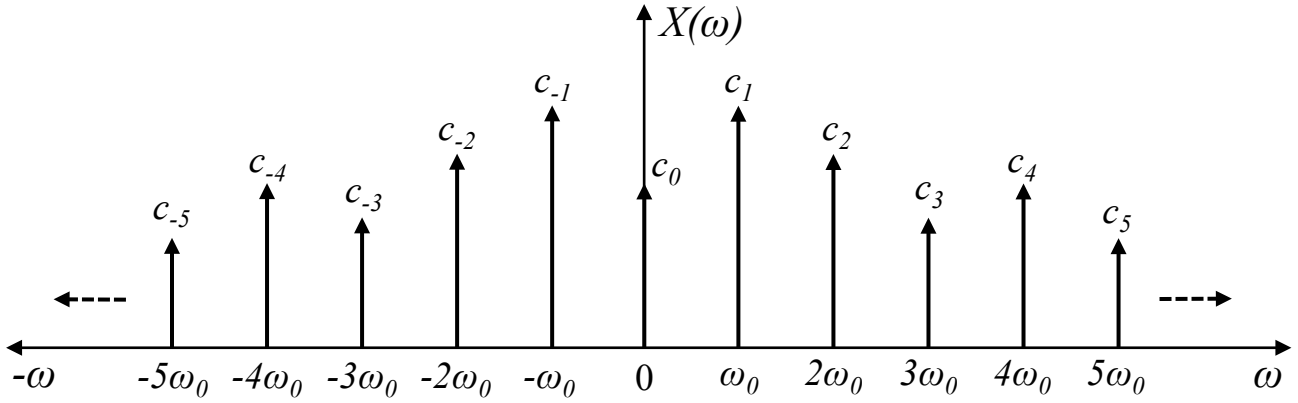
$$e^{j\theta} + e^{-j\theta} = 2\cos\theta$$

Fourier Series – "Double-Sided" Spectrum...

To get from the "double-sided" frequency spectrum to a "single-sided" spectrum...

(i.e. the way a spectrum analyzer will display it)

$$x(t) = c_0 + \sum_{n=1}^{\infty} c_n e^{jn\omega_0 t} + \sum_{n=1}^{\infty} c_{-n} e^{-jn\omega_0 t}$$



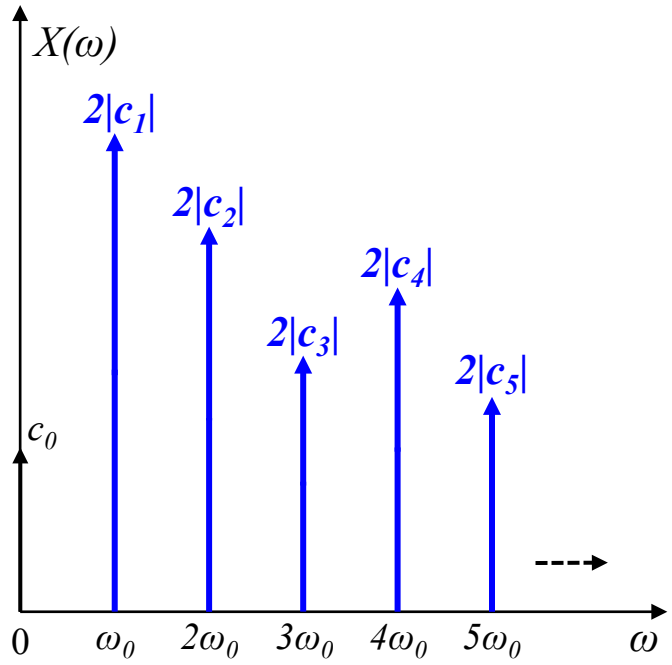
Fourier Series – "Single-Sided" Spectrum

$$x(t) = c_0 + \sum_{n=1}^{\infty} 2|c_n| \cos(n\omega_0 t + \angle c_n)$$

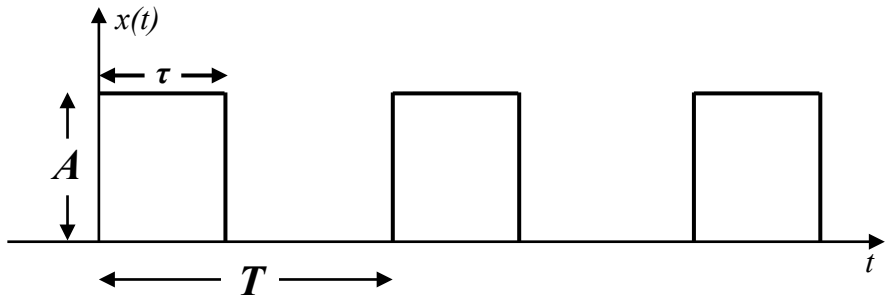
...the magnitude of each frequency component is doubled

DC component not affected

(I'll take multiplying by that factor of 2 at the end as a small price to pay for the much simpler approach of going through the analysis with exponentials instead of sinusoids...)



Rectangular Pulse Train (Digital/Clock Type Waveform)



$$\omega_0 = 2\pi f_0 = \frac{2\pi}{T}$$

$$c_n = \frac{1}{T}\int_{t_1}^{t_1+T} x(t)e^{-jn\omega_0 t}dt$$

$$\frac{\tau}{T} = Duty\ cycle$$

Positive frequencies:

$$c_n = \frac{1}{T}\int_0^{\tau} Ae^{-jn\omega_0 t}dt + \frac{1}{T}\int_{\tau}^{T} 0 \cdot e^{-jn\omega_0 t}dt$$

$$c_n = \frac{A}{-jn\omega_0 T}\left(e^{-jn\omega_0\tau} - e^{0}\right)$$

$$c_n = \frac{A}{jn\omega_0 T}\left(1 - e^{-jn\omega_0\tau}\right)$$

$$c_n = \frac{A}{jn\omega_0 T}e^{-jn\omega_0\tau/2}\left(e^{jn\omega_0\tau/2} - e^{-jn\omega_0\tau/2}\right)$$

Recall:

$$j\sin\theta = \tfrac{1}{2}\left(e^{j\theta} - e^{-j\theta}\right)$$

$$e^{j\theta} - e^{-j\theta} = 2j\sin\theta$$

$$c_n = \frac{A}{jn\omega_0 T}e^{-jn\omega_0\tau/2} \cdot 2j\sin\left(\tfrac{1}{2}n\omega_0\tau\right)$$

$$c_n = \frac{A\tau}{T}\left[\frac{\sin\left(\frac{1}{2}n\omega_0\tau\right)}{\frac{1}{2}n\omega_0\tau}\right]e^{-jn\omega_0\tau/2}$$

Negative frequencies:

$$c_{-n} = \frac{1}{T}\int_0^{\tau} Ae^{jn\omega_0 t}dt + \frac{1}{T}\int_{\tau}^{T} 0 \cdot e^{jn\omega_0 t}dt$$

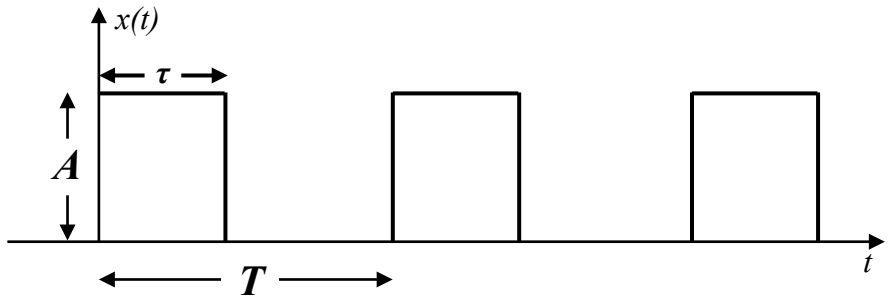
$$c_{-n} = \frac{A}{jn\omega_0 T}\left(e^{jn\omega_0\tau} - e^{0}\right)$$

$$c_{-n} = \frac{A}{jn\omega_0 T}e^{jn\omega_0\tau/2}\left(e^{jn\omega_0\tau/2} - e^{-jn\omega_0\tau/2}\right)$$

$$c_{-n} = \frac{A}{jn\omega_0 T}e^{jn\omega_0\tau/2} \cdot 2j\sin\left(\tfrac{1}{2}n\omega_0\tau\right)$$

$$c_{-n} = \frac{A\tau}{T}\left[\frac{\sin\left(\frac{1}{2}n\omega_0\tau\right)}{\frac{1}{2}n\omega_0\tau}\right]e^{jn\omega_0\tau/2}$$

Rectangular Pulse Train (cont.)



$$\omega_0 = 2\pi f_0 = \frac{2\pi}{T}$$

$$c_n = \frac{1}{T}\int_{t_1}^{t_1+T} x(t)e^{-jn\omega_0 t}dt$$

$$\frac{\tau}{T} = Duty\ cycle$$

Positive frequencies:

$$c_n = \frac{A\tau}{T}\left[\frac{\sin\left(\frac{1}{2}n\omega_0\tau\right)}{\frac{1}{2}n\omega_0\tau}\right]e^{-jn\omega_0\tau/2}$$

$$c_n = \underbrace{\frac{A\tau}{T}\left[\frac{\sin(n\pi f_0\tau)}{n\pi f_0\tau}\right]}_{|c_n|}\underbrace{e^{-jn\pi f_0\tau}}_{\angle c_n} \qquad \omega_0 = 2\pi f_0$$

$$|c_n| = A\frac{\tau}{T}\left[\frac{\sin(n\pi f_0\tau)}{n\pi f_0\tau}\right]$$

$$\angle c_n = -n\pi f_0\tau$$

Negative frequencies:

$$c_{-n} = \frac{A\tau}{T}\left[\frac{\sin\left(\frac{1}{2}n\omega_0\tau\right)}{\frac{1}{2}n\omega_0\tau}\right]e^{jn\omega_0\tau/2}$$

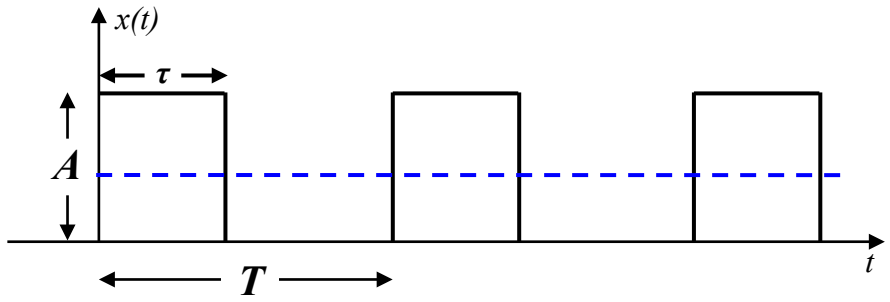
$$c_{-n} = \frac{A\tau}{T}\left[\frac{\sin(n\pi f_0\tau)}{n\pi f_0\tau}\right]e^{jn\pi f_0\tau}$$

$$|c_{-n}| = A\frac{\tau}{T}\left[\frac{\sin(n\pi f_0\tau)}{n\pi f_0\tau}\right] = |c_n|$$

$$\angle c_{-n} = n\pi f_0\tau = -\angle c_n$$

Complex conjugates

Rectangular Pulse Train (cont.)



$$\omega_0 = 2\pi f_0 = \frac{2\pi}{T}$$

$$c_n = \frac{1}{T}\int_{t_1}^{t_1+T} x(t)e^{-jn\omega_0 t}dt$$

$$\frac{\tau}{T} = Duty\ cycle$$

DC component:

$$c_0 = \frac{1}{T}\int_0^{\tau} A dt + \frac{1}{T}\int_{\tau}^{T} 0 \cdot dt \qquad c_0 = A\frac{\tau}{T}$$

Putting it all together:

$$x(t) = c_0 + \sum_{n=1}^{\infty} c_n e^{jn\omega_0 t} + \sum_{n=1}^{\infty} c_{-n} e^{-jn\omega_0 t}$$

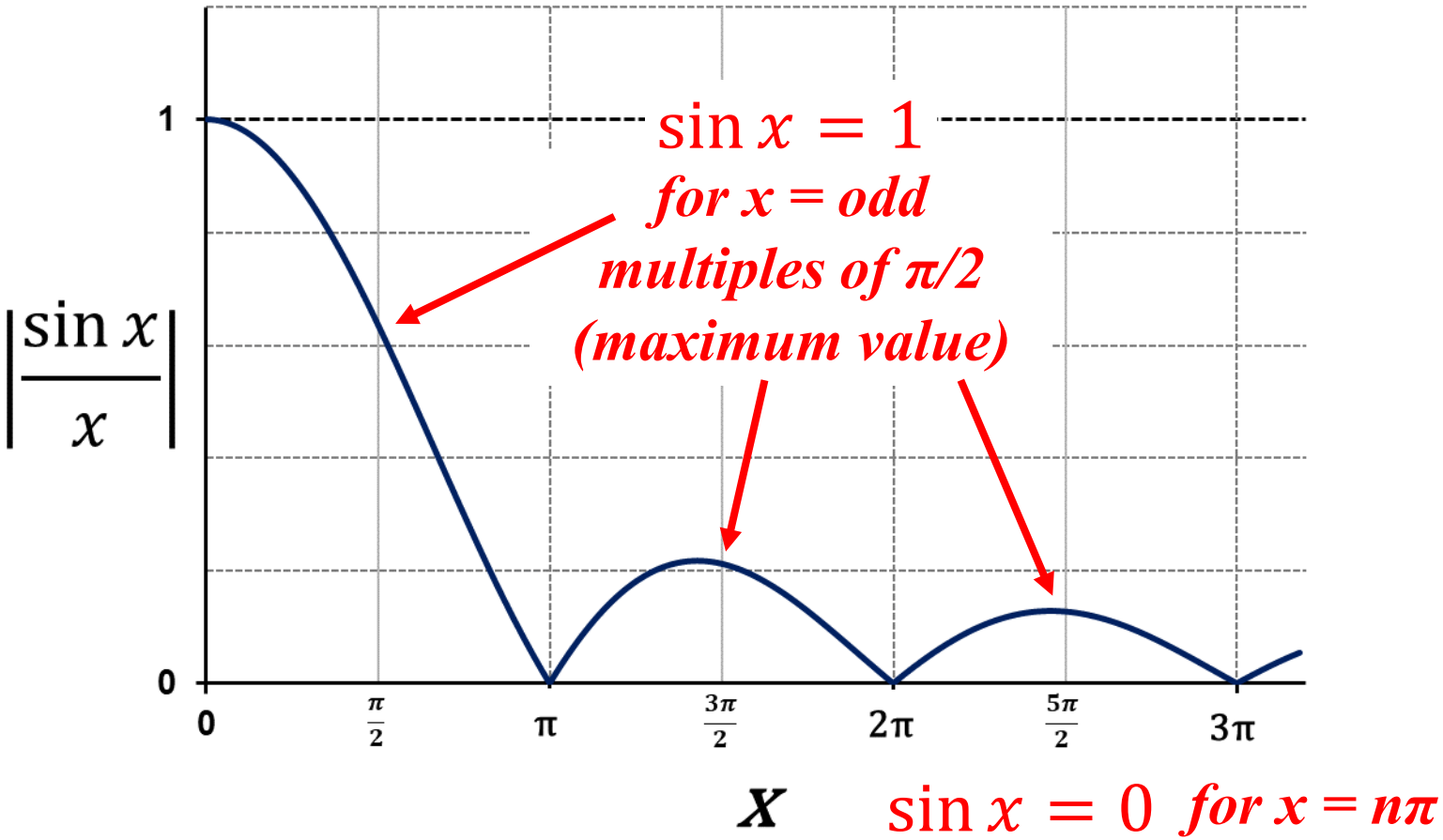
$$x(t) = c_0 + \sum_{n=1}^{\infty} 2|c_n|\cos(n\omega_0 t + \angle c_n)$$

Fourier series expansion of rectangular pulse train:

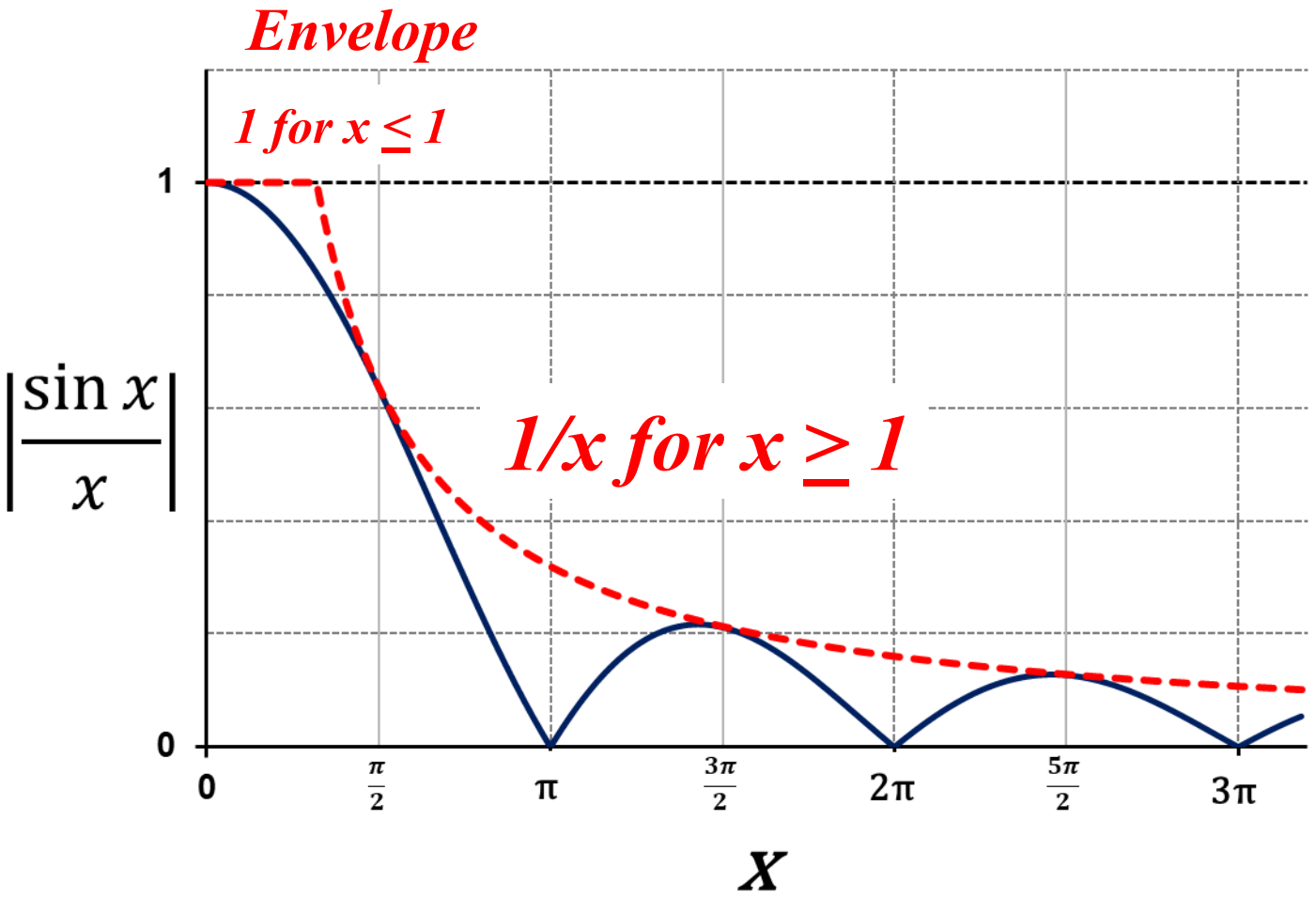
$$x(t) = A\frac{\tau}{T} + \sum_{n=1}^{\infty} 2A\frac{\tau}{T}\left[\frac{\sin(n\pi f_0\tau)}{n\pi f_0\tau}\right]\cos(2\pi n f_0 t - n\pi f_0\tau)$$

follows envelope of $\frac{sin(\pi f\tau)}{\pi f\tau} \rightarrow \frac{sin(x)}{x}$

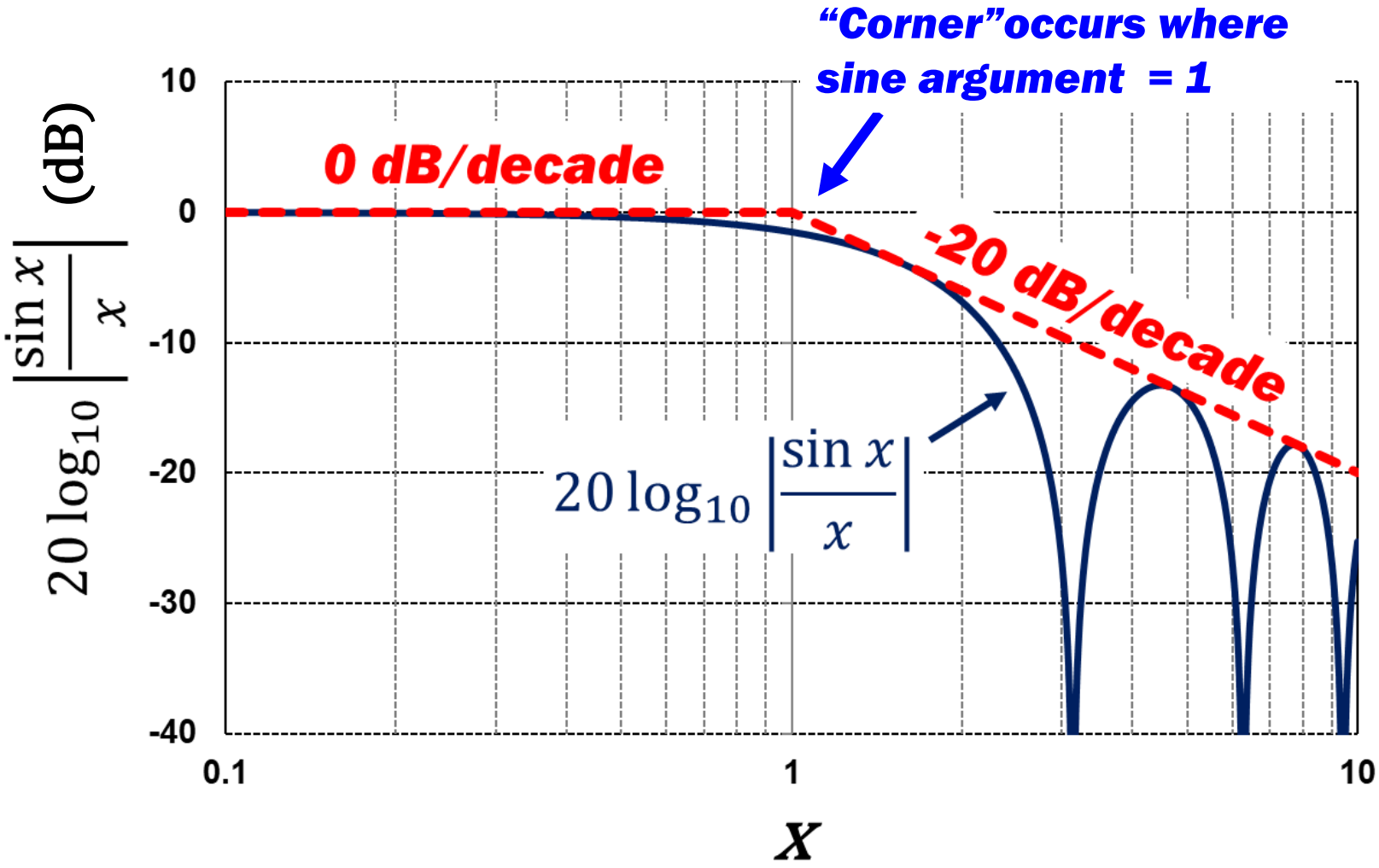
Response of |sin(x)/x|



Envelope of |sin(x)/x|



|sin(x)/x| in "dB-space"



1/x (1/f) rolloff in linear space equates to -20 dB/decade rolloff in "dB-space"

$$20\log_{10}\frac{1}{10} = -20\ dB$$

This example:

Corner occurs when $\pi f\tau = 1$

Corner frequency: $f = \frac{1}{\pi\tau}$

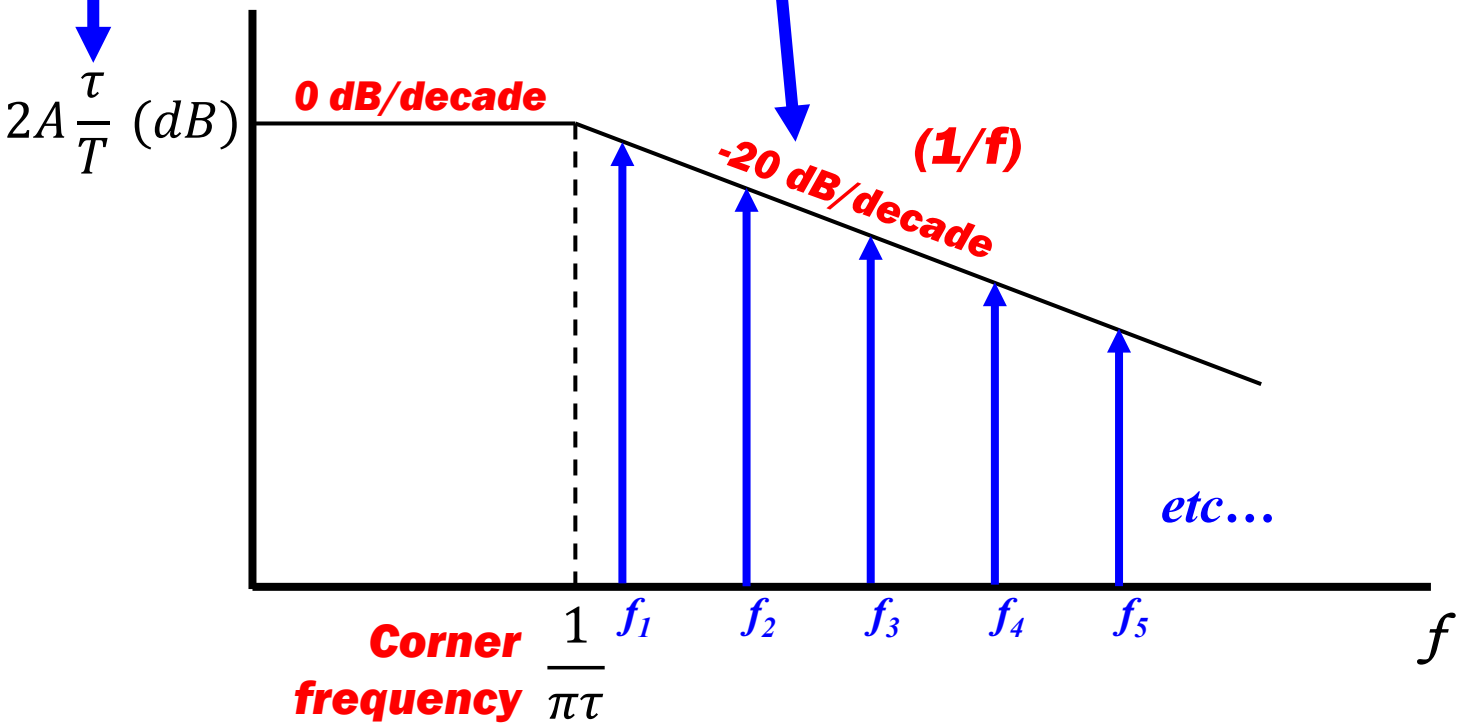
Envelope of Rectangular Pulse Train Spectrum

$$x(t) = A\frac{\tau}{T} + \sum_{n=1}^{\infty} 2A\frac{\tau}{T}\left[\frac{\sin(n\pi f_0\tau)}{n\pi f_0\tau}\right]\cos(2\pi n f_0 t - n\pi f_0\tau)$$

DC component

Low frequency "plateau"

Harmonic response



Square Wave (50% Duty Cycle)

$$x(t) = A\frac{\tau}{T} + \sum_{n=1}^{\infty} 2A\frac{\tau}{T}\left[\frac{\sin(n\pi f_0\tau)}{n\pi f_0\tau}\right]\cos(2\pi n f_0 t - n\pi f_0\tau)$$

Put in terms of duty cycle:

$f_0 = \frac{1}{T}$

$$x(t) = A\frac{\tau}{T} + \sum_{n=1}^{\infty} 2A\frac{\tau}{T}\left[\frac{\sin\left(n\pi\frac{\tau}{T}\right)}{n\pi\frac{\tau}{T}}\right]\cos\left(2\pi n f_0 t - n\pi\frac{\tau}{T}\right)$$

For perfect square wave with 50% duty cycle (τ/T = ½):

$$x(t) = \frac{A}{2} + \sum_{n=1}^{\infty} A\left[\frac{\sin\left(\frac{n\pi}{2}\right)}{\frac{n\pi}{2}}\right]\cos\left(2\pi n f_0 t - \frac{n\pi}{2}\right)$$

For even n: $\sin\left(\frac{n\pi}{2}\right) = 0$

No even harmonics in perfect square wave

Even harmonics will occur when duty cycle ≠ 50%

Square Wave (cont.)

$$x(t) = \frac{A}{2} + \sum_{n=1}^{\infty} A \left[\frac{\sin\left(\frac{n\pi}{2}\right)}{\frac{n\pi}{2}} \right] \cos\left(2\pi n f_0 t - \frac{n\pi}{2}\right)$$

$$x(t) = \frac{A}{2} + \frac{2A}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi}{2}\right) \cos\left(2\pi n f_0 t - \frac{n\pi}{2}\right)$$

Fourier series expansion of square wave:

$$x(t) = \frac{A}{2} + \frac{2A}{\pi} \sum_{n=1,3,5..}^{\infty} \frac{1}{n} \sin(2\pi n f_0 t)$$

Odd harmonics only

n = 1: $\sin\left(\frac{\pi}{2}\right) \cos\left(2\pi n f_0 t - \frac{\pi}{2}\right) = \underline{\sin(2\pi n f_0 t)}$

n = 3: $\sin\left(\frac{3\pi}{2}\right) \cos\left(2\pi n f_0 t - \frac{3\pi}{2}\right) = (-1)[-\sin(2\pi n f_0 t)]$

$= \underline{\sin(2\pi n f_0 t)}$

n = 5: $\sin\left(\frac{5\pi}{2}\right) \cos\left(2\pi n f_0 t - \frac{5\pi}{2}\right) = \underline{\sin(2\pi n f_0 t)}$

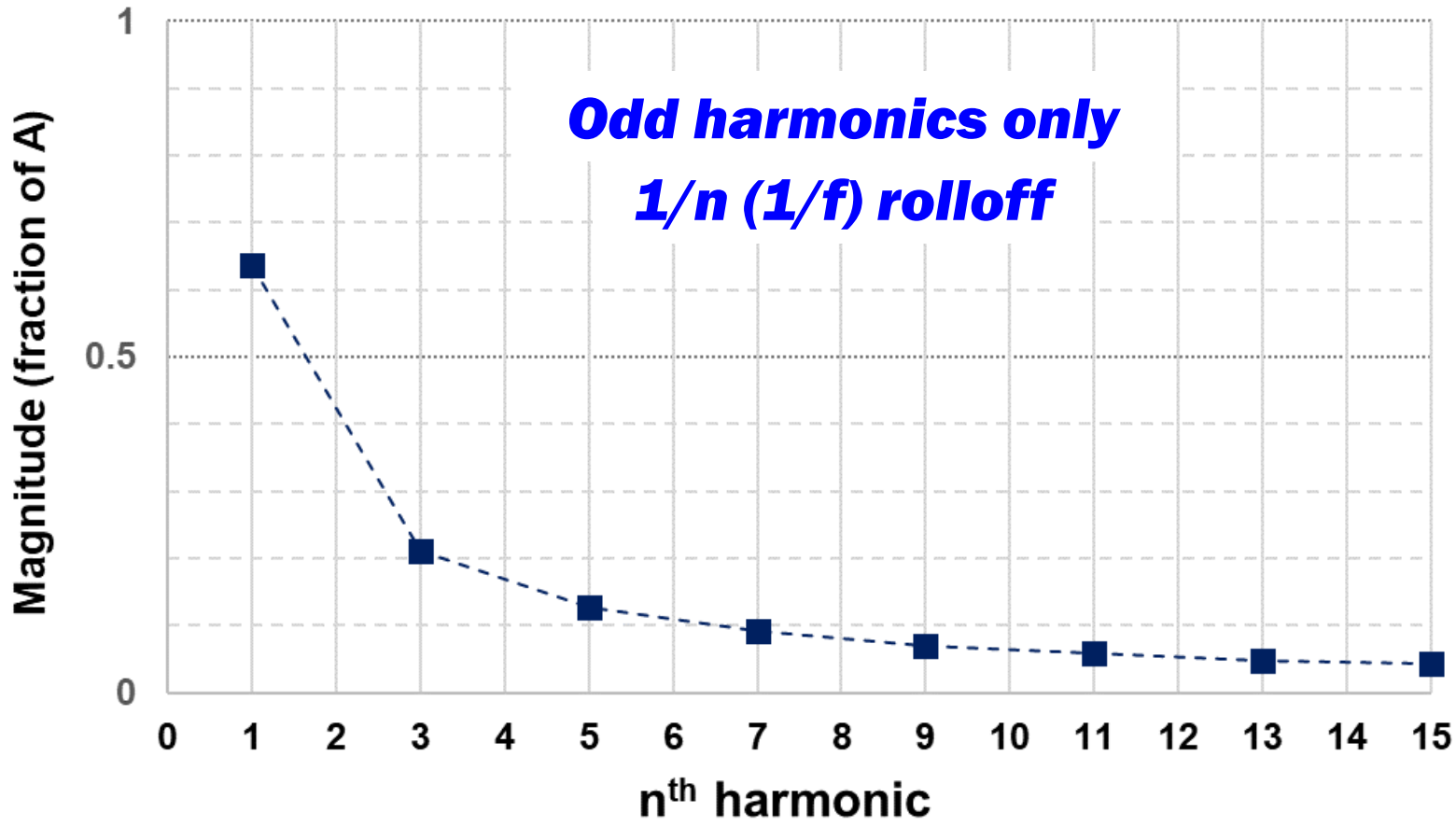
n = 7: $\sin\left(\frac{7\pi}{2}\right) \cos\left(2\pi n f_0 t - \frac{7\pi}{2}\right) = (-1)[-\sin(2\pi n f_0 t)]$

$= \underline{\sin(2\pi n f_0 t)}$

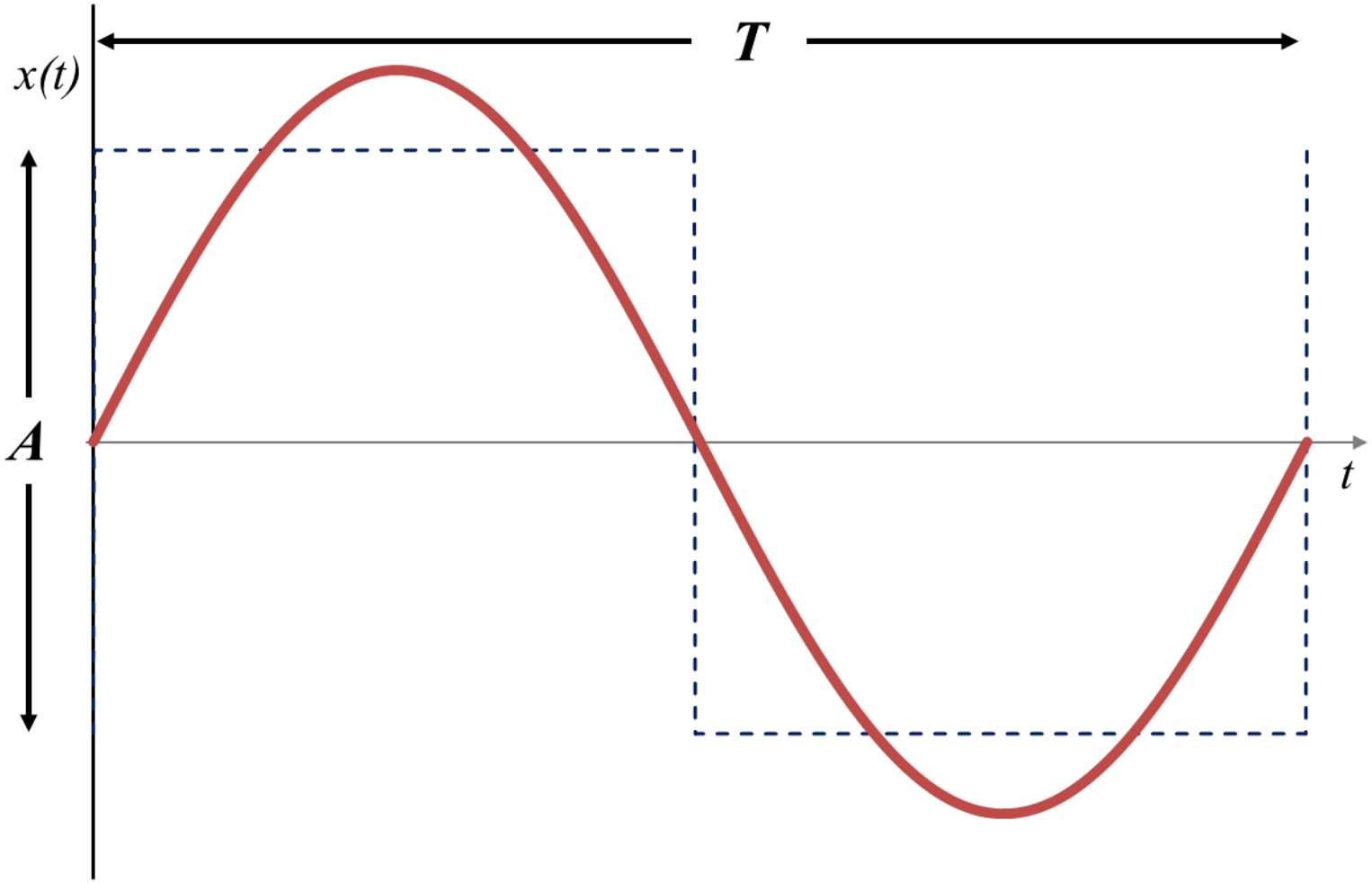
Square Wave Harmonics

$$x(t) = \frac{A}{2} + \frac{2A}{\pi} \sum_{n=1,3,5..}^{\infty} \frac{1}{n} \sin(2\pi n f_0 t)$$

← **No phase term**
(all components in phase)



Square Wave Expansion – Fundamental Only

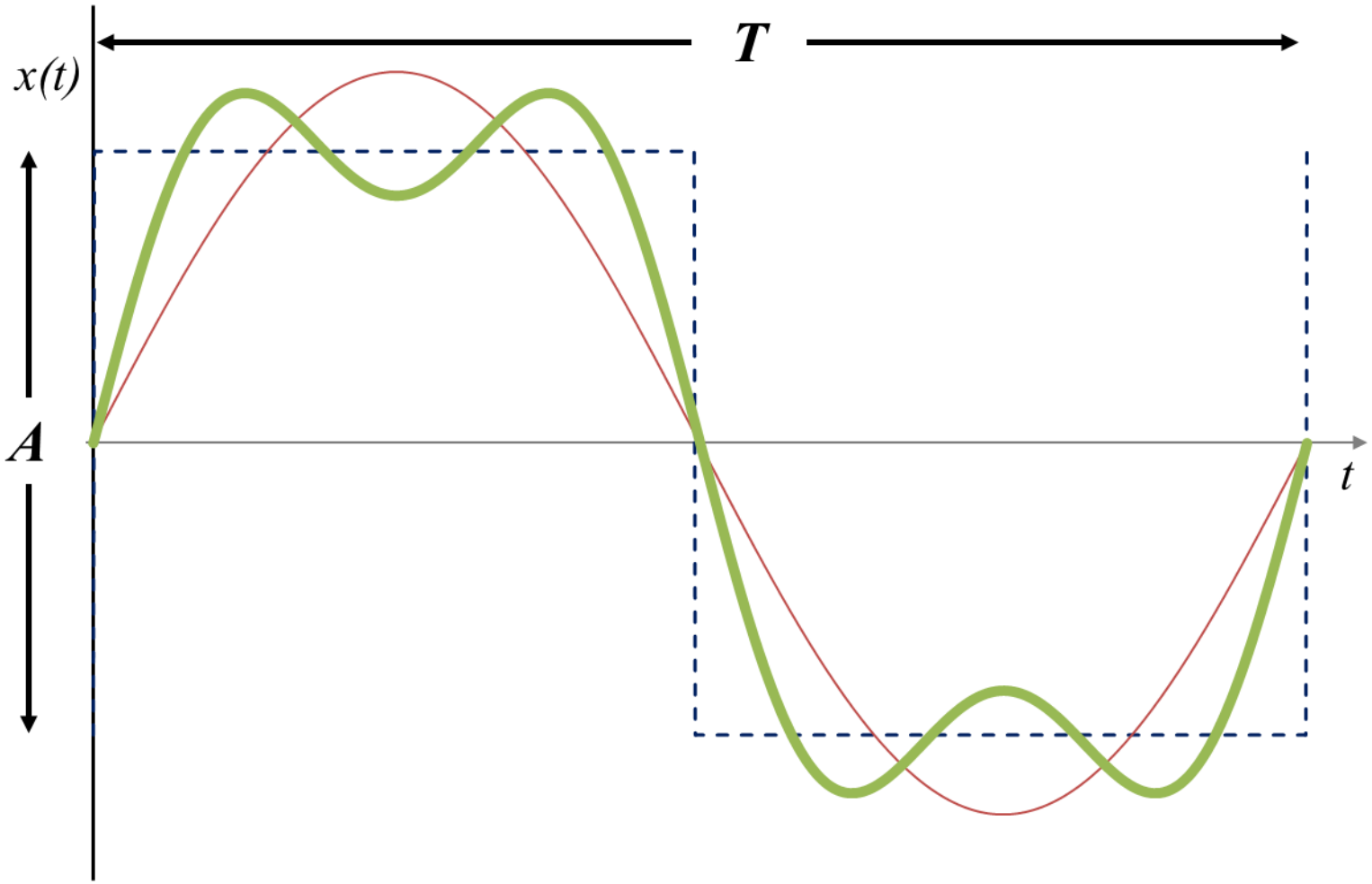


$$x(t) = \frac{A}{2} + \frac{2A}{\pi} \sum_{n=1,3,5..}^{\infty} \frac{1}{n} \sin(2\pi n f_0 t)$$

Square wave

Fundamental

Square Wave Expansion – 1st and 3rd harmonics



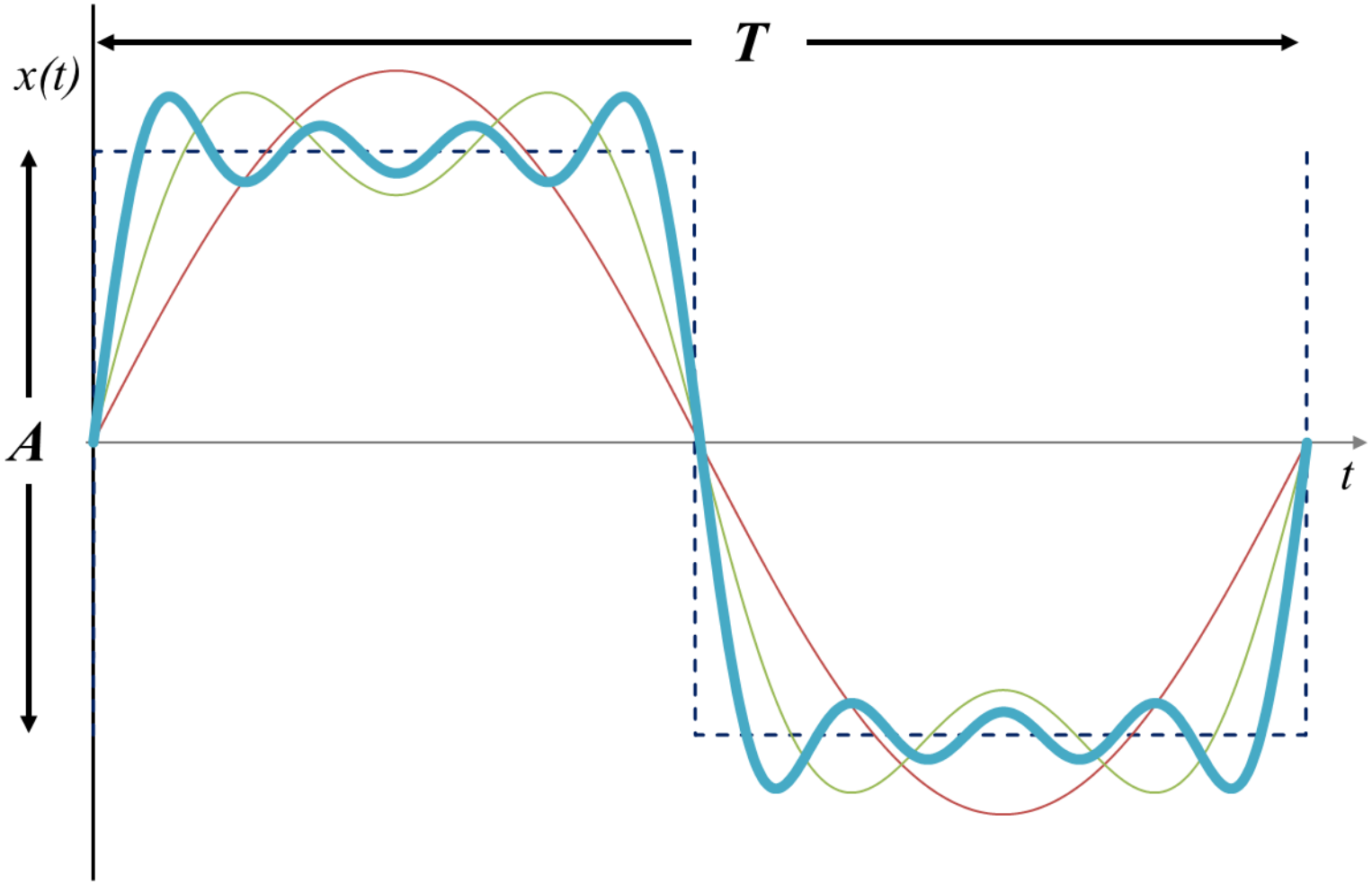
$$x(t) = \frac{A}{2} + \frac{2A}{\pi} \sum_{n=1,3,5..}^{\infty} \frac{1}{n} \sin(2\pi n f_0 t)$$

Square wave

Fundamental

1st & 3rd harmonics

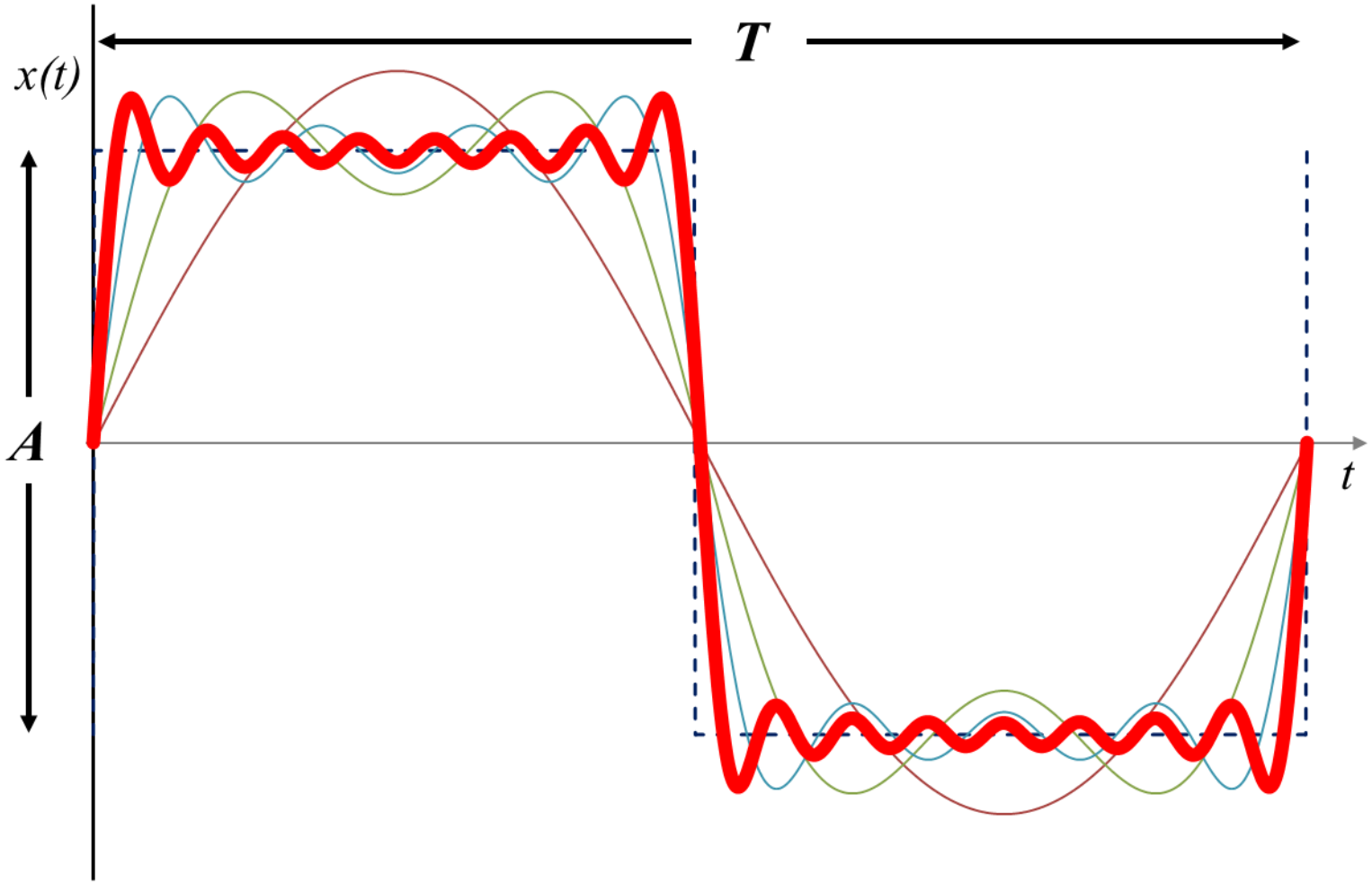
Square Wave Expansion – First 7 harmonics (odd only)



$$x(t) = \frac{A}{2} + \frac{2A}{\pi} \sum_{n=1,3,5..}^{\infty} \frac{1}{n} \sin(2\pi n f_0 t)$$

- Square wave
- Fundamental
- 1st & 3rd harmonics
- First 7 harmonics (odd only)

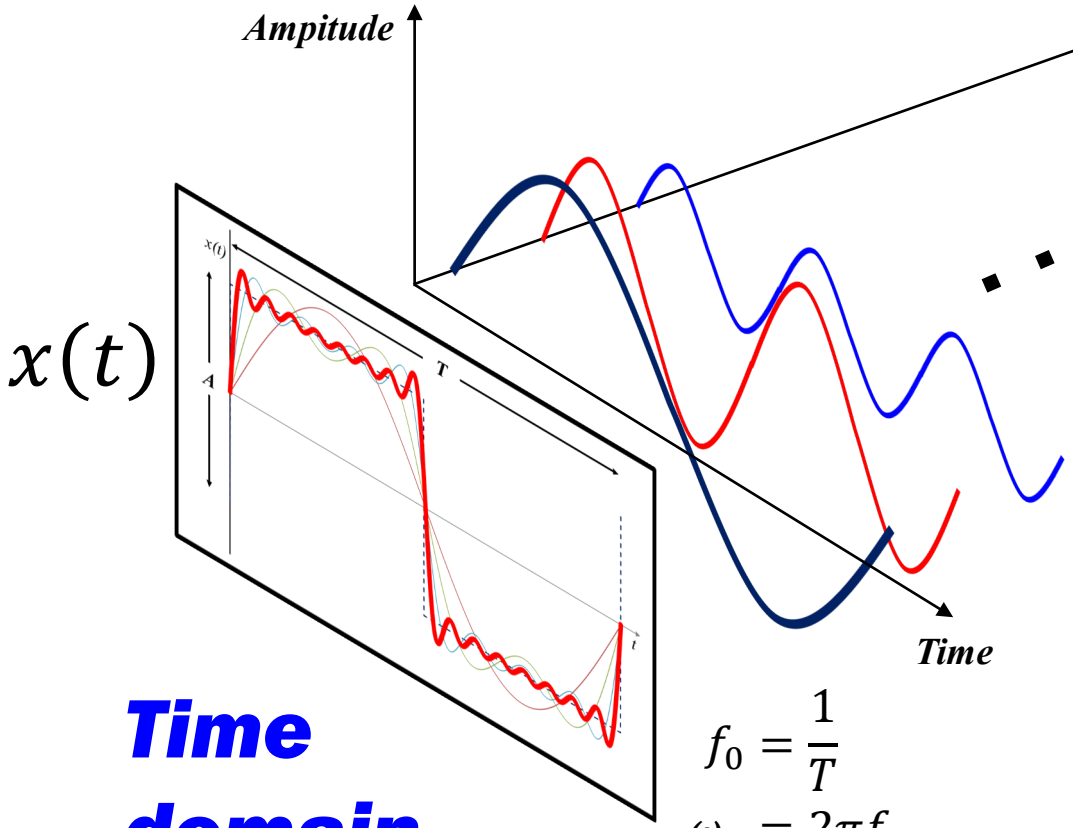
Square Wave Expansion – First 15 harmonics (odd only)



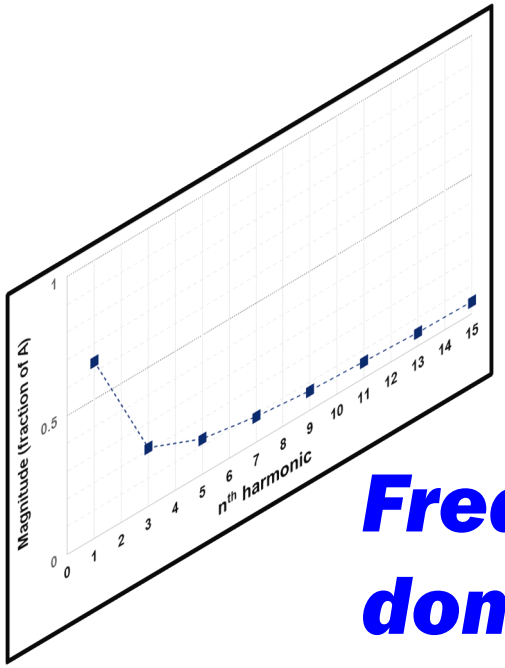
$$x(t) = \frac{A}{2} + \frac{2A}{\pi} \sum_{n=1,3,5..}^{\infty} \frac{1}{n} \sin(2\pi n f_0 t)$$

- Square wave
- Fundamental
- 1st & 3rd harmonics
- First 7 harmonics (odd only)
- First 15 harmonics (odd only)

Square Wave Expansion: Time and Frequency Domain



$$x(t) = \frac{A}{2} + \frac{2A}{\pi} \sum_{n=1,3,5..}^{\infty} \frac{1}{n} \sin(2\pi n f_0 t)$$



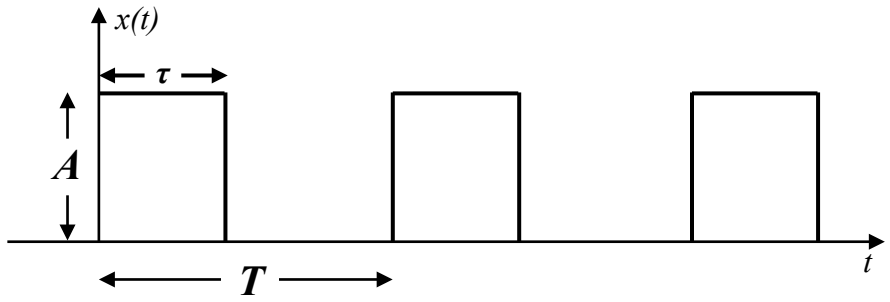
Time domain

$$f_0 = \frac{1}{T}$$

$$\omega_0 = 2\pi f_0$$

Frequency domain

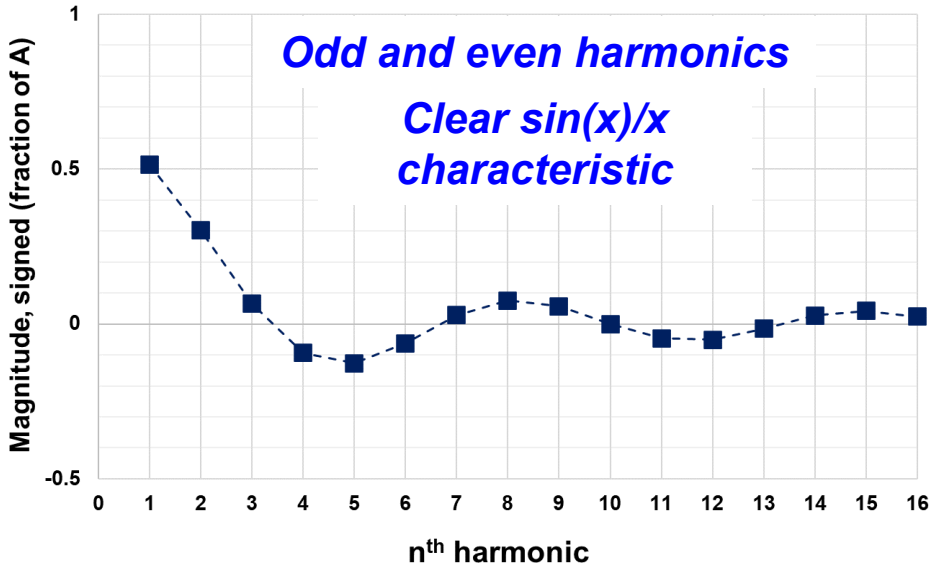
Rectangular Wave Expansion



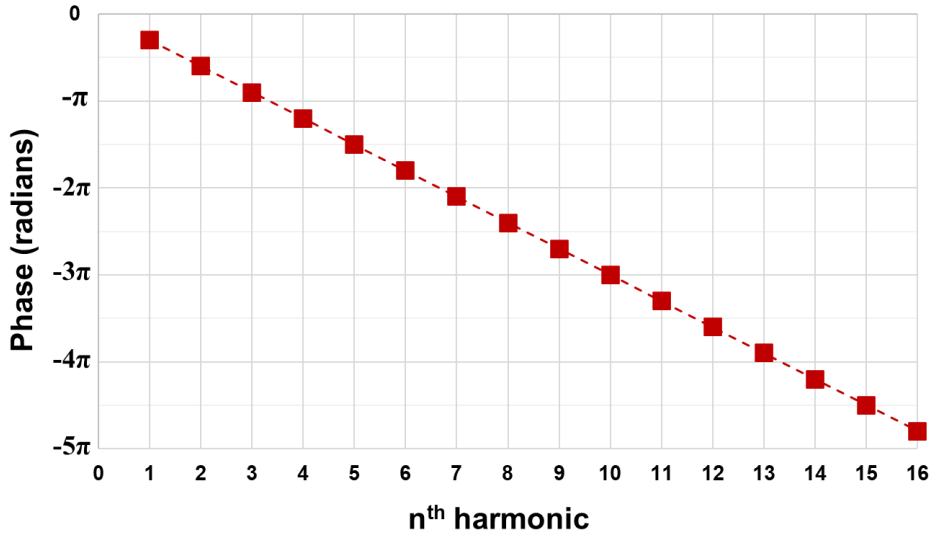
$$x(t) = A\frac{\tau}{T} + \sum_{n=1}^{\infty} 2A\frac{\tau}{T}\left[\frac{\sin(n\pi f_0\tau)}{n\pi f_0\tau}\right]\cos(2\pi n f_0 t - n\pi f_0\tau)$$

Example:
30% duty cycle
($\tau/T = 0.3$)

Rectangular Wave Harmonics – Magnitude and Phase

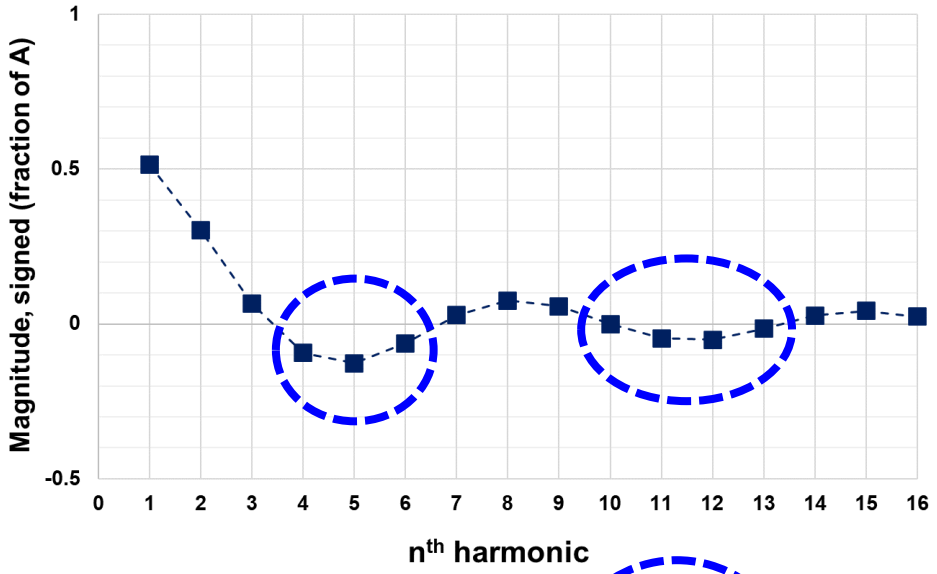


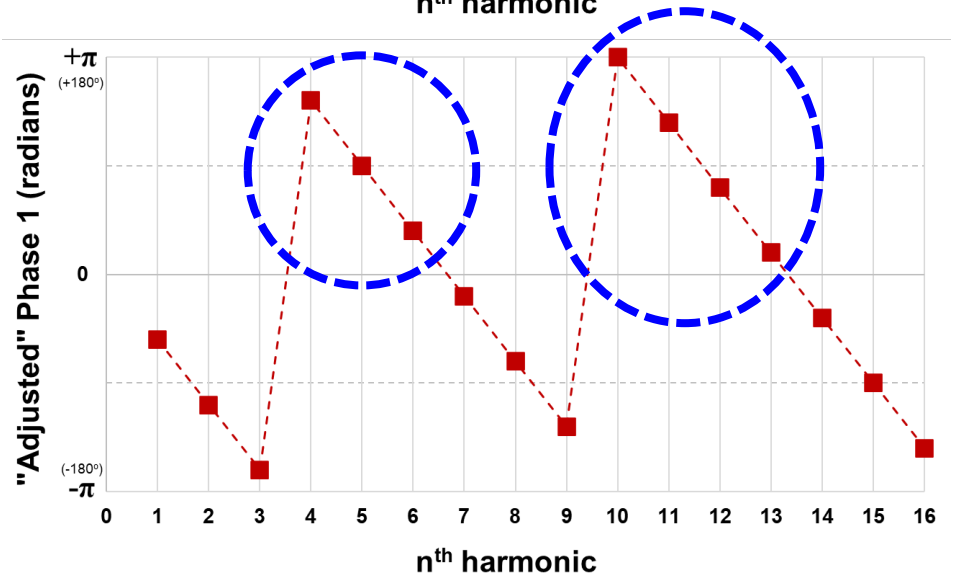
$$x(t) = A\frac{\tau}{T} + \sum_{n=1}^{\infty} \underbrace{2A\frac{\tau}{T}\left[\frac{\sin(n\pi f_0\tau)}{n\pi f_0\tau}\right]}_{\text{Magnitude (signed)}} \cos(2\pi n f_0 t - \underbrace{n\pi f_0\tau}_{\text{Phase}})$$



We like our phases to stay between –π and +π (-180° and +180°) →

Rectangular Wave Harmonics – Magnitude and Phase (cont.)



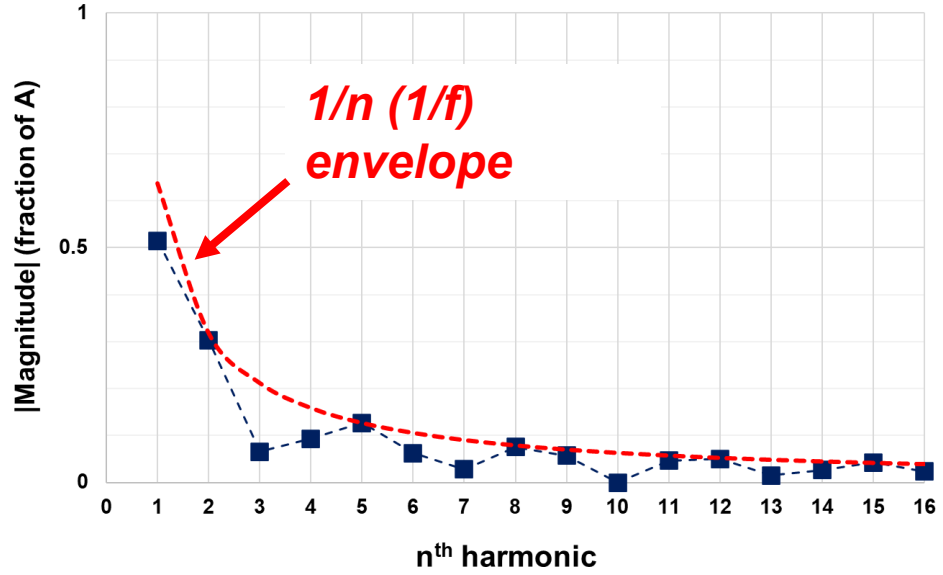


$$x(t) = A\frac{\tau}{T} + \sum_{n=1}^{\infty} 2A\frac{\tau}{T}\left[\frac{\sin(n\pi f_0\tau)}{n\pi f_0\tau}\right]\cos(2\pi n f_0 t - n\pi f_0\tau)$$

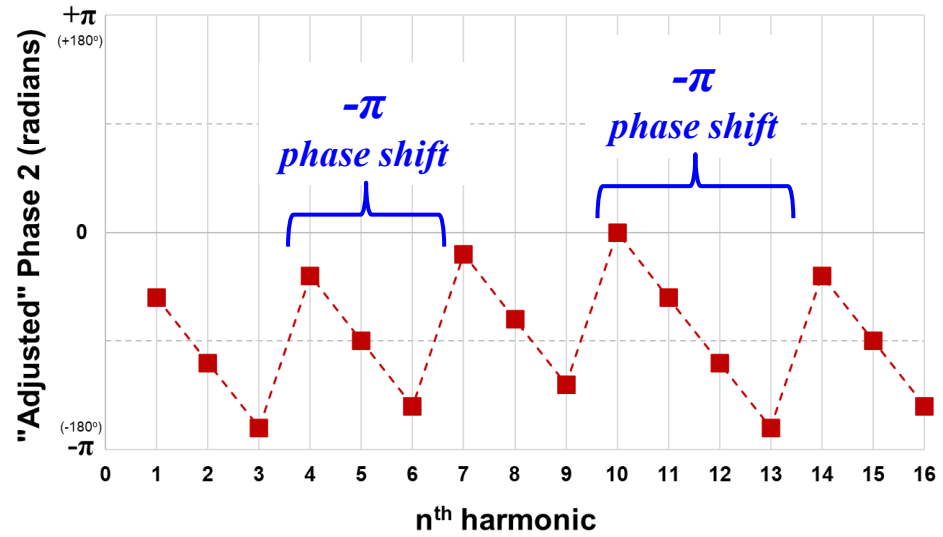
We like our magnitudes to be positive numbers (take absolute value)...

...which includes phase shift of +/- π (+/-180°) →

Rectangular Wave Harmonics – Magnitude and Phase (cont.)



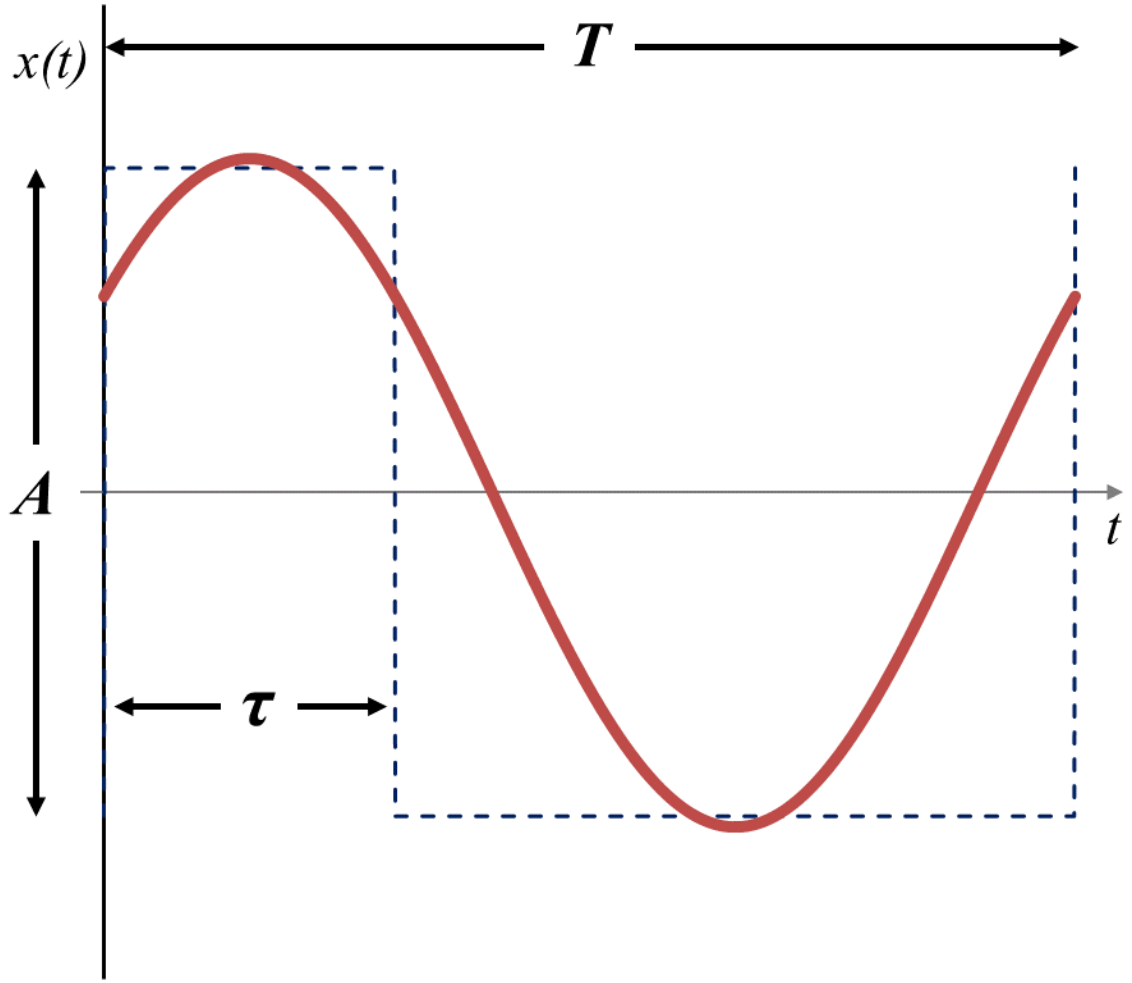
$$x(t) = A\frac{\tau}{T} + \sum_{n=1}^{\infty} 2A\frac{\tau}{T}\left[\frac{\sin(n\pi f_0\tau)}{n\pi f_0\tau}\right]\cos(2\pi n f_0 t - n\pi f_0\tau)$$



Abrupt phase changes are indication that the measurement device (e.g. spectrum analyzer) is making these adjustments

They do NOT occur naturally

Rectangular Wave Expansion – Fundamental Only

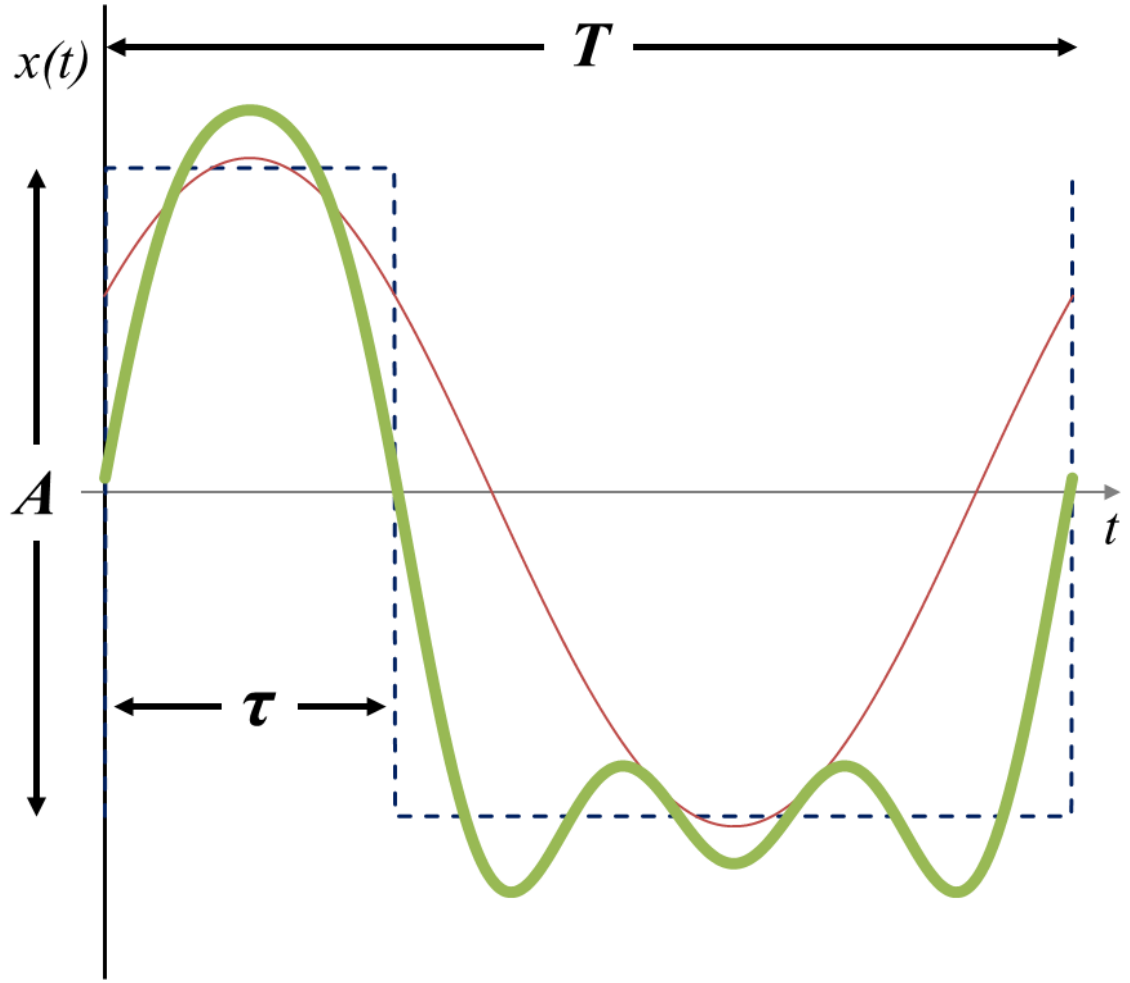


$$x(t) = A\frac{\tau}{T} + \sum_{n=1}^{\infty} 2A\frac{\tau}{T}\left[\frac{\sin(n\pi f_0\tau)}{n\pi f_0\tau}\right]\cos(2\pi n f_0 t - n\pi f_0\tau)$$

Rectangular Pulse

Fundamental

Rectangular Wave Expansion – First 4 harmonics (odd & even)



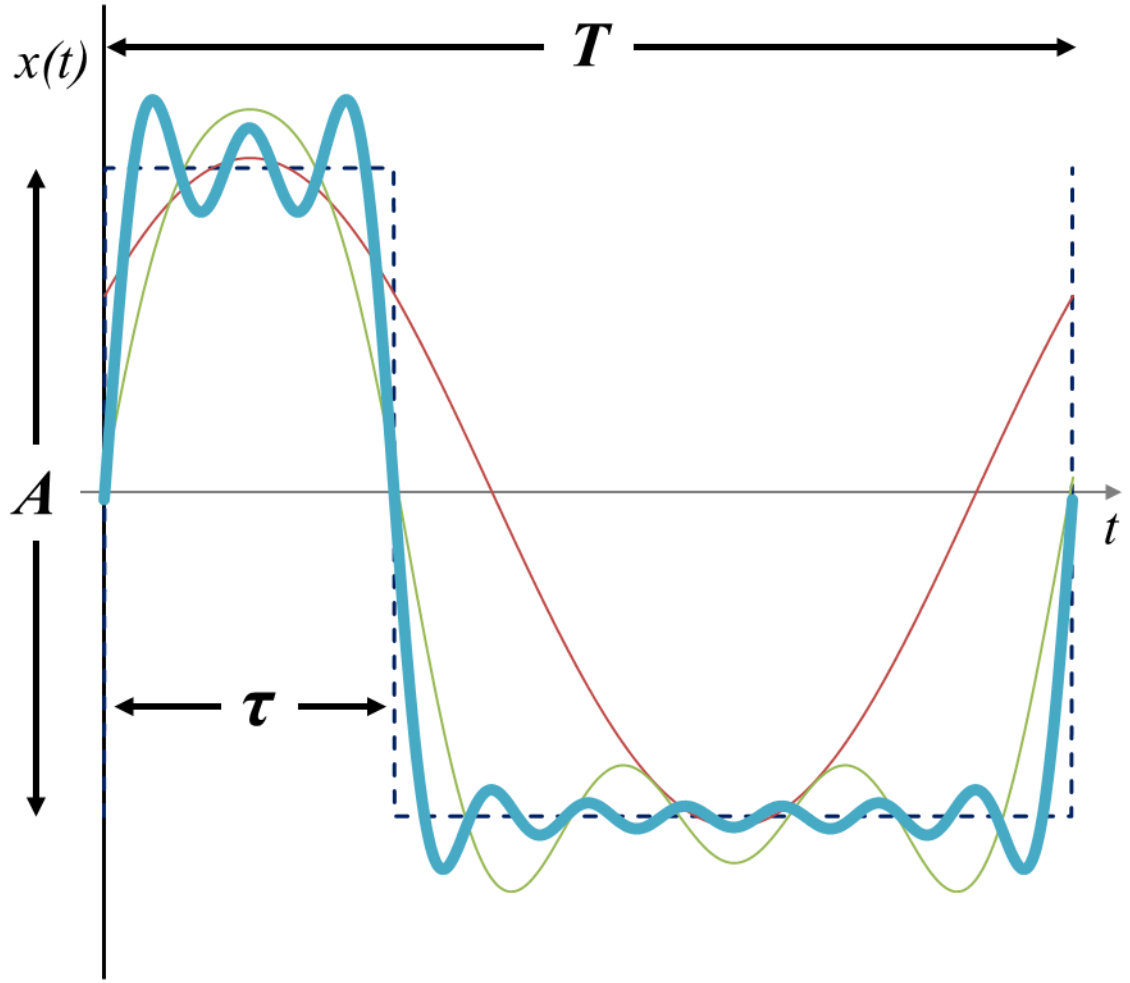
$$x(t) = A\frac{\tau}{T} + \sum_{n=1}^{\infty} 2A\frac{\tau}{T}\left[\frac{\sin(n\pi f_0\tau)}{n\pi f_0\tau}\right]\cos(2\pi n f_0 t - n\pi f_0\tau)$$

Rectangular Pulse

Fundamental

First 4 harmonics

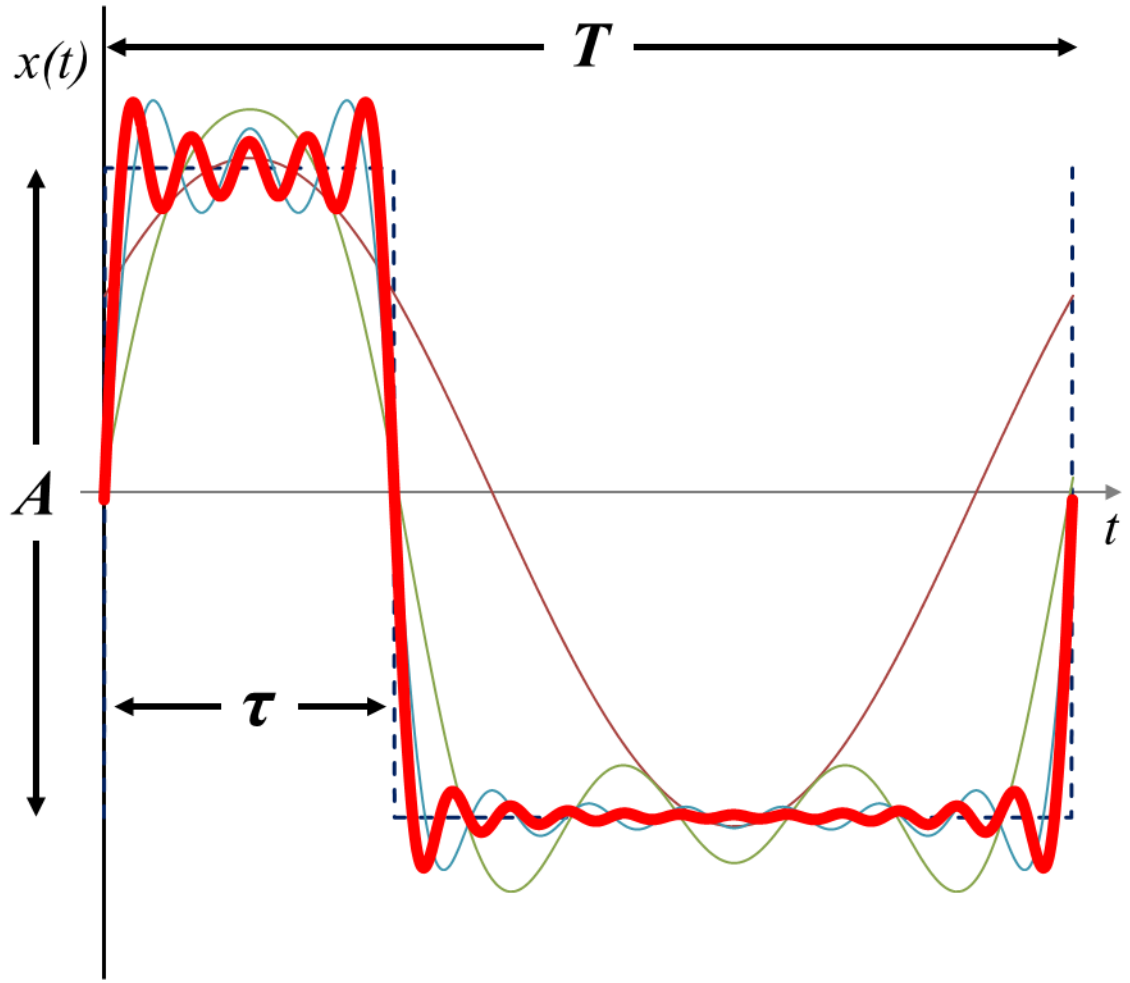
Rectangular Wave Expansion – First 10 harmonics (odd & even)



$$x(t) = A\frac{\tau}{T} + \sum_{n=1}^{\infty} 2A\frac{\tau}{T}\left[\frac{\sin(n\pi f_0\tau)}{n\pi f_0\tau}\right]\cos(2\pi n f_0 t - n\pi f_0\tau)$$

- Rectangular Pulse
- Fundamental
- First 4 harmonics
- First 10 harmonics

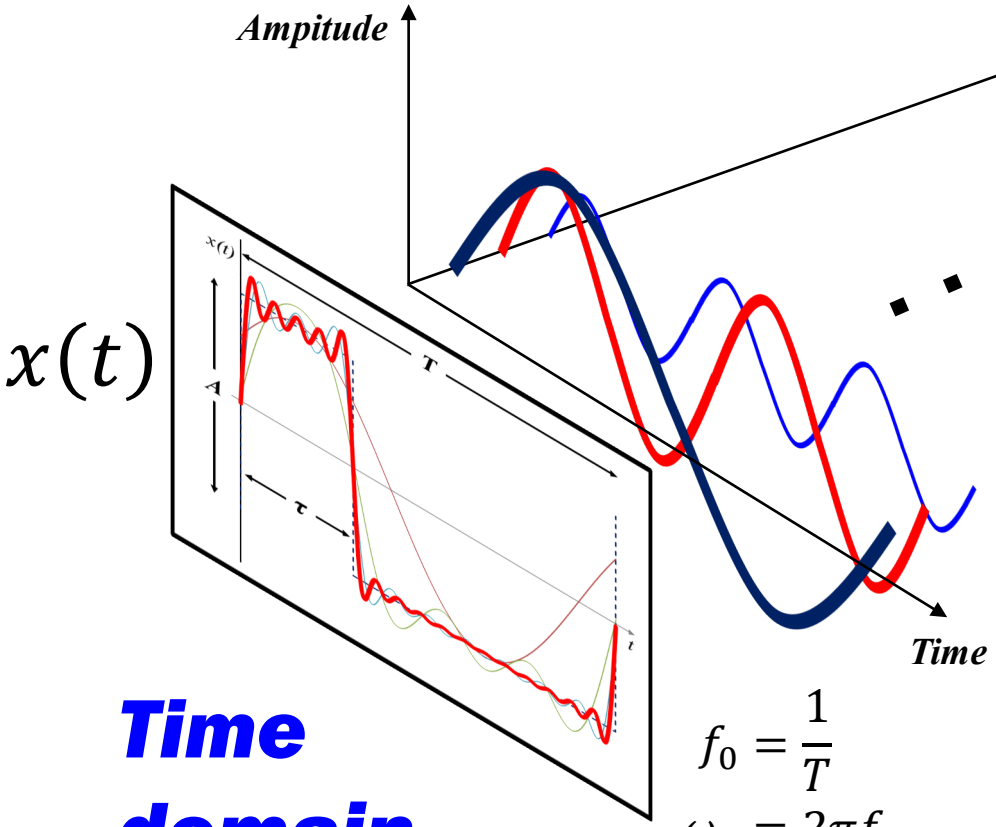
Rectangular Wave Expansion – First 16 harmonics (odd & even)



$$x(t) = A\frac{\tau}{T} + \sum_{n=1}^{\infty} 2A\frac{\tau}{T}\left[\frac{\sin(n\pi f_0\tau)}{n\pi f_0\tau}\right]\cos(2\pi n f_0 t - n\pi f_0\tau)$$

- Rectangular Pulse
- Fundamental
- First 4 harmonics
- First 10 harmonics
- First 16 harmonics

Rectangular Wave Expansion: Time and Frequency Domain

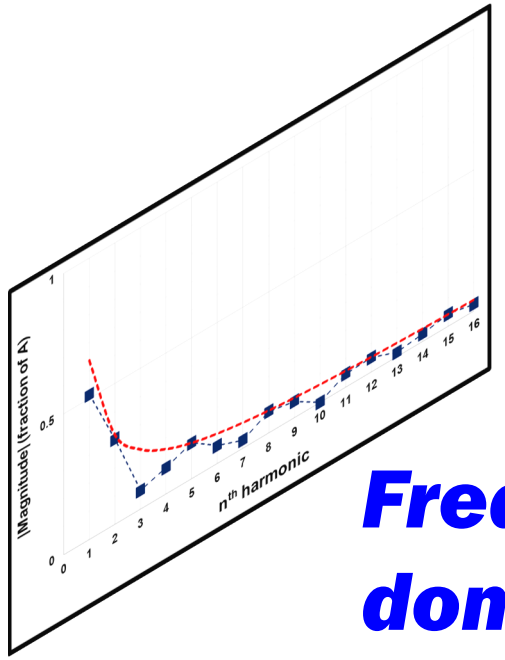


$$x(t) = A\frac{\tau}{T} + \sum_{n=1}^{\infty} 2A\frac{\tau}{T}\left[\frac{\sin(n\pi f_0\tau)}{n\pi f_0\tau}\right]\cos(2\pi n f_0 t - n\pi f_0\tau)$$

Time domain

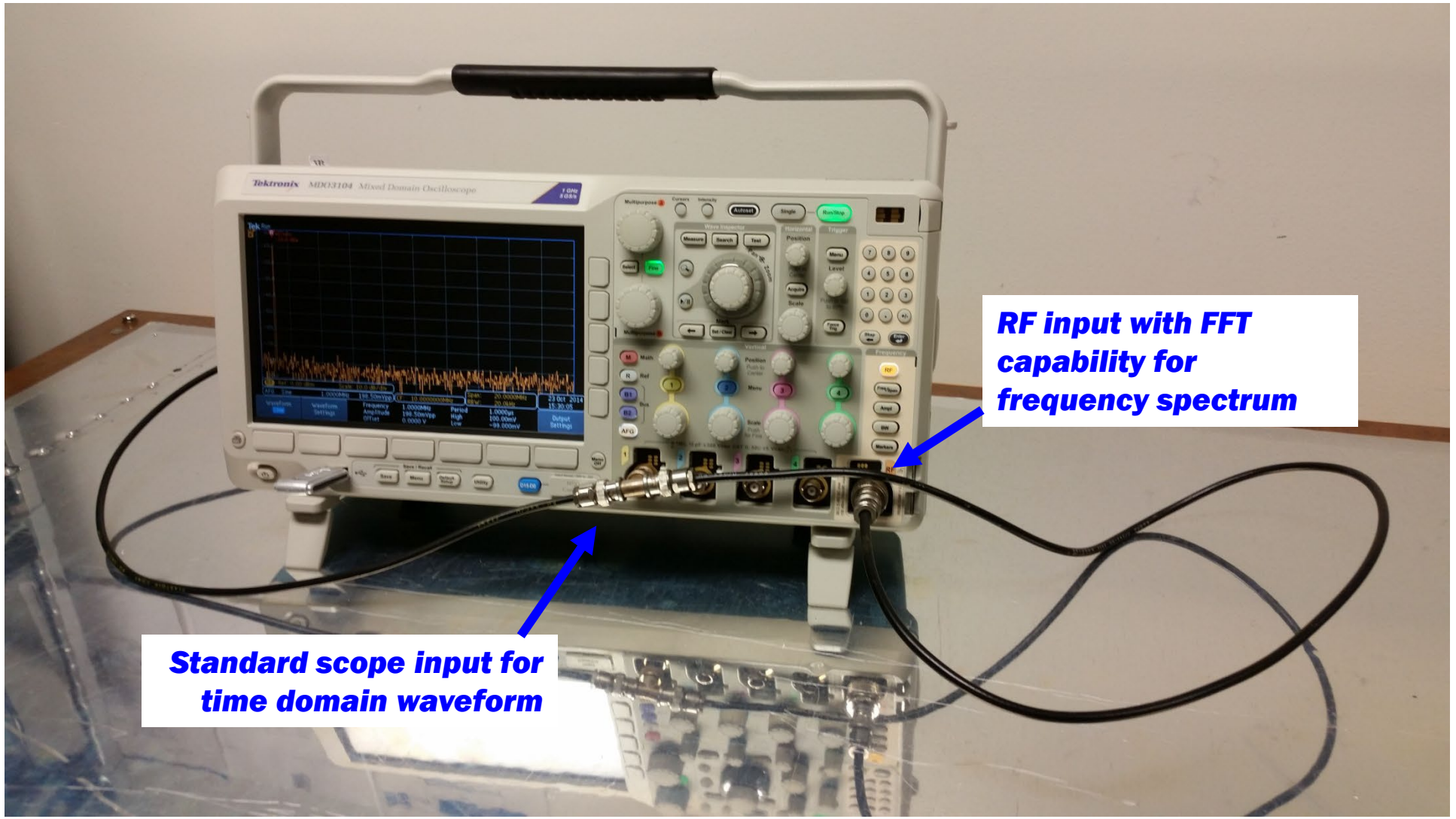
$$f_0 = \frac{1}{T}$$

$$\omega_0 = 2\pi f_0$$

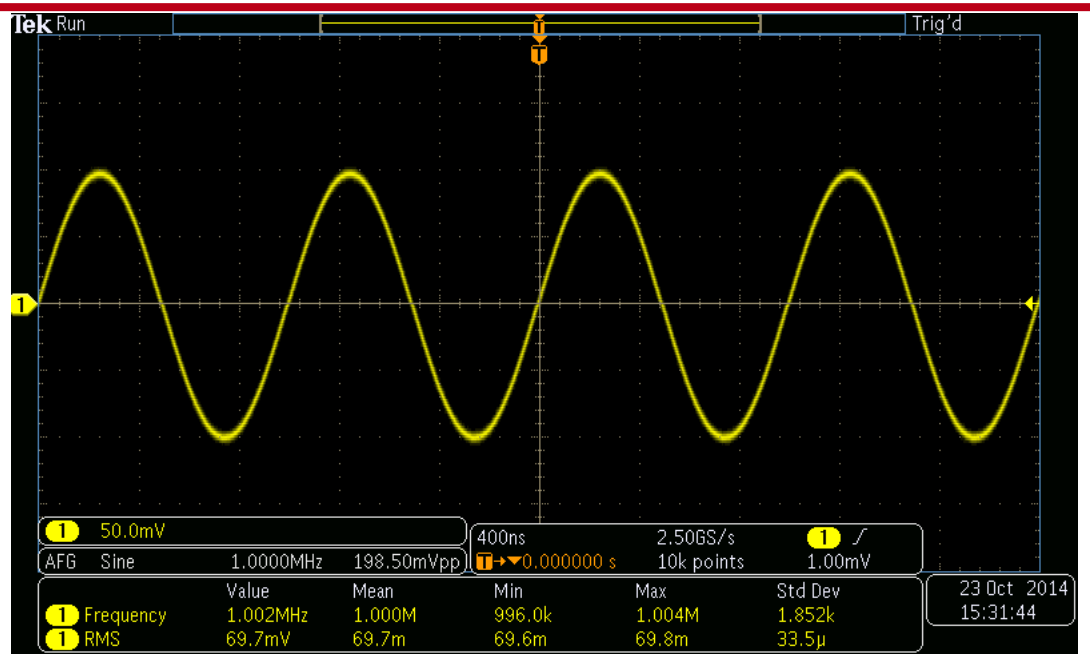


Frequency domain

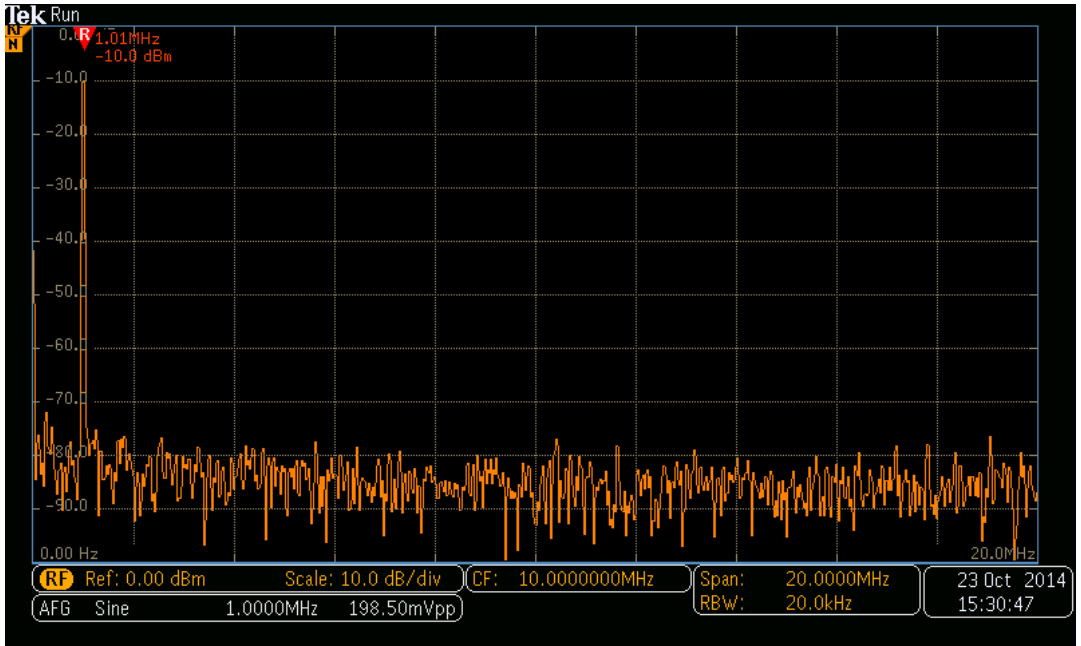
Hook it up and see ‘em for yourself...



Sinusoid

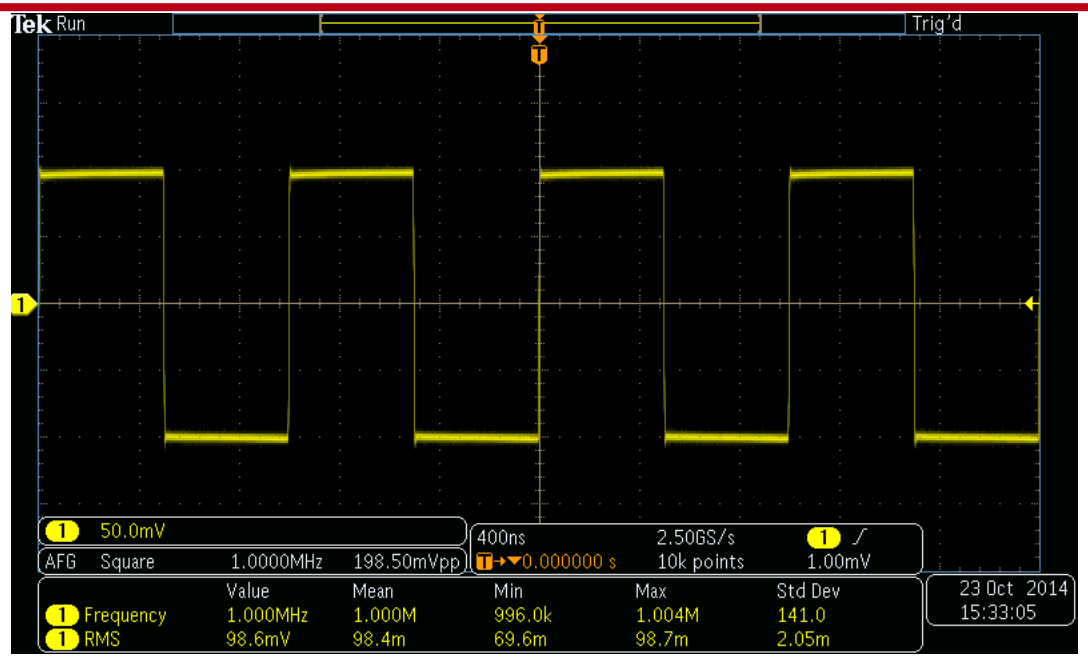


Single peak

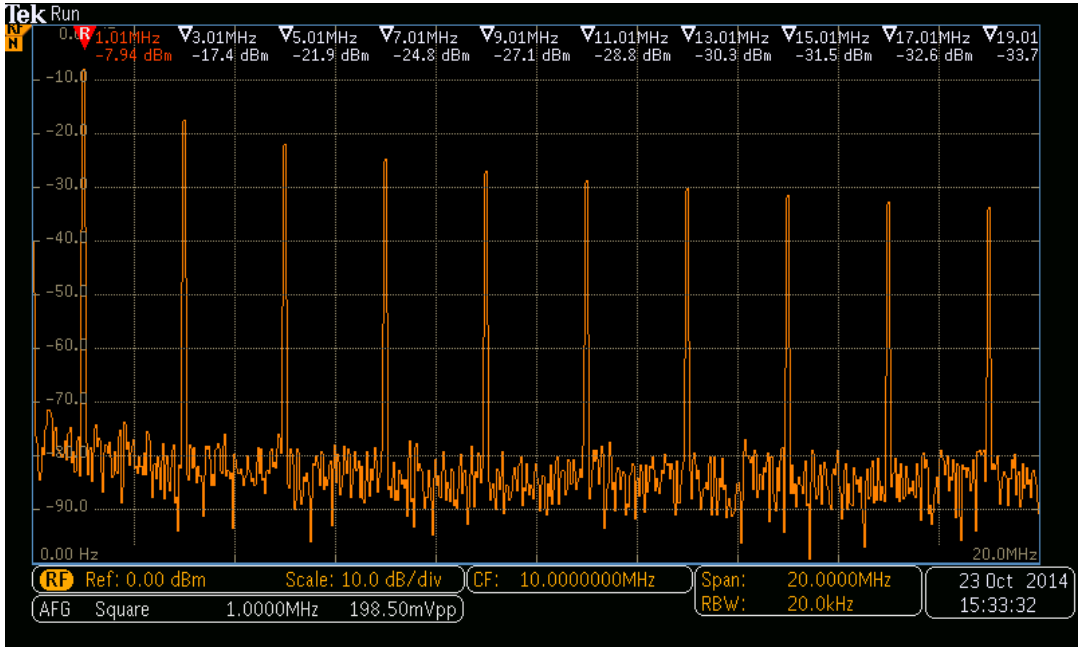


Square Wave

Duty cycle = 50%)

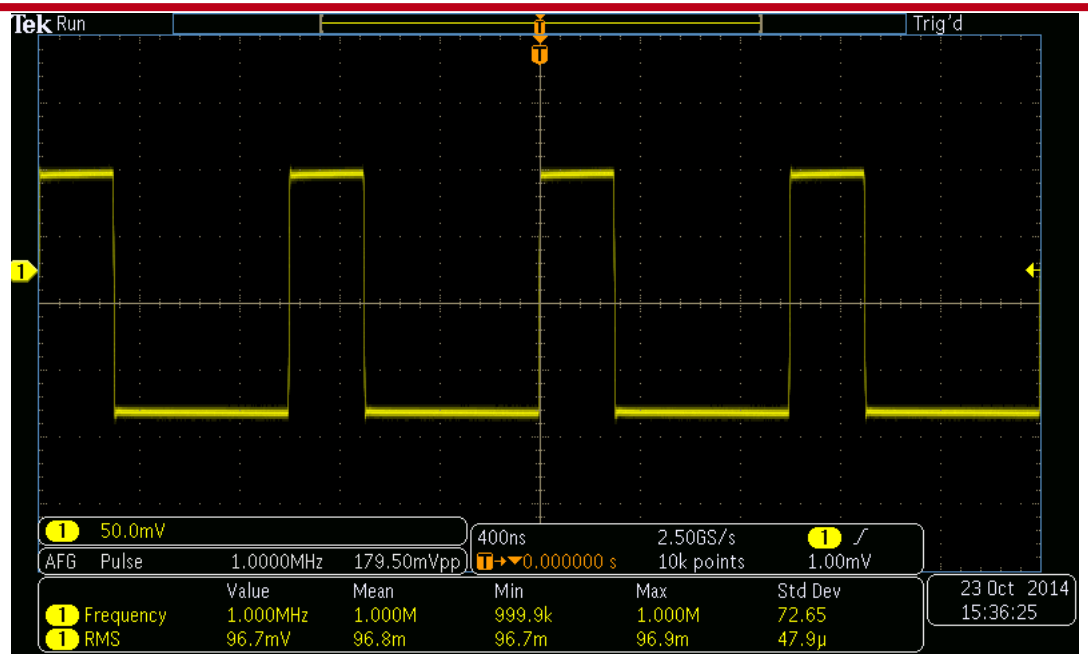


Odd harmonics only



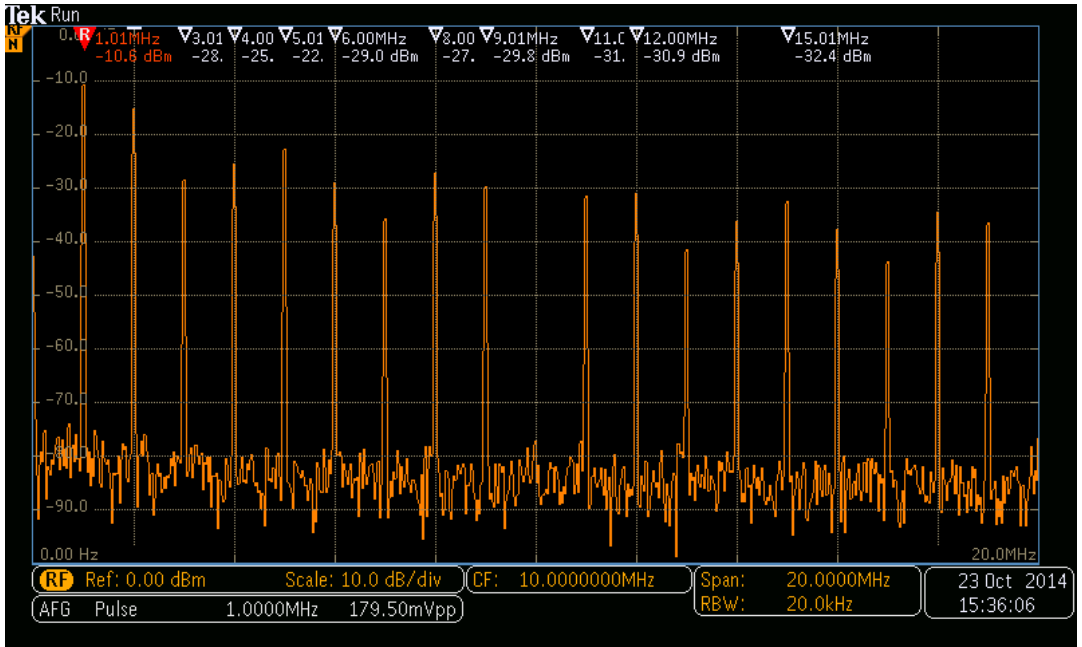
Rectangular Pulse Train

Duty cycle ≠ 50%)



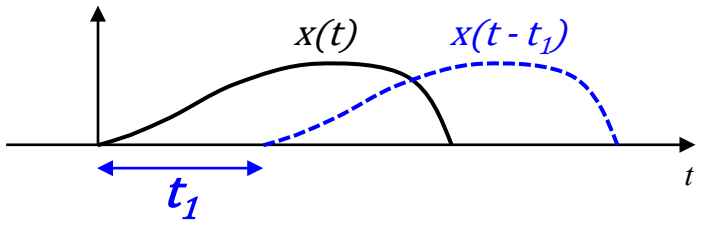
Odd & even harmonics

(duty cycle ≠ 50%)



Useful Properties of Fourier Series

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$



Time shift:

$$x(t - t_1) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 (t - t_1)}$$

$$x(t - t_1) = \sum_{n=-\infty}^{\infty} c_n e^{-jn\omega_0 t_1} e^{jn\omega_0 t}$$

Phase shift

$\theta = -n\omega_0 t_1$

Useful Properties of Fourier Series (cont.)

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

Derivative:

$$\frac{dx(t)}{dt} = \sum_{n=-\infty}^{\infty} jn\omega_0 c_n e^{jn\omega_0 t}$$

$$\frac{d^2x(t)}{dt^2} = \sum_{n=-\infty}^{\infty} (jn\omega_0)^2 c_n e^{jn\omega_0 t}$$

$$\frac{d^k x(t)}{dt^k} = \sum_{n=-\infty}^{\infty} \underbrace{(jn\omega_0)^k c_n}_{(c_n)_k} e^{jn\omega_0 t}$$

Coefficients for k^{th} order derivative

$$c_n = \frac{1}{(jn\omega_0)^k}(c_n)_k$$

Coefficients for x(t)

Integral:

$$\int x(t) = \sum_{n=-\infty}^{\infty} \frac{1}{jn\omega_0} c_n e^{jn\omega_0 t}$$

$$\iint x(t) = \sum_{n=-\infty}^{\infty} \frac{1}{(jn\omega_0)^2} c_n e^{jn\omega_0 t}$$

$$\ldots \iiint x(t) = \sum_{n=-\infty}^{\infty} \underbrace{\frac{1}{(jn\omega_0)^k} c_n}_{(c_n)_k} e^{jn\omega_0 t}$$

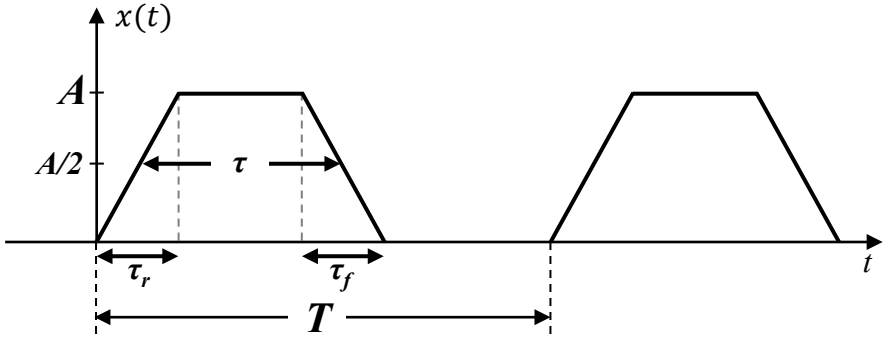
k^{th} order integral

Coefficients for k^{th} order integral

$$c_n = (jn\omega_0)^k (c_n)_k$$

Coefficients for x(t)

Trapezoidal Pulse Train



Similar parameters as rectangular pulse train

A = peak-to-peak amplitude

T = period

τ = pulse width
(measured between 50% points of rising and falling edges)

Adding finite rise and fall times τ_r and τ_f

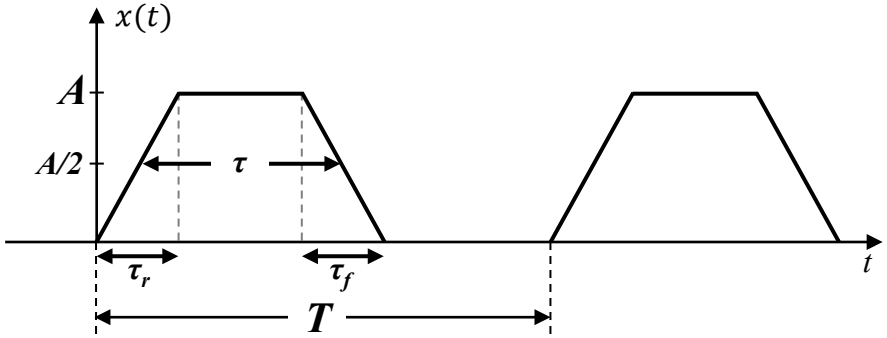
τ_r and τ_f are generally measured between 10% and 90% of minimum and maximum values

In this analysis, we will measure them between 0% and 100% to simplify calculations

Clearly much simpler than rise/fall characteristics of real-world waveforms, but facilitates simpler analysis to provide insight into effects of rise/fall times on signal spectra

→ "Crux of the biscuit"

Trapezoidal Pulse Train (cont.)



DC component (average over a period):

Area under rising edge: $A_1 = \frac{1}{2}A\tau_r$

Area under falling edge: $A_3 = \frac{1}{2}A\tau_f$

Area under flat portion: $A_2 = A\left(\tau - \frac{\tau_r}{2} - \frac{\tau_f}{2}\right) = A\tau - \frac{1}{2}A\tau_r - \frac{1}{2}A\tau_f$

Total area under pulse: $A_{TOT} = A_1 + A_2 + A_3$

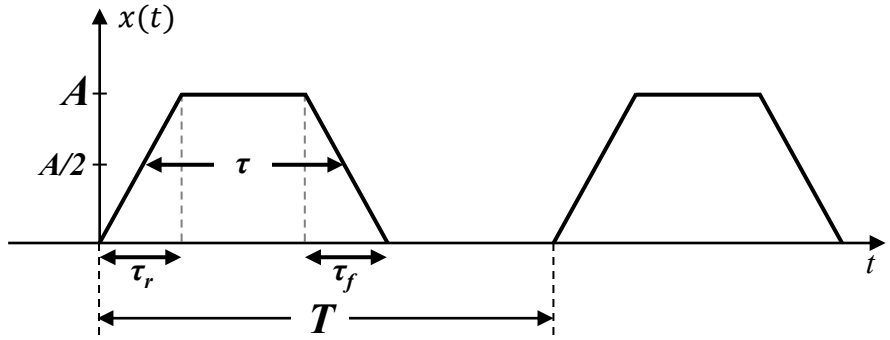
$= A\tau$

Average over period T: $x(t)_{av} = \frac{A\tau}{T}$

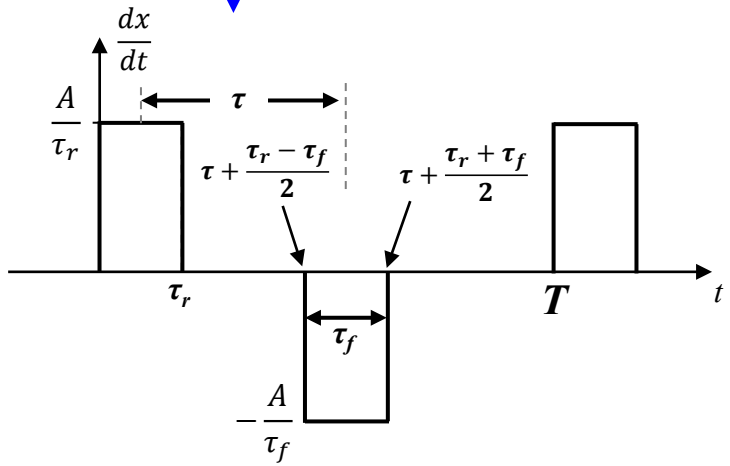
DC component $c_0 = \frac{A\tau}{T}$

(same as rectangular pulse train)

Trapezoidal Pulse Train (cont.)



$$\frac{dx(t)}{dt}$$



Coefficients of dx(t)/dt:

$$(c_n)_1 = \frac{1}{T}\int_0^{\tau_r} \frac{A}{\tau_r} e^{-jn\omega_0 t}\,dt + \frac{1}{T}\int_{\tau+(\tau_r-\tau_f)/2}^{\tau+(\tau_r+\tau_f)/2} \left(-\frac{A}{\tau_f}\right) e^{-jn\omega_0 t}\,dt$$

$$(c_n)_1 = \frac{1}{-jn\omega_0 T}\frac{A}{\tau_r}\left(e^{-jn\omega_0\tau_r} - 1\right) - \frac{1}{(-jn\omega_0)T}\frac{A}{\tau_f}\left[e^{-jn\omega_0[\tau+(\tau_r+\tau_f)/2]} - e^{-jn\omega_0[\tau+(\tau_r-\tau_f)/2]}\right]$$

Factoring out ½ of exponential terms (for reasons that will become clear)

$$(c_n)_1 = \frac{-1}{jn\omega_0\tau_r}\frac{A}{T}e^{-jn\omega_0\tau_r/2}\left(e^{jn\omega_0\tau_r/2} - e^{-jn\omega_0\tau_r/2}\right) + \frac{1}{jn\omega_0\tau_f}\frac{A}{T}e^{-jn\omega_0\tau}e^{-jn\omega_0\tau_r/2}\left[e^{-jn\omega_0\tau_f/2} - e^{jn\omega_0\tau_f/2}\right]$$

Reverse order and change sign

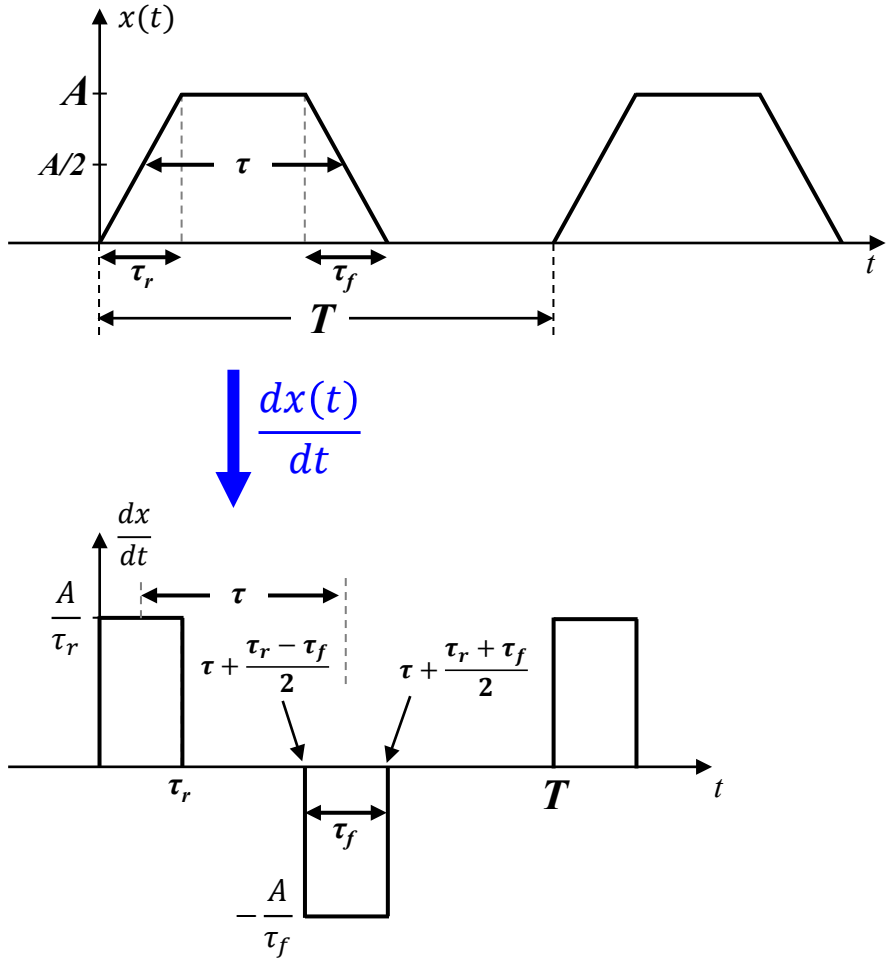
$$= \frac{-1}{jn\omega_0\tau_r}\frac{A}{T}e^{-jn\omega_0\tau_r/2}\left(e^{jn\omega_0\tau_r/2} - e^{-jn\omega_0\tau_r/2}\right) - \frac{1}{jn\omega_0\tau_f}\frac{A}{T}e^{-jn\omega_0\tau}e^{-jn\omega_0\tau_r/2}\left[e^{jn\omega_0\tau_f/2} - e^{-jn\omega_0\tau_f/2}\right]$$

$$= \frac{-1}{jn\omega_0\tau_r}\frac{A}{T}e^{-jn\omega_0\tau_r/2}\left[2j\sin\left(\tfrac{1}{2}n\omega_0\tau_r\right)\right] - \frac{1}{jn\omega_0\tau_f}\frac{A}{T}e^{-jn\omega_0\tau}e^{-jn\omega_0\tau_r/2}\left[2j\sin\left(\tfrac{1}{2}n\omega_0\tau_f\right)\right]$$

$$= \frac{A}{T}e^{-jn\omega_0\tau_r/2}\left[\frac{\sin\left(\frac{1}{2}n\omega_0\tau_r\right)}{\frac{1}{2}n\omega_0\tau_r}\right] - \frac{A}{T}e^{-jn\omega_0\tau}e^{-jn\omega_0\tau_r/2}\left[\frac{\sin\left(\frac{1}{2}n\omega_0\tau_f\right)}{\frac{1}{2}n\omega_0\tau_f}\right]$$

$$= \frac{A}{T}e^{-jn\omega_0\tau_r/2}\left[\frac{\sin\left(\frac{1}{2}n\omega_0\tau_r\right)}{\frac{1}{2}n\omega_0\tau_r} - \frac{\sin\left(\frac{1}{2}n\omega_0\tau_f\right)}{\frac{1}{2}n\omega_0\tau_f}e^{-jn\omega_0\tau}\right]$$

Trapezoidal Pulse Train (cont.)



Coefficients of dx(t)/dt:

$$(c_n)_1 = \frac{A}{T} e^{-jn\omega_0\tau_r/2}\left[\frac{\sin\left(\frac{1}{2}n\omega_0\tau_r\right)}{\frac{1}{2}n\omega_0\tau_r} - \frac{\sin\left(\frac{1}{2}n\omega_0\tau_f\right)}{\frac{1}{2}n\omega_0\tau_f} e^{-jn\omega_0\tau}\right]$$

Coefficients of x(t):

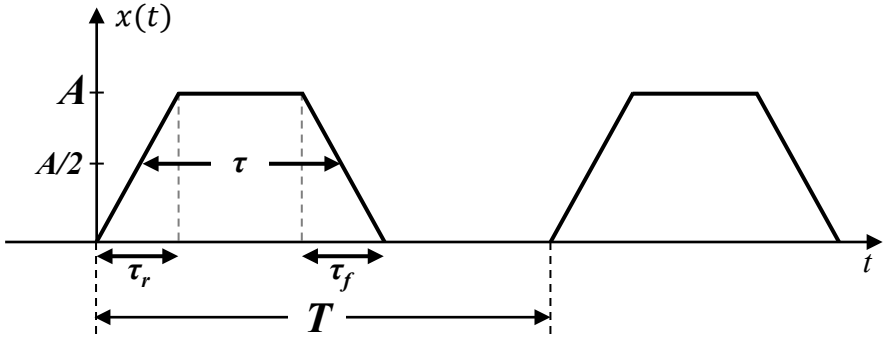
$$c_n = \frac{1}{jn\omega_0}(c_n)_1$$

$$c_n = \frac{A}{jn\omega_0 T} e^{-jn\omega_0\tau_r/2}\left[\frac{\sin\left(\frac{1}{2}n\omega_0\tau_r\right)}{\frac{1}{2}n\omega_0\tau_r} - \frac{\sin\left(\frac{1}{2}n\omega_0\tau_f\right)}{\frac{1}{2}n\omega_0\tau_f} e^{-jn\omega_0\tau}\right]$$

$$c_n = -j\frac{A}{2\pi n} e^{-jn\omega_0\tau_r/2} e^{-jn\omega_0\tau/2}\left[\frac{\sin\left(\frac{1}{2}n\omega_0\tau_r\right)}{\frac{1}{2}n\omega_0\tau_r} e^{jn\omega_0\tau/2} - \frac{\sin\left(\frac{1}{2}n\omega_0\tau_f\right)}{\frac{1}{2}n\omega_0\tau_f} e^{-jn\omega_0\tau/2}\right]$$

End of the road? Hah!

Trapezoidal Pulse Train (cont.)



Coefficients of x(t):

$$c_n = -j\frac{A}{2\pi n}e^{-jn\omega_0\tau_r/2}e^{-jn\omega_0\tau/2}\left[\frac{\sin\left(\frac{1}{2}n\omega_0\tau_r\right)}{\frac{1}{2}n\omega_0\tau_r}e^{jn\omega_0\tau/2} - \frac{\sin\left(\frac{1}{2}n\omega_0\tau_f\right)}{\frac{1}{2}n\omega_0\tau_f}e^{-jn\omega_0\tau/2}\right]$$

Simplifying assumption:

$\tau_r = \tau_f$

$$c_n = -j\frac{A}{2\pi n}e^{-jn\omega_0\tau_r/2}e^{-jn\omega_0\tau/2}\left[\frac{\sin\left(\frac{1}{2}n\omega_0\tau_r\right)}{\frac{1}{2}n\omega_0\tau_r}e^{jn\omega_0\tau/2} - \frac{\sin\left(\frac{1}{2}n\omega_0\tau_r\right)}{\frac{1}{2}n\omega_0\tau_r}e^{-jn\omega_0\tau/2}\right]$$

$$= -j\frac{A}{2\pi n}e^{-jn\omega_0\tau_r/2}e^{-jn\omega_0\tau/2}\left[\frac{\sin\left(\frac{1}{2}n\omega_0\tau_r\right)}{\frac{1}{2}n\omega_0\tau_r}\right]\left[e^{jn\omega_0\tau/2} - e^{-jn\omega_0\tau/2}\right]$$

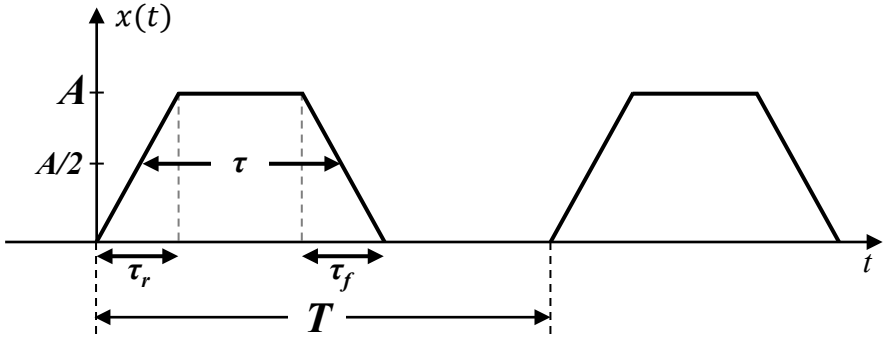
$$= -j\frac{A}{2\pi n}e^{-jn\omega_0\tau_r/2}e^{-jn\omega_0\tau/2}\left[\frac{\sin\left(\frac{1}{2}n\omega_0\tau_r\right)}{\frac{1}{2}n\omega_0\tau_r}\right]\left[2j\sin\left(\frac{1}{2}n\omega_0\tau\right)\right]$$

Rearrange terms

$$= \frac{A\tau\omega_0}{2\pi}e^{-jn\omega_0\tau_r/2}e^{-jn\omega_0\tau/2}\left[\frac{\sin\left(\frac{1}{2}n\omega_0\tau_r\right)}{\frac{1}{2}n\omega_0\tau_r}\right]\left[\frac{\sin\left(\frac{1}{2}n\omega_0\tau\right)}{\frac{1}{2}n\omega_0\tau}\right]$$

$$= \frac{A\tau}{T}e^{-jn\omega_0\tau_r/2}e^{-jn\omega_0\tau/2}\left[\frac{\sin\left(\frac{1}{2}n\omega_0\tau_r\right)}{\frac{1}{2}n\omega_0\tau_r}\right]\left[\frac{\sin\left(\frac{1}{2}n\omega_0\tau\right)}{\frac{1}{2}n\omega_0\tau}\right]$$

Trapezoidal Pulse Train (cont.)



Coefficients of x(t):

$$c_n = \frac{A\tau}{T} e^{-jn\omega_0\tau_r/2} e^{-jn\omega_0\tau/2} \left[\frac{\sin\left(\frac{1}{2}n\omega_0\tau_r\right)}{\frac{1}{2}n\omega_0\tau_r}\right]\left[\frac{\sin\left(\frac{1}{2}n\omega_0\tau\right)}{\frac{1}{2}n\omega_0\tau}\right]$$

$$c_n = \frac{A\tau}{T}\left[\frac{\sin\left(\frac{1}{2}n\omega_0\tau_r\right)}{\frac{1}{2}n\omega_0\tau_r}\right]\left[\frac{\sin\left(\frac{1}{2}n\omega_0\tau\right)}{\frac{1}{2}\pi n\omega_0\tau}\right] e^{-jn\omega_0(\tau+\tau_r)/2}$$

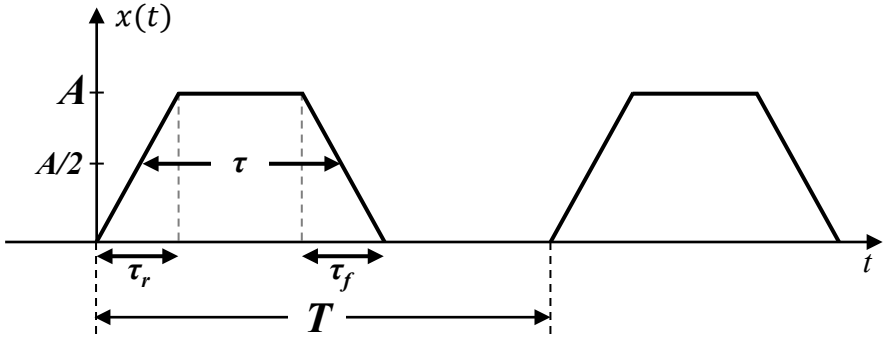
$\omega_0 = 2\pi f_0$:

$$c_n = \underbrace{\frac{A\tau}{T}\left[\frac{\sin(n\pi f_0\tau_r)}{n\pi f_0\tau_r}\right]\left[\frac{\sin(n\pi f_0\tau)}{n\pi f_0\tau}\right]}_{|c_n|} \underbrace{e^{-jn\pi f_0(\tau+\tau_r)}}_{\angle c_n}$$

Fourier series expansion of (simplified) trapezoidal pulse train:

$$x(t) = A\frac{\tau}{T} + \sum_{n=1}^{\infty} 2A\frac{\tau}{T}\left[\frac{\sin(n\pi f_0\tau)}{n\pi f_0\tau}\right]\left[\frac{\sin(n\pi f_0\tau_r)}{n\pi f_0\tau_r}\right]\cos[2\pi f_0 t - n\pi f_0(\tau+\tau_r)]$$

Trapezoidal Pulse Train (cont.)



Compare to Fourier series expansion of rectangular pulse train:

$$x(t) = A\frac{\tau}{T} + \sum_{n=1}^{\infty} 2A\frac{\tau}{T}\left[\frac{\sin(n\pi f_0\tau)}{n\pi f_0\tau}\right]\cos(2\pi n f_0 t - n\pi f_0\tau)$$

$$x(t) = A\frac{\tau}{T} + \sum_{n=1}^{\infty} 2A\frac{\tau}{T}\left[\frac{\sin(n\pi f_0\tau)}{n\pi f_0\tau}\right]\left[\frac{\sin(n\pi f_0\tau_r)}{n\pi f_0\tau_r}\right]\cos[2\pi f_0 t - n\pi f_0(\tau+\tau_r)]$$

Same DC component

Same low frequency response

New high frequency term Follows envelope of $\frac{\sin(\pi f\tau_r)}{\pi f\tau_r} \rightarrow \frac{sin(x)}{x}$

Corner occurs when $\pi f\tau_r = 1$

Corner frequency: $f = \frac{1}{\pi\tau_r}$

Envelope of Trapezoidal Pulse Train Spectrum

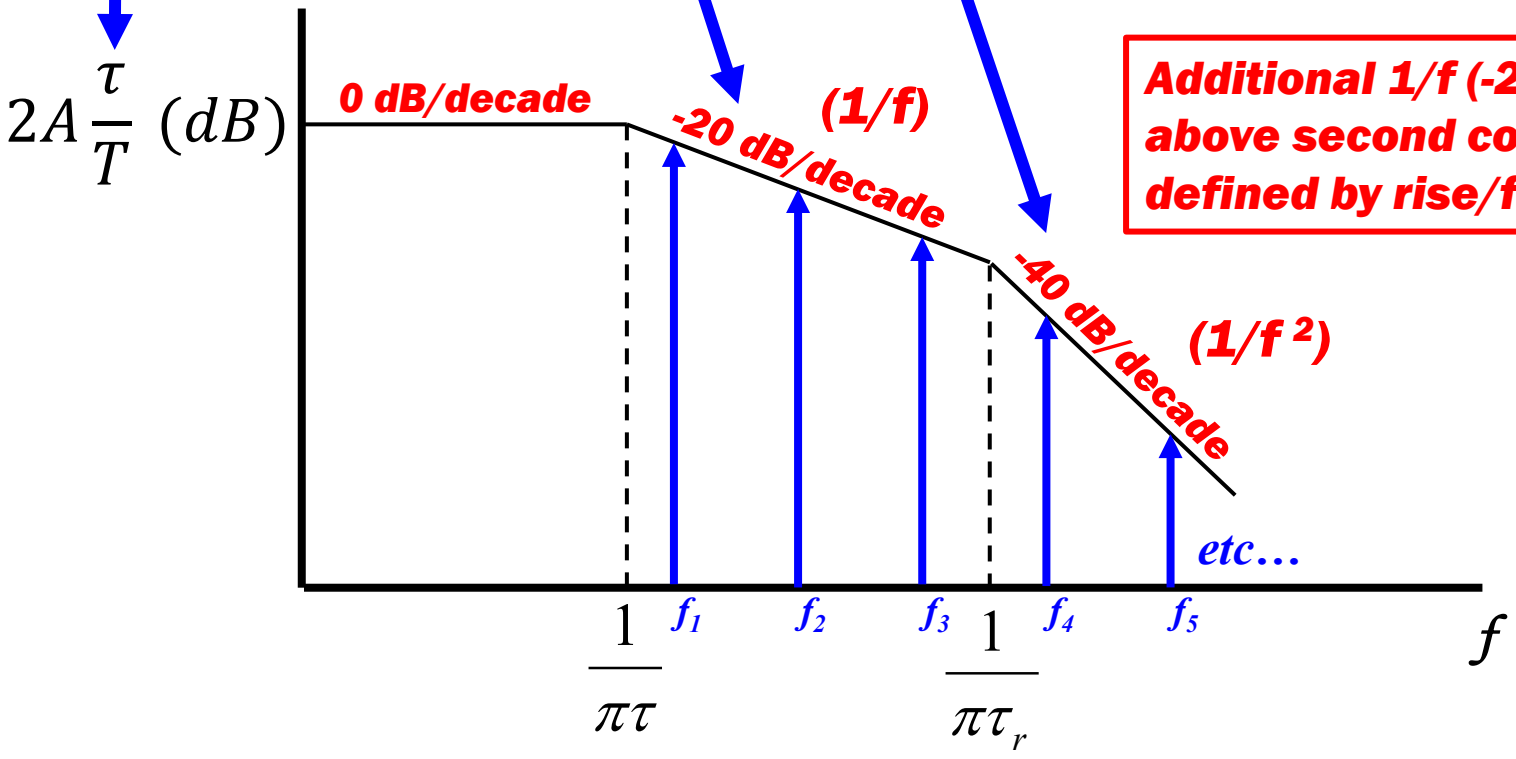
$$x(t) = A\frac{\tau}{T} + \sum_{n=1}^{\infty} 2A\frac{\tau}{T}\left[\frac{\sin(n\pi f_0\tau)}{n\pi f_0\tau}\right]\left[\frac{\sin(n\pi f_0\tau_r)}{n\pi f_0\tau_r}\right]\cos[2\pi f_0 t - n\pi f_0(\tau+\tau_r)]$$

DC component

Low frequency "plateau"

Harmonic response (pulse width)

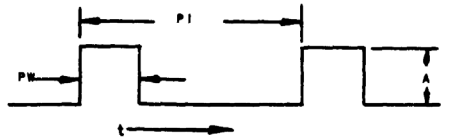
Harmonic response (rise/fall time)



Additional 1/f (-20 dB/decade) rolloff above second corner frequency defined by rise/fall times

Spectra for Various Pulse Shapes

From U.S. Army Electronics Laboratories
Interference Reduction Guide for Design Engineers, 1 August 1964:



where: PW = Pulse width
PI = Pulse interval = $\frac{1}{PRR}$

$$An = 2A\frac{(PW)}{PI} \cdot \left[\frac{\sin \frac{n\pi (PW)}{PI}}{\frac{n\pi (PW)}{PI}}\right]$$

where: An = amplitude of the n^{th} harmonic
n = 0, 1, 2, 3, 4, . . .

IN1212-76

Figure 1-4. Fourier Analysis of a Repetitive Rectangular Pulse

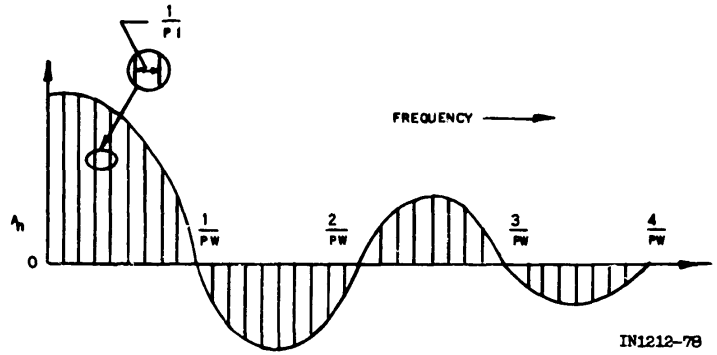


Figure 1-6. Doubling the Pulse Repetition Rate of Figure 1-5 Plot (Spacing Between Harmonics Doubled and Amplitude of Harmonics Increased)

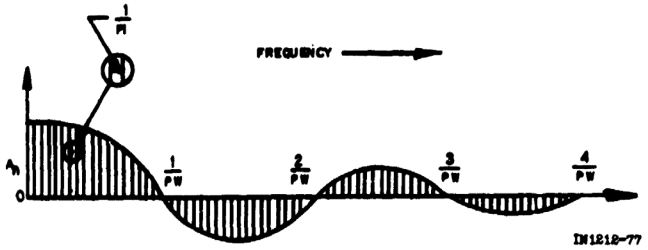


Figure 1-5. A Plot of Harmonic Frequency Versus Harmonic Amplitude for the Pulse Shown on Figure 1-4

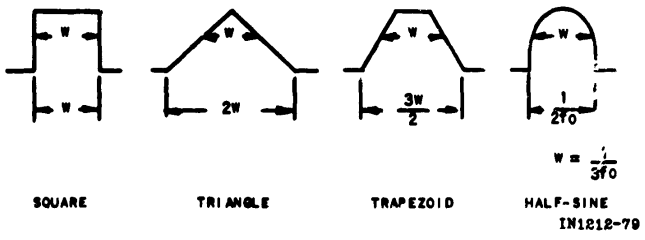


Figure 1-7. Amplitude Curves For Various Waveforms

Spectra for Various Pulse Shapes (cont.)

From U.S. Army Electronics Laboratories
Interference Reduction Guide for Design Engineers, 1 August 1964:

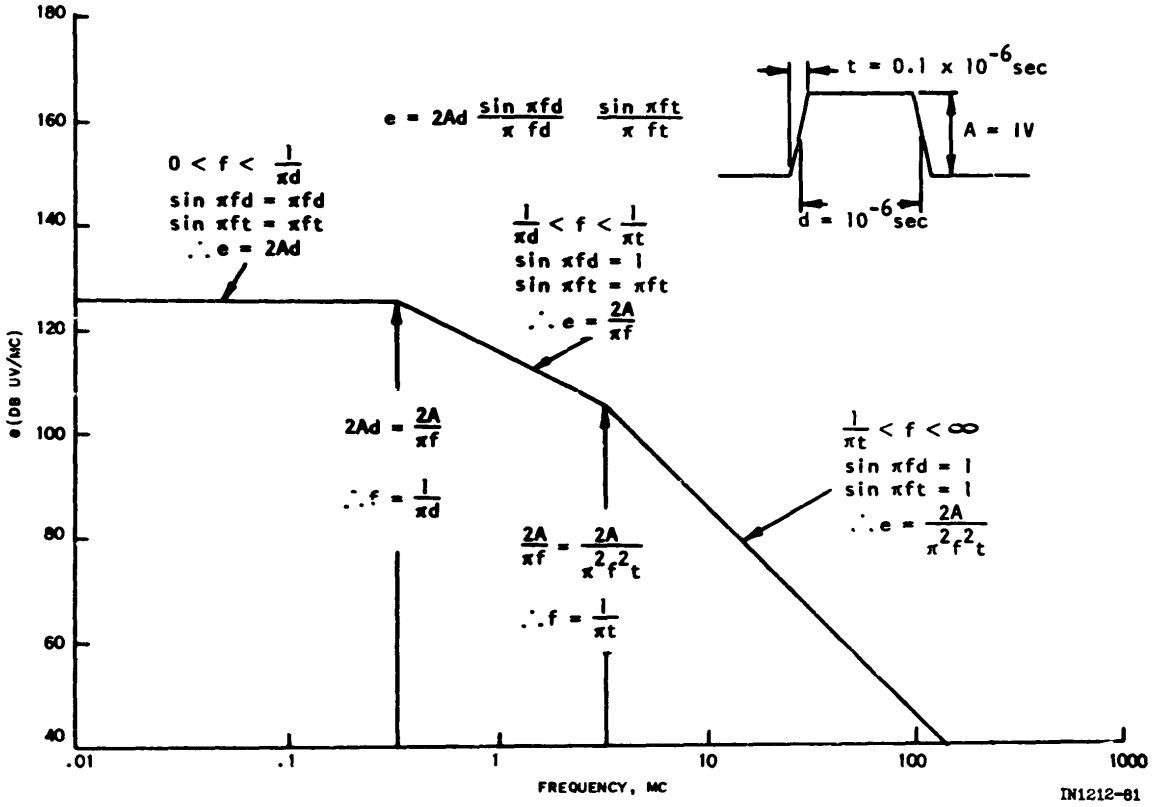


Figure 1-9. Interference Level for a 1 Volt, 1 μsec Trapezoidal Pulse

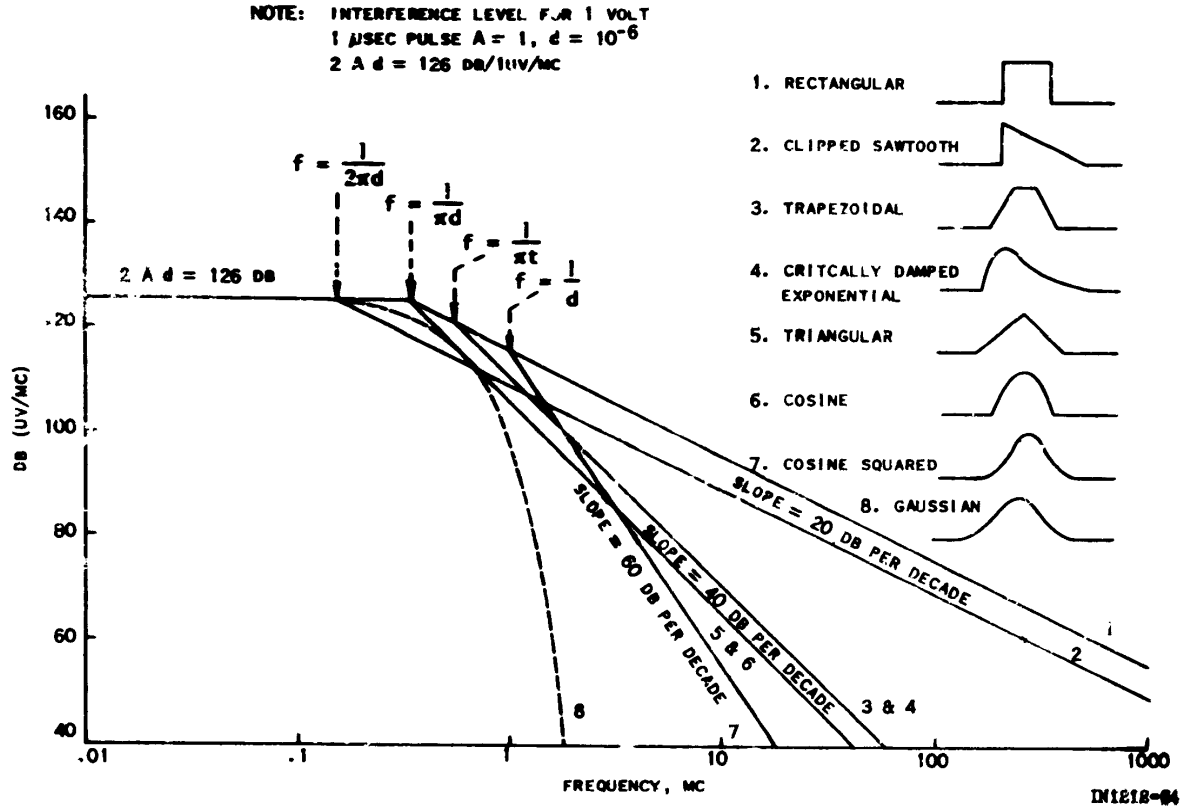
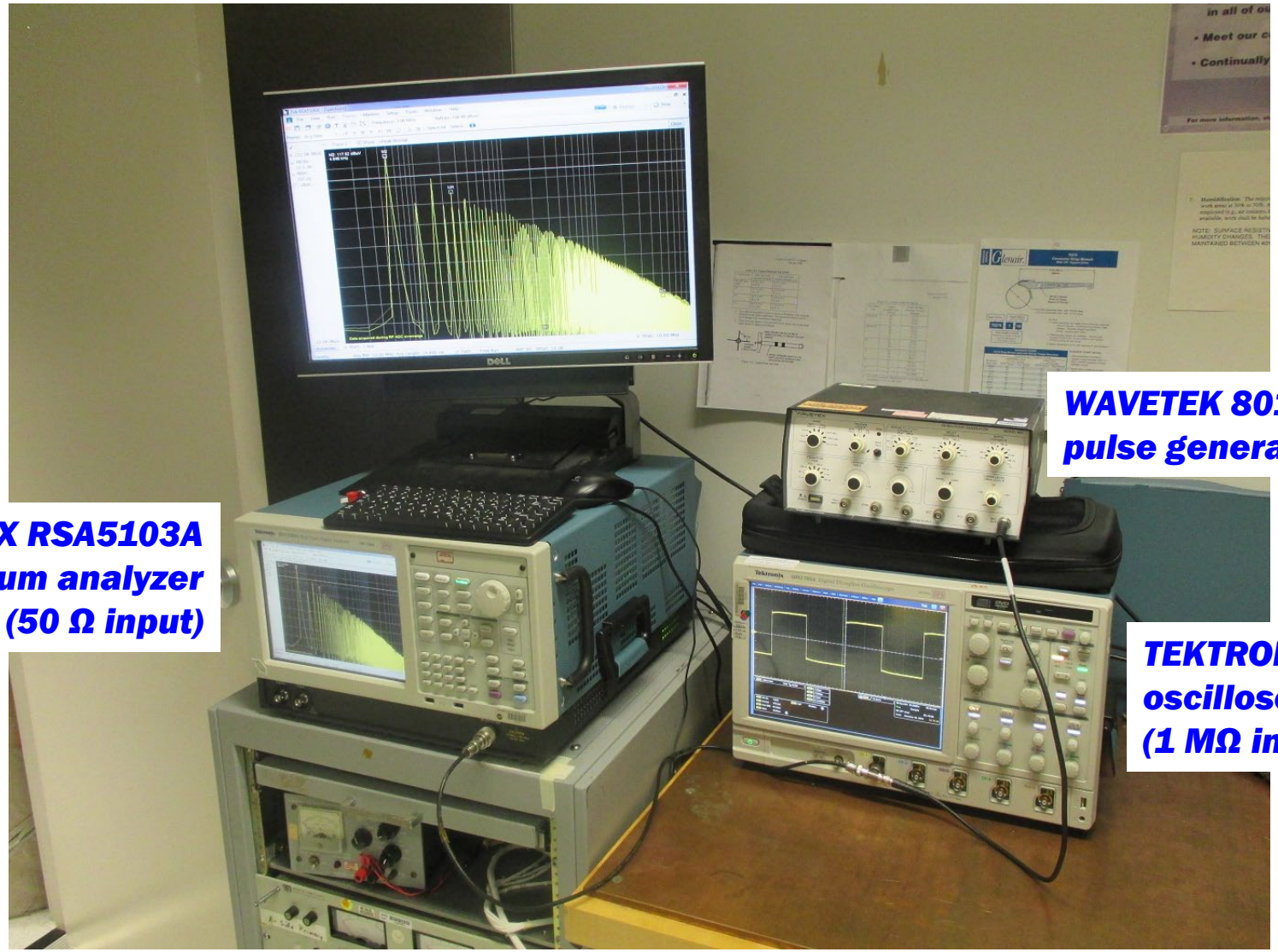
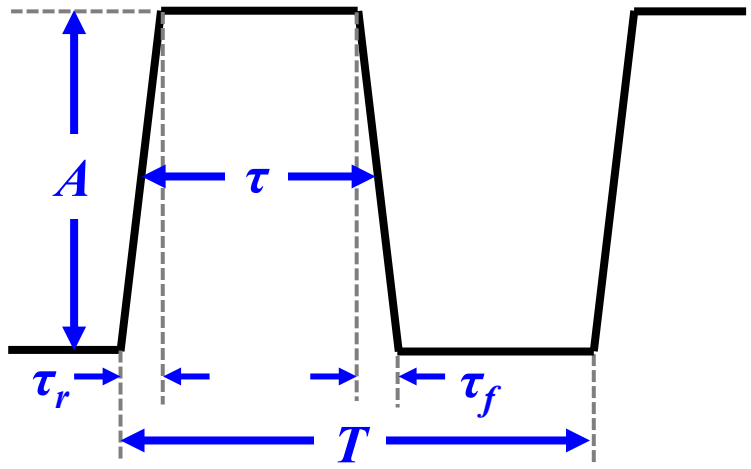


Figure 1-12. Interference Levels for Eight Common Pulses

Time Domain vs. Frequency Domain Demonstration Setup



Applied Waveform

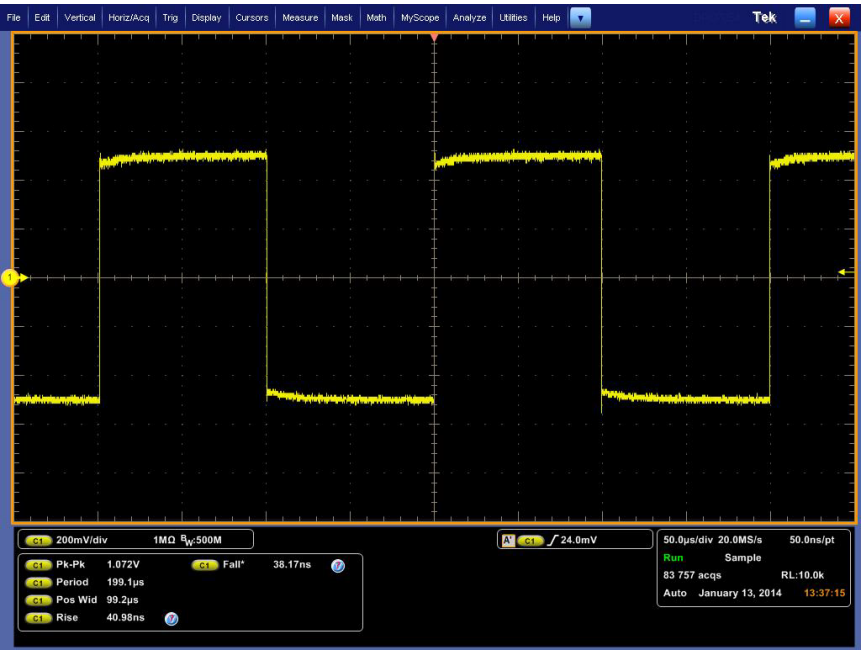


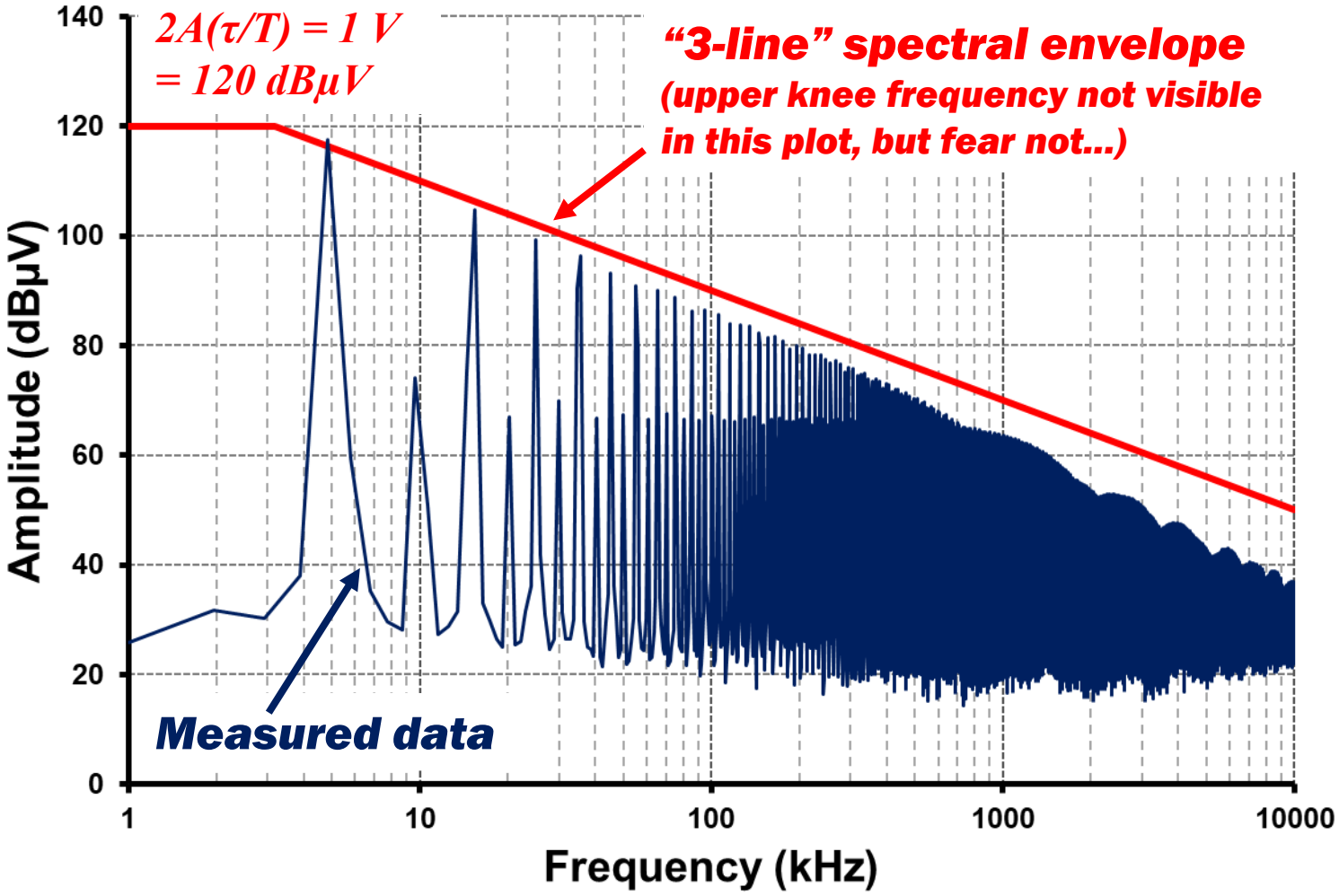
$$A = 1\ V$$

$$T = 200\ \mu s \quad (f_0 = 5\ kHz)$$

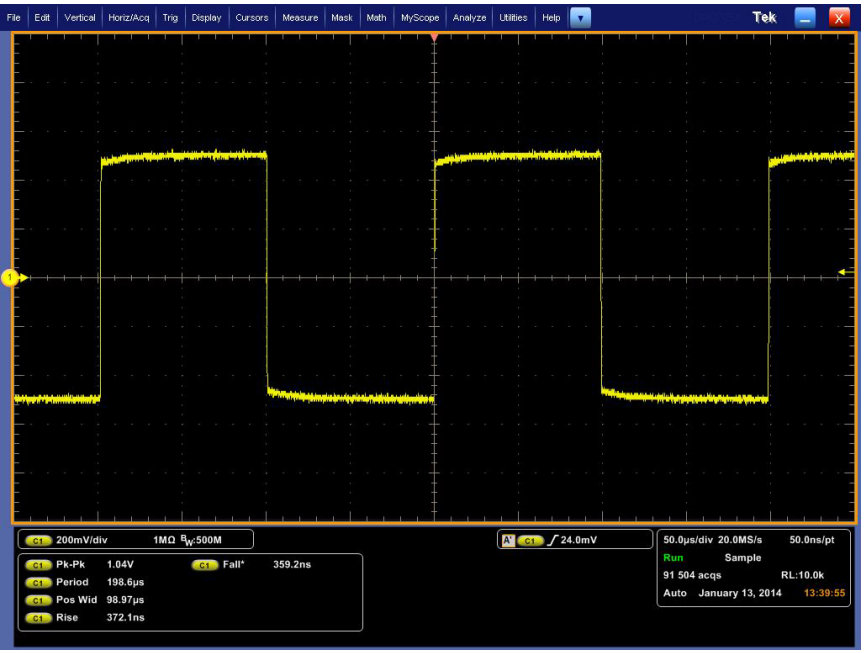
τ, τ_r, & τ_f varied as indicated on following slides

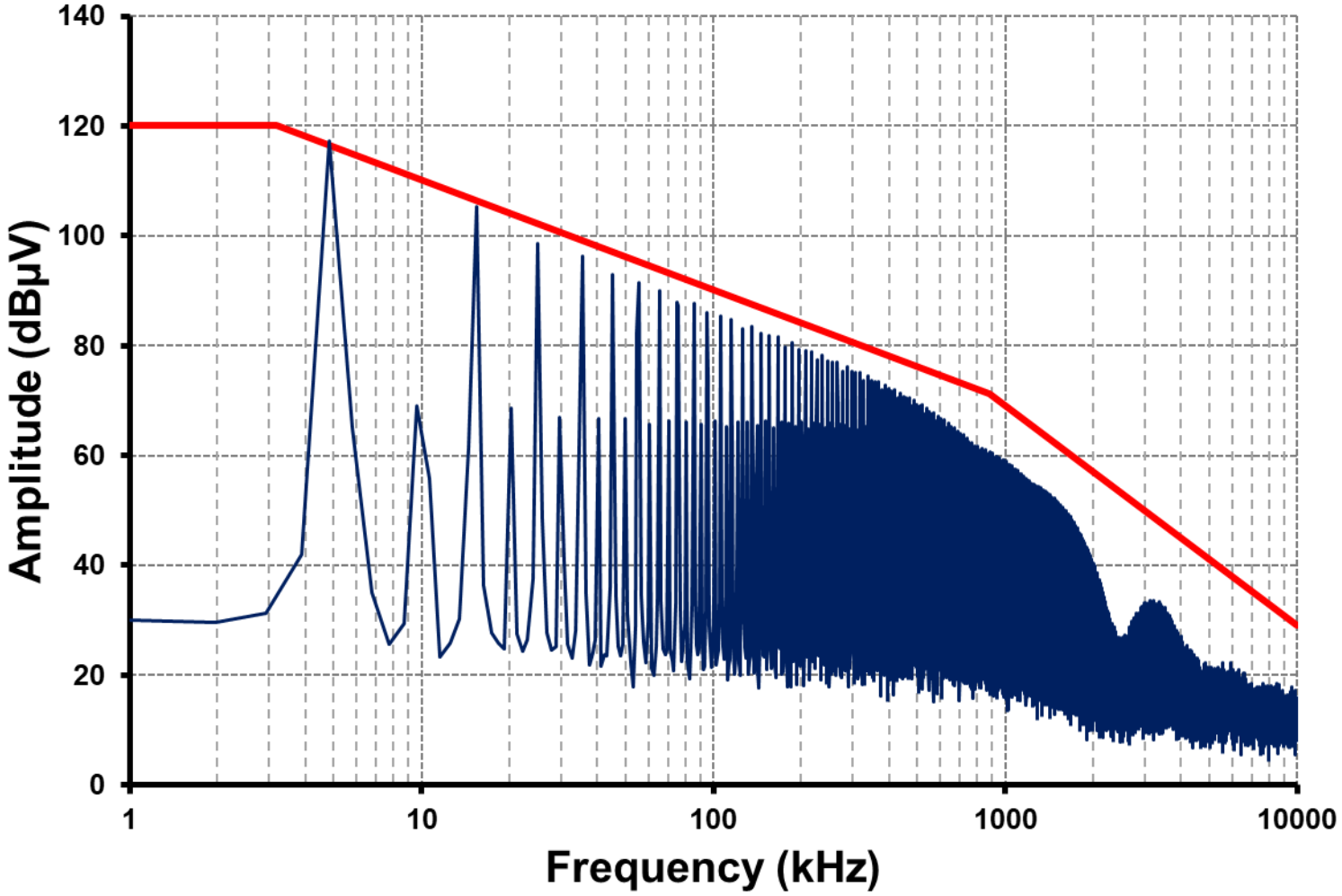
$\tau = 100\ \mu s\ (\tau/T = 50\%);\ \tau_r = \tau_f = 40\ ns$



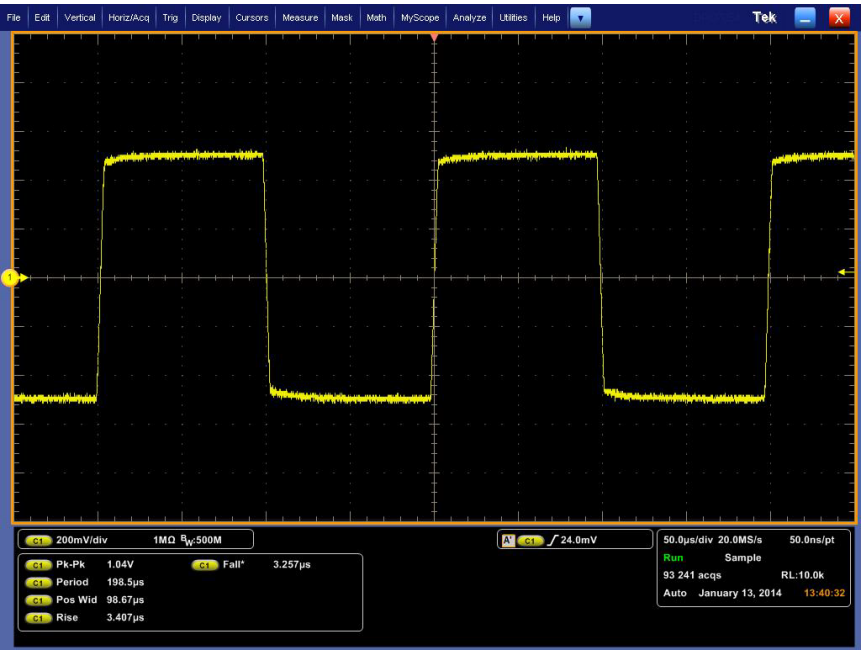


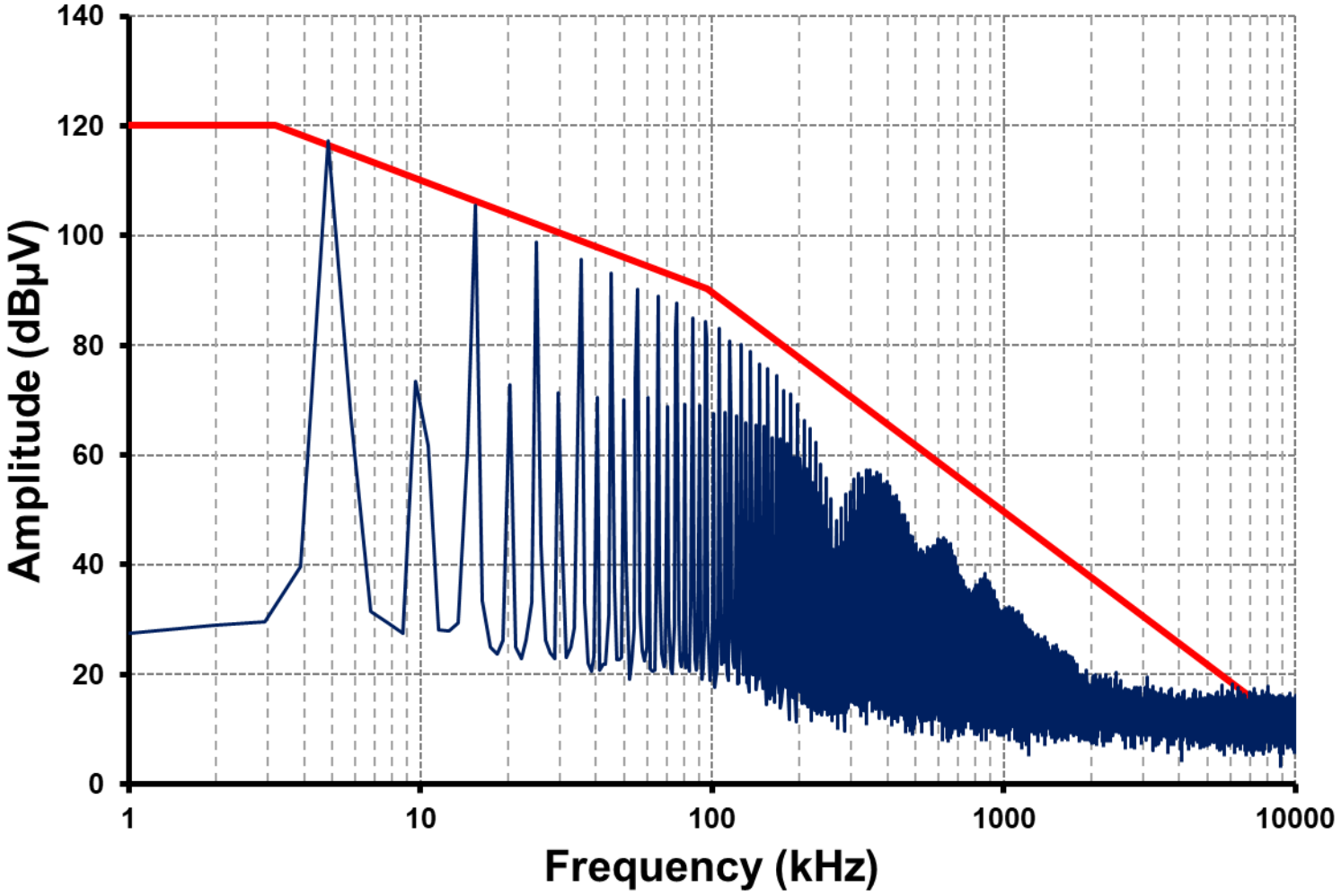
$\tau = 100\ \mu s\ (\tau/T = 50\%);\ \tau_r = \tau_f = 360\ ns$



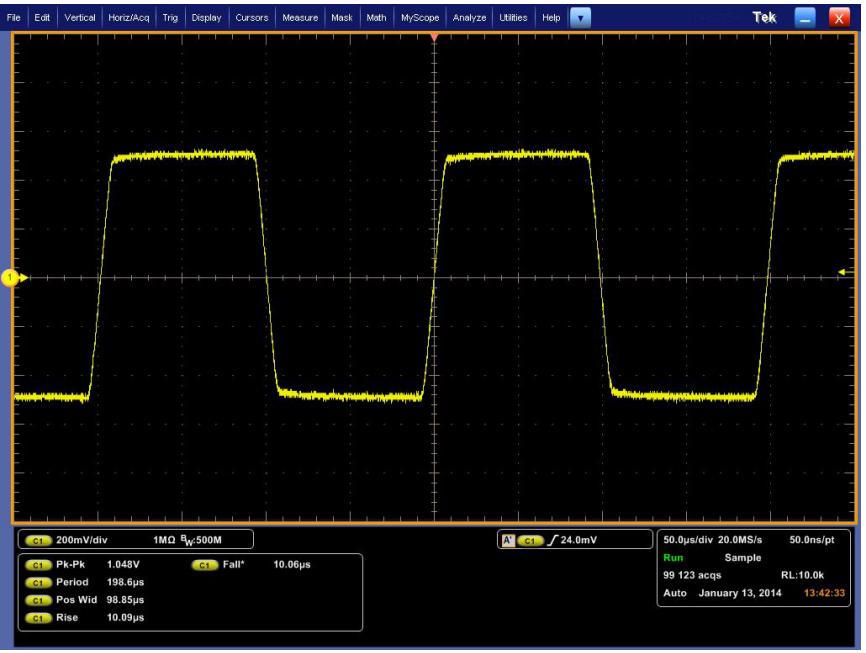


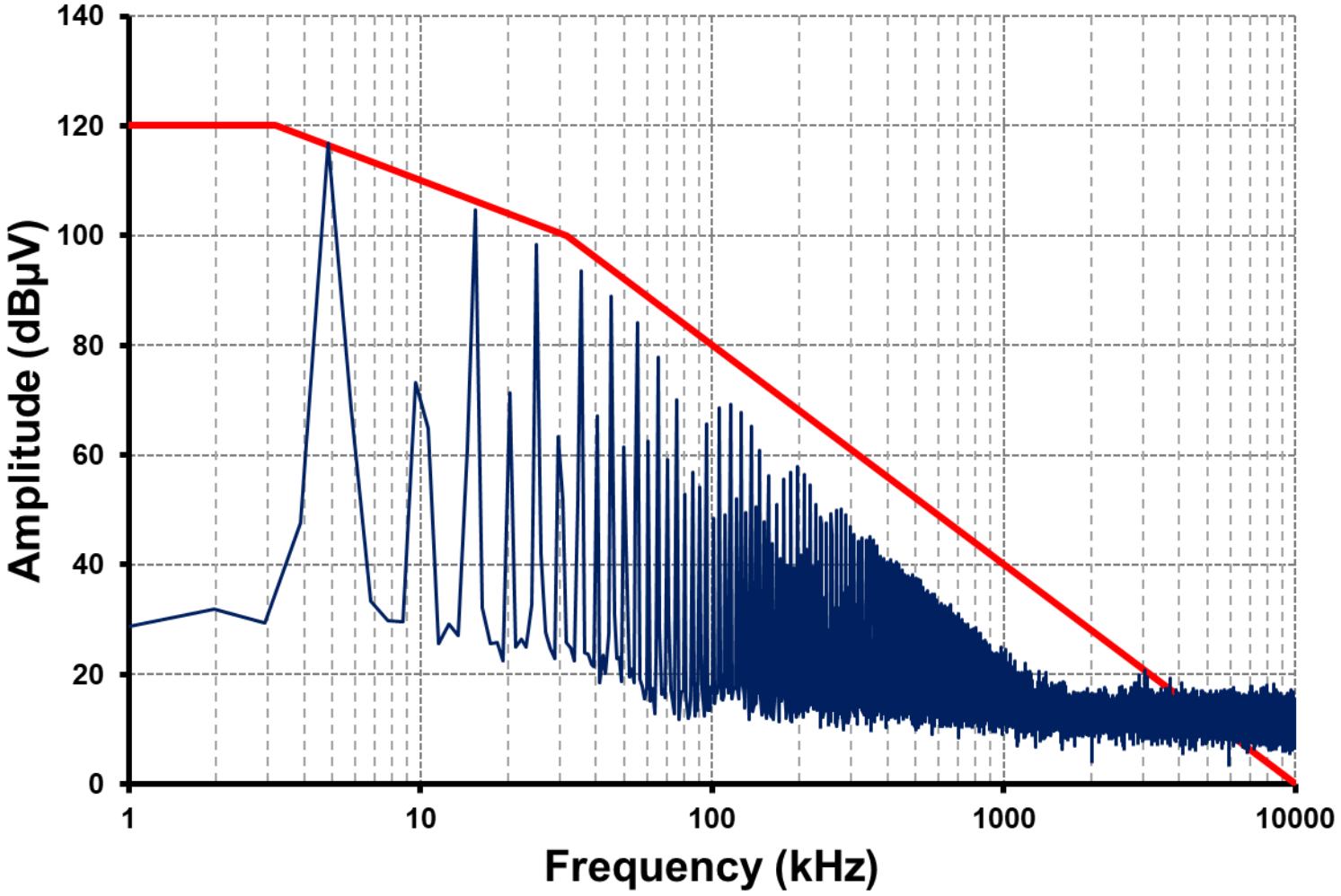
$\tau = 100\ \mu s\ (\tau/T = 50\%);\ \tau_r = \tau_f = 3.6\ \mu s$



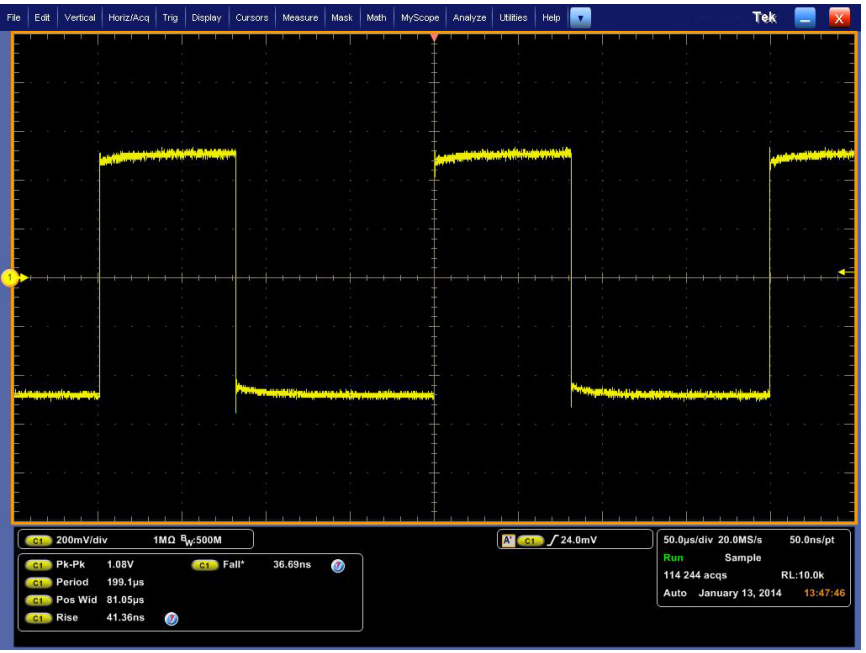


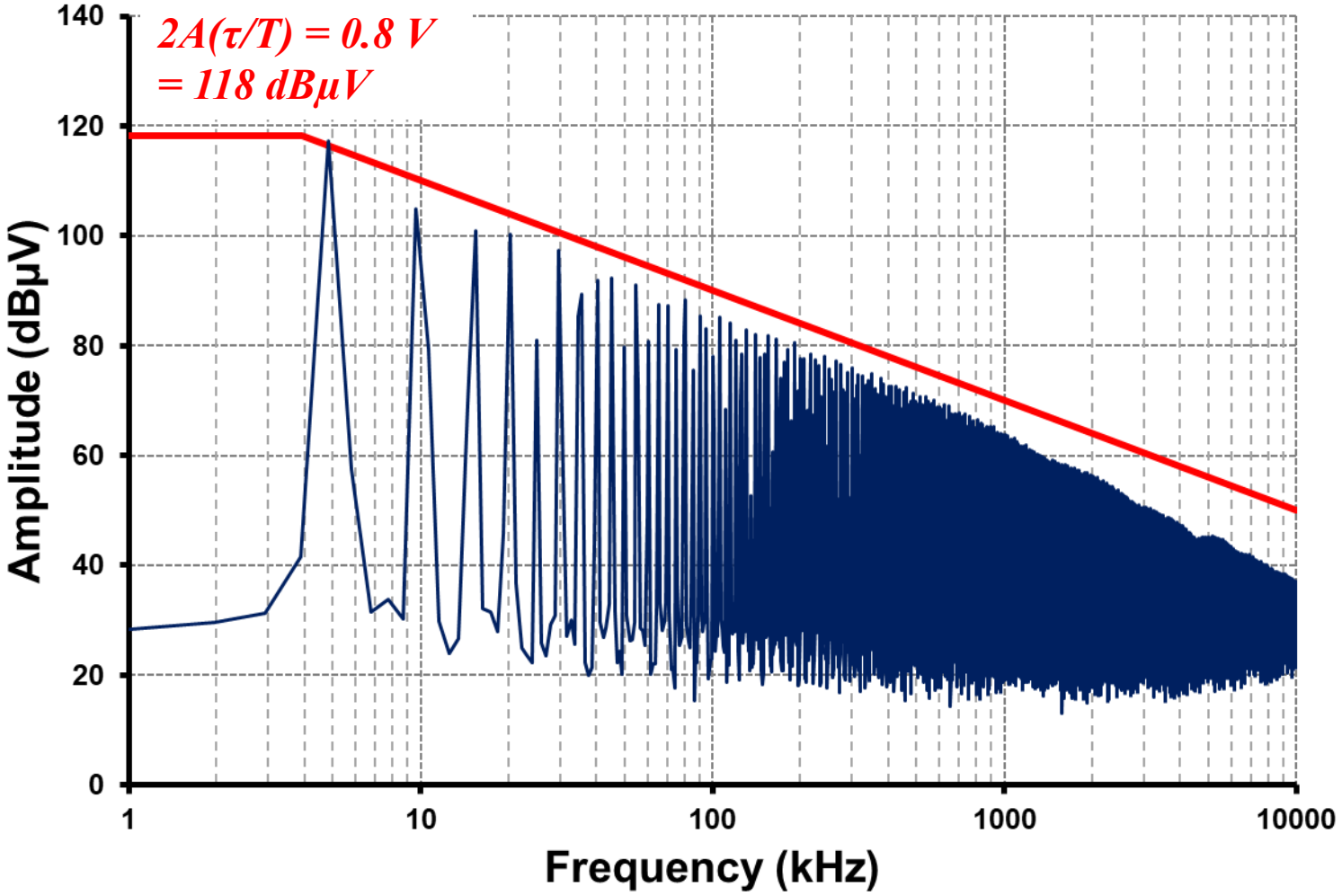
$\tau = 100\ \mu s\ (\tau/T = 50\%);\ \tau_r = \tau_f = 10\ \mu s$



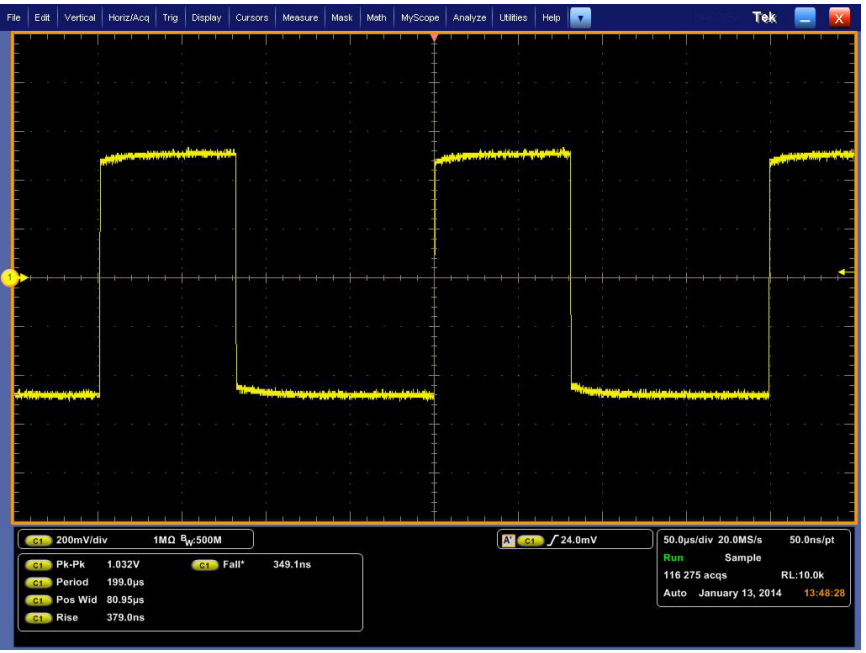


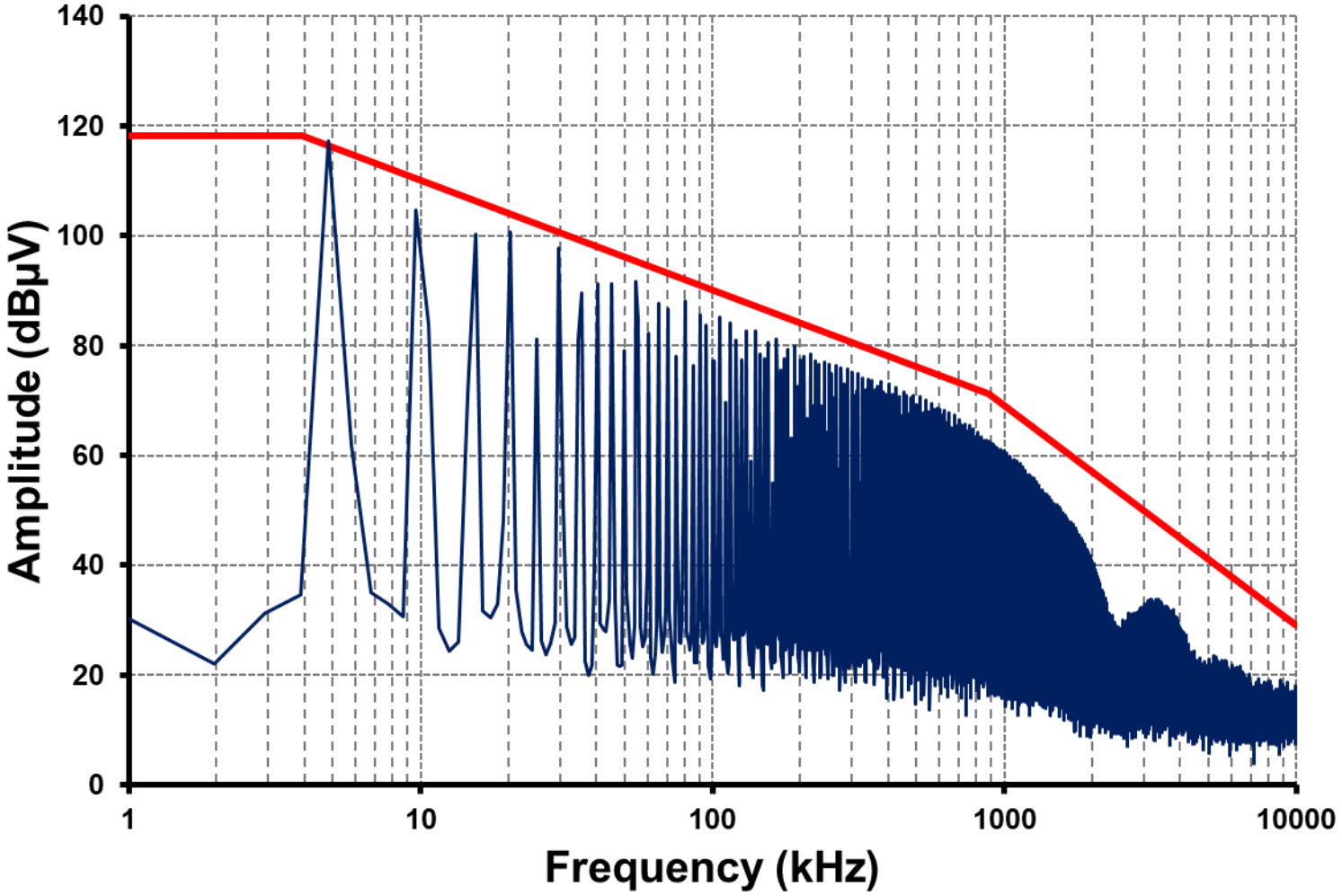
$\tau = 80\ \mu s\ (\tau/T = 40\%);\ \tau_r = \tau_f = 40\ ns$



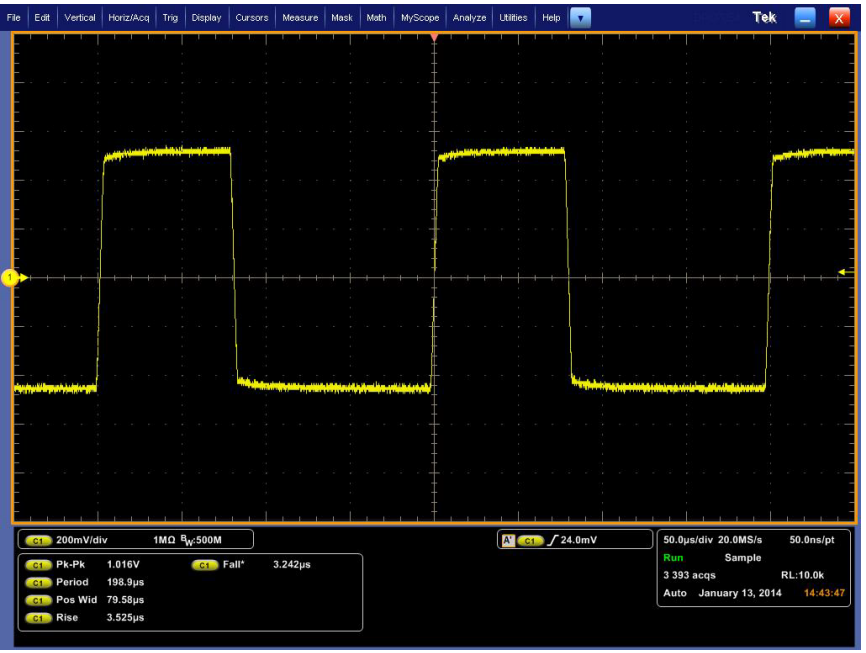


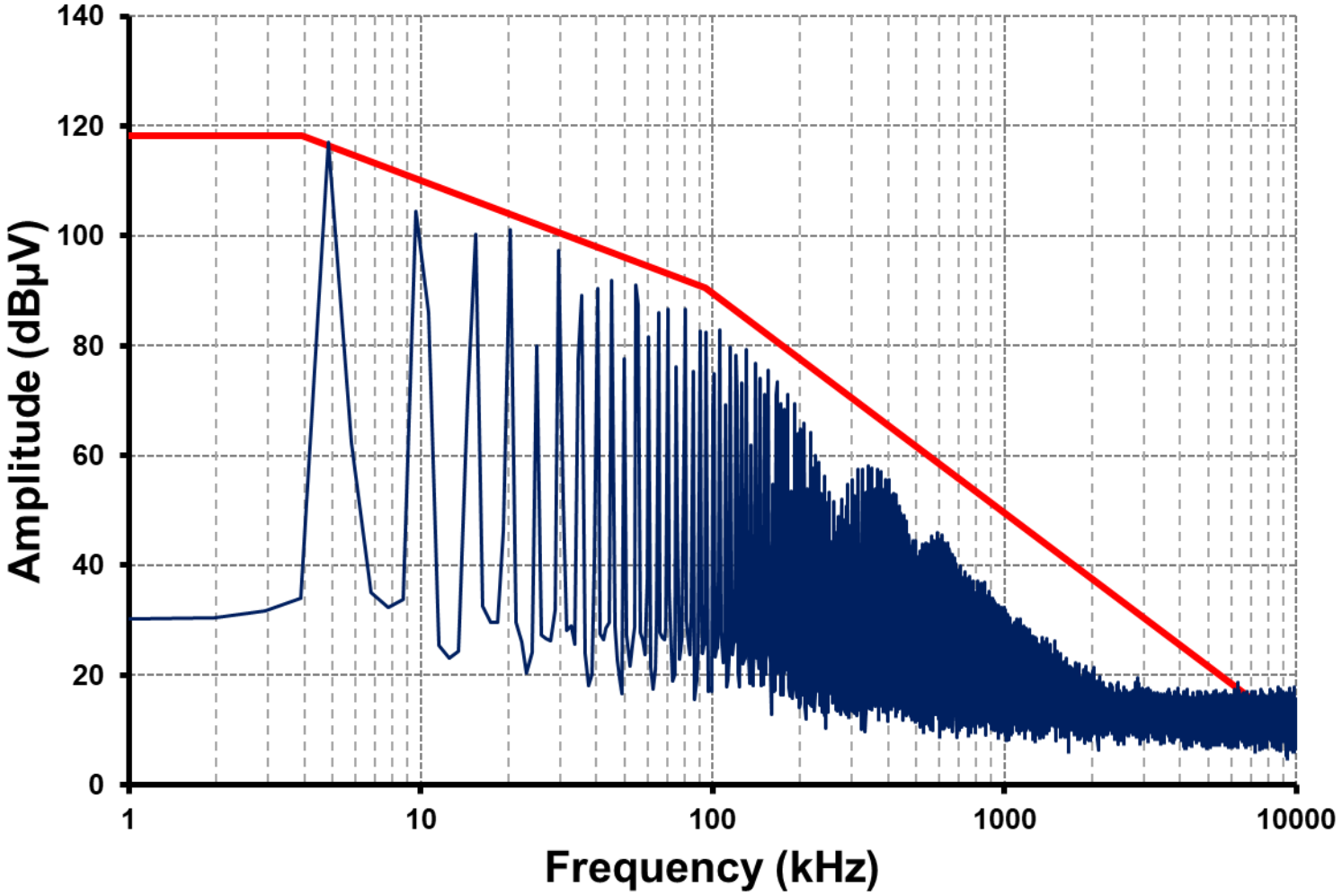
$\tau = 80\ \mu s\ (\tau/T = 40\%);\ \tau_r = \tau_f = 360\ ns$



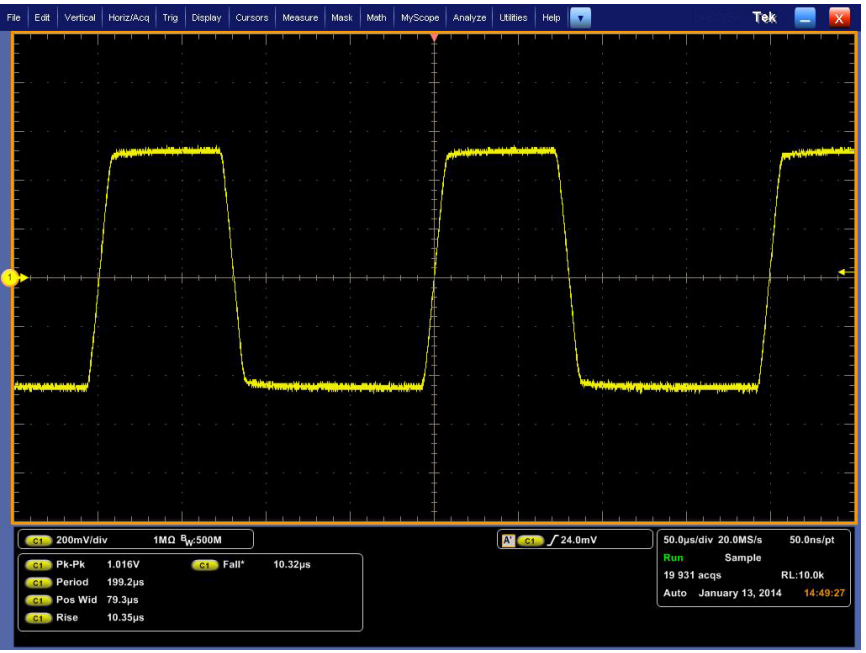


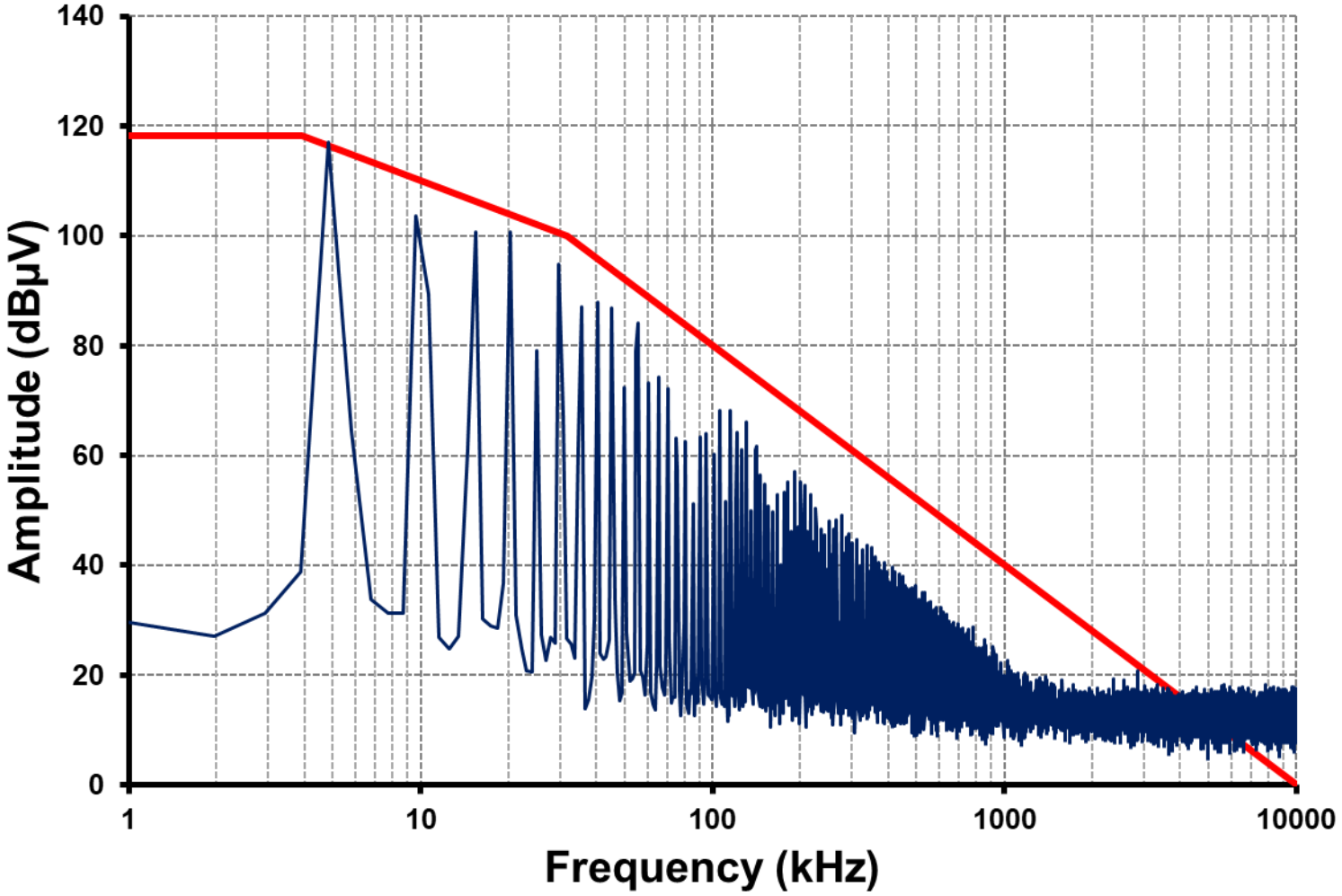
$\tau = 80\ \mu s\ (\tau/T = 40\%);\ \tau_r = \tau_f = 3.3\ \mu s$



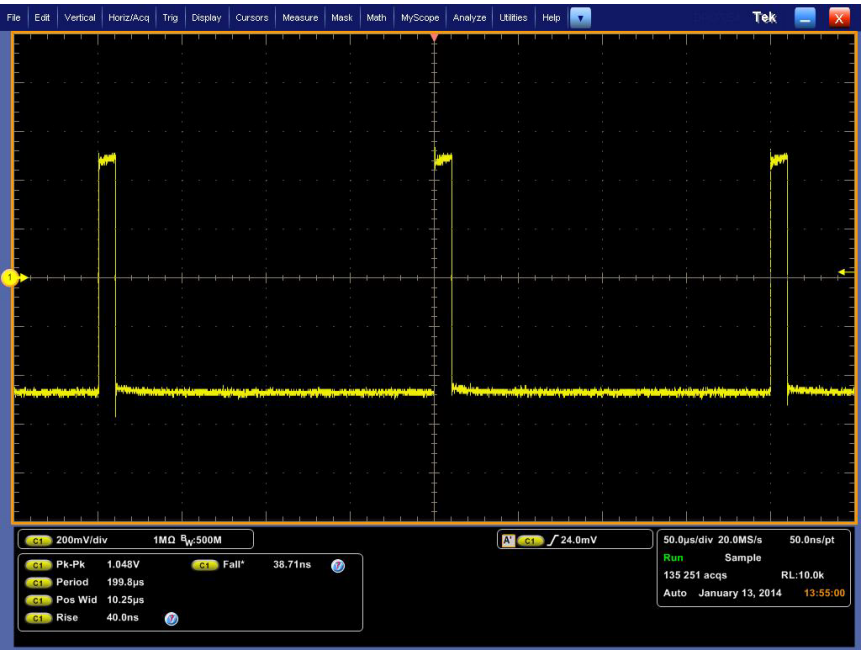


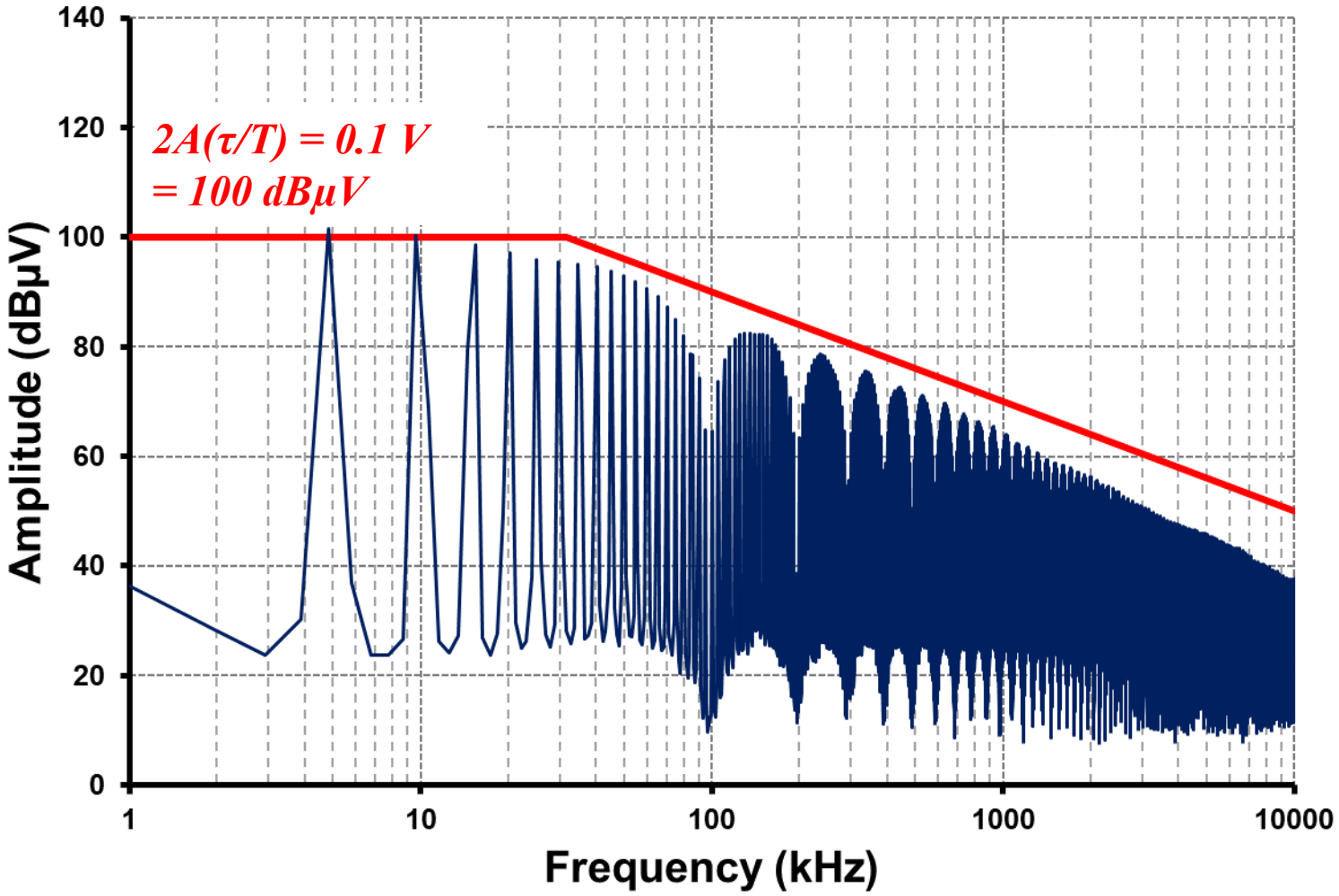
$\tau = 80\ \mu s\ (\tau/T = 40\%);\ \tau_r = \tau_f = 10\ \mu s$



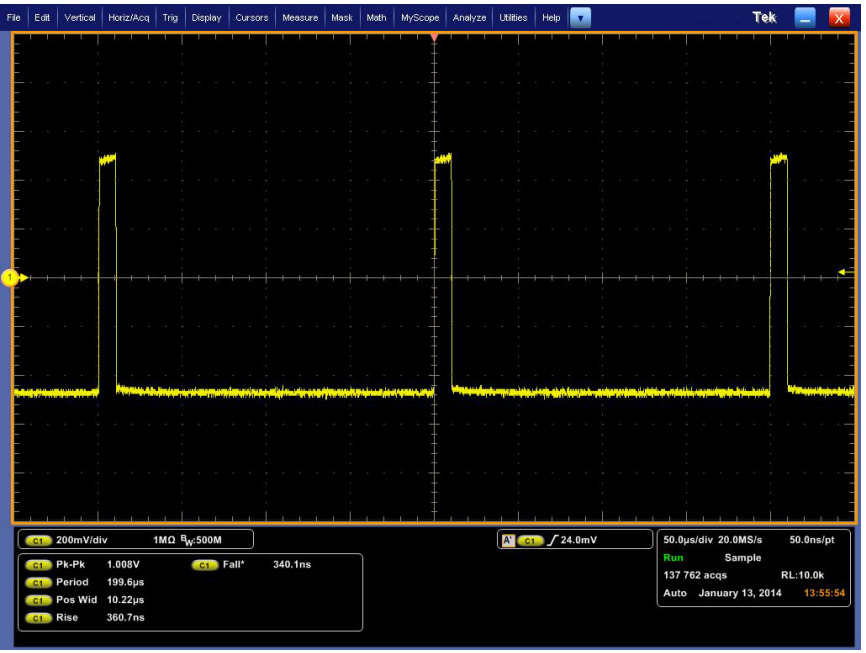


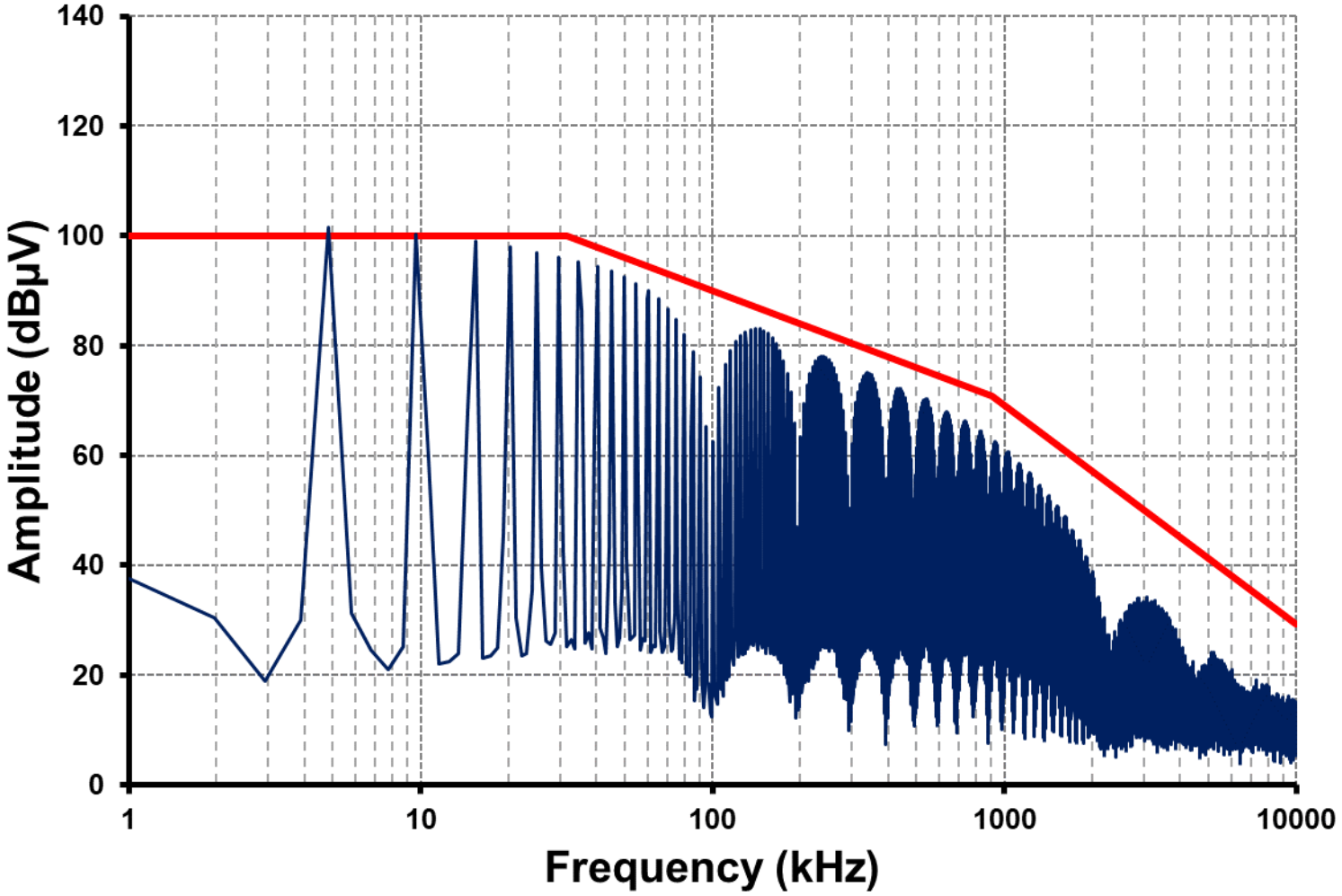
$\tau = 10\ \mu s\ (\tau/T = 5\%);\ \tau_r = \tau_f = 40\ ns$



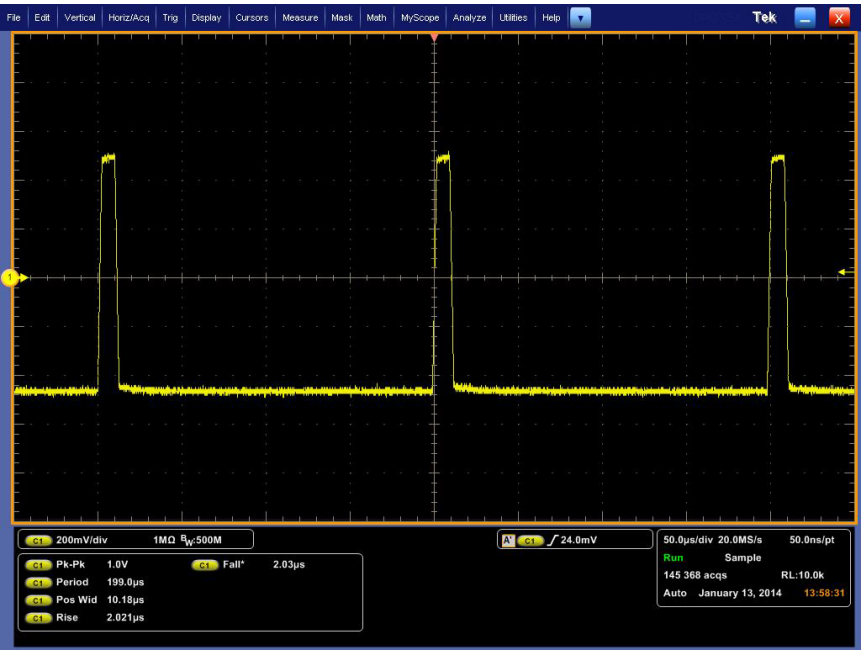


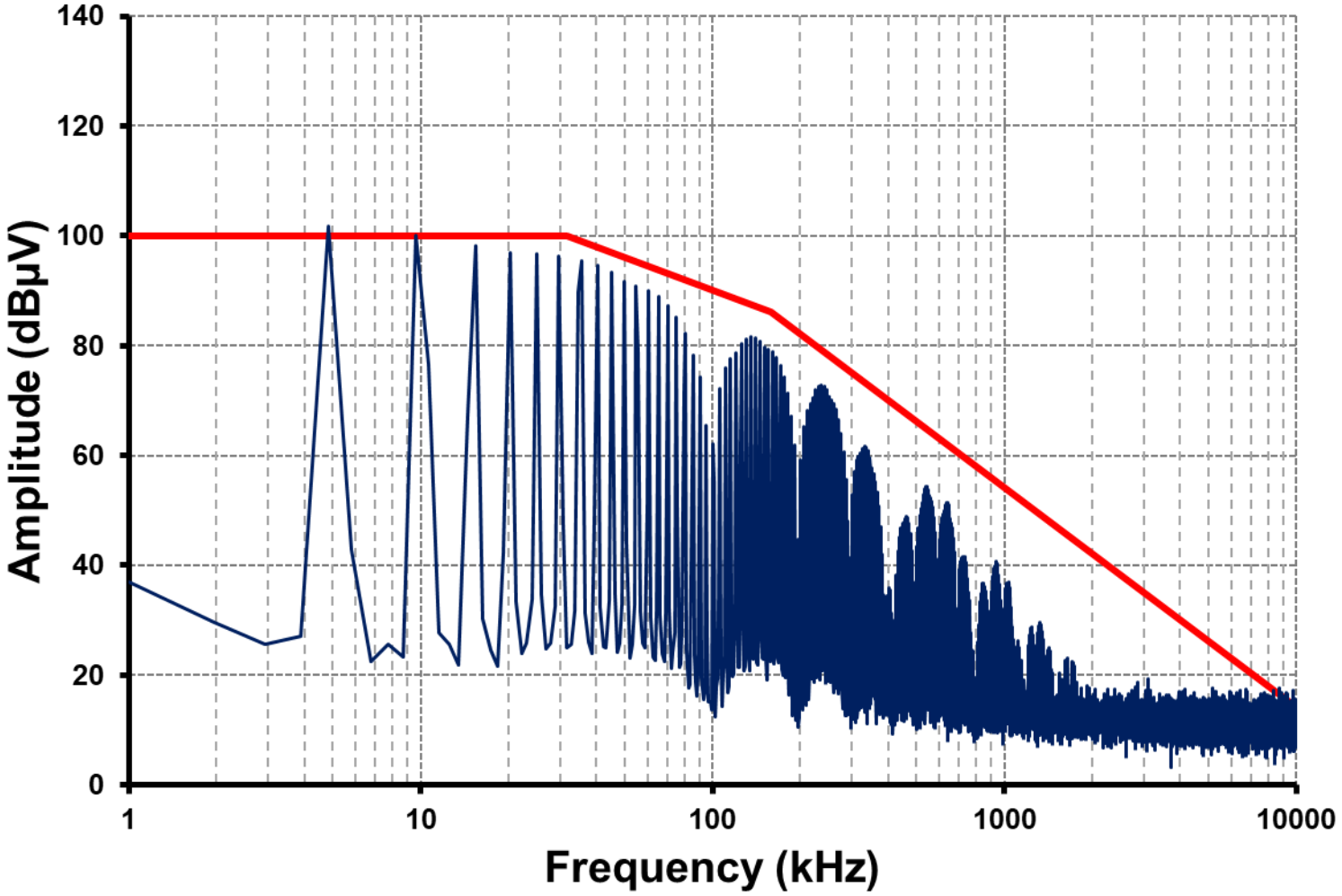
$\tau = 10\ \mu s\ (\tau/T = 5\%);\ \tau_r = \tau_f = 350\ ns$



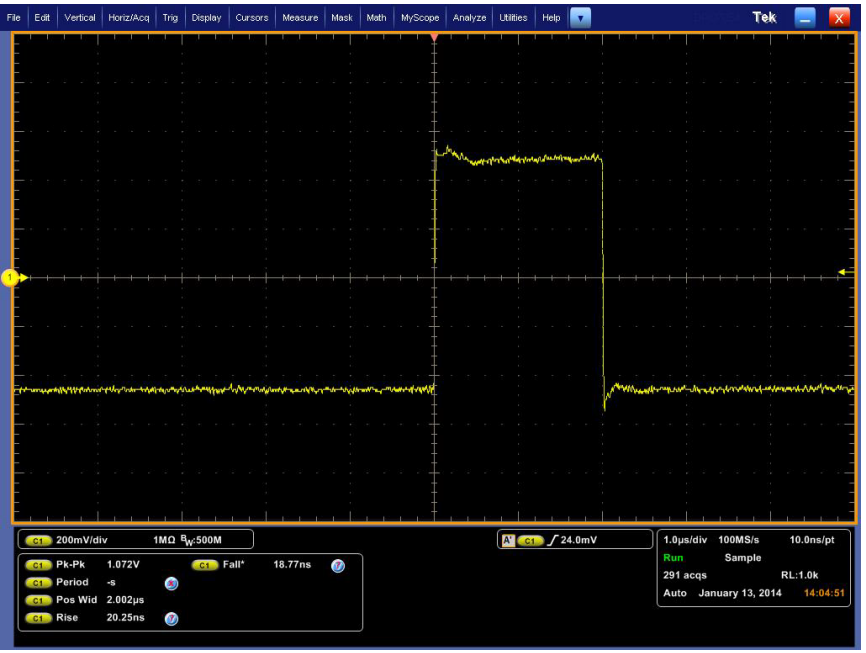


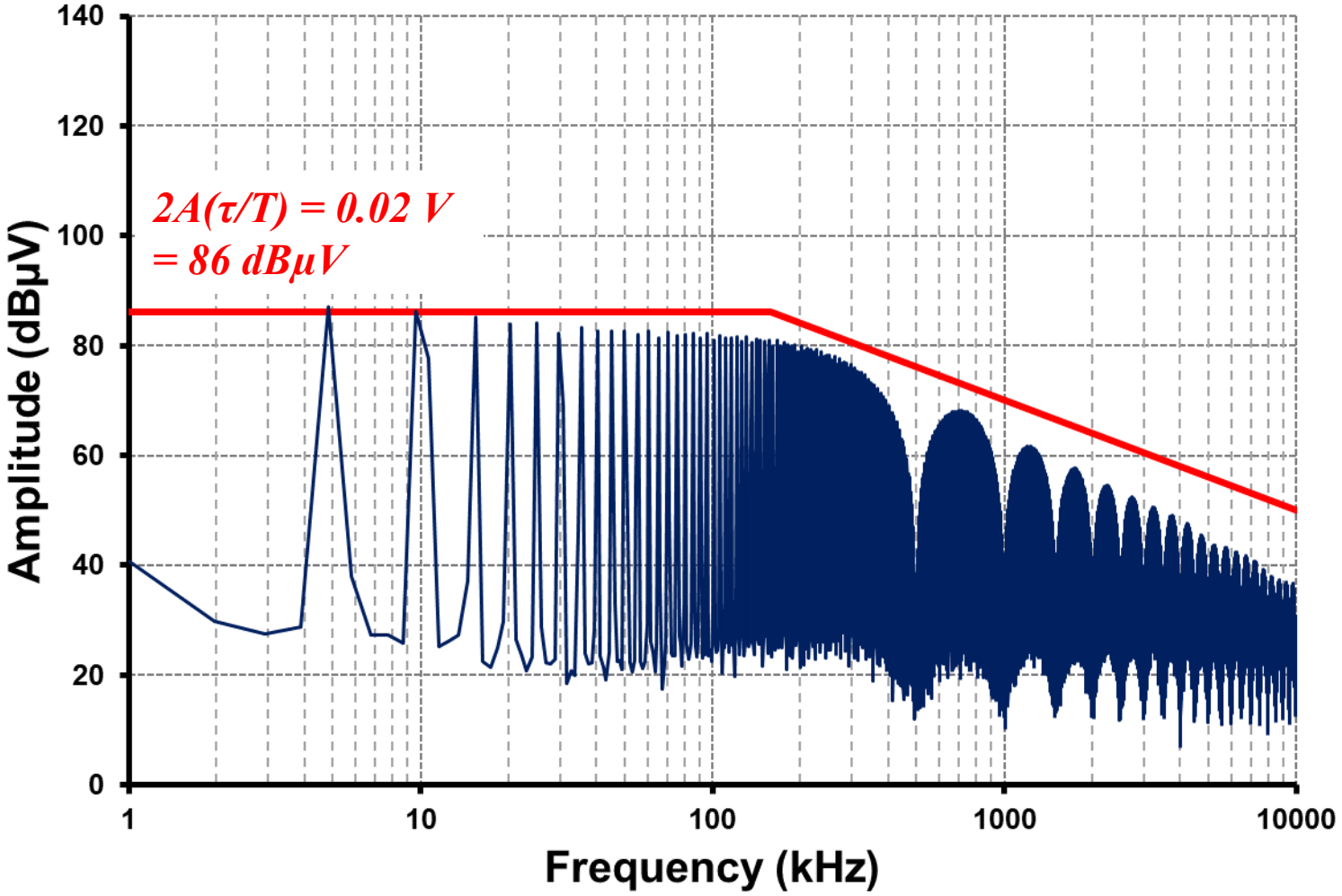
$\tau = 10\ \mu s\ (\tau/T = 5\%);\ \tau_r = \tau_f = 2\ \mu s$



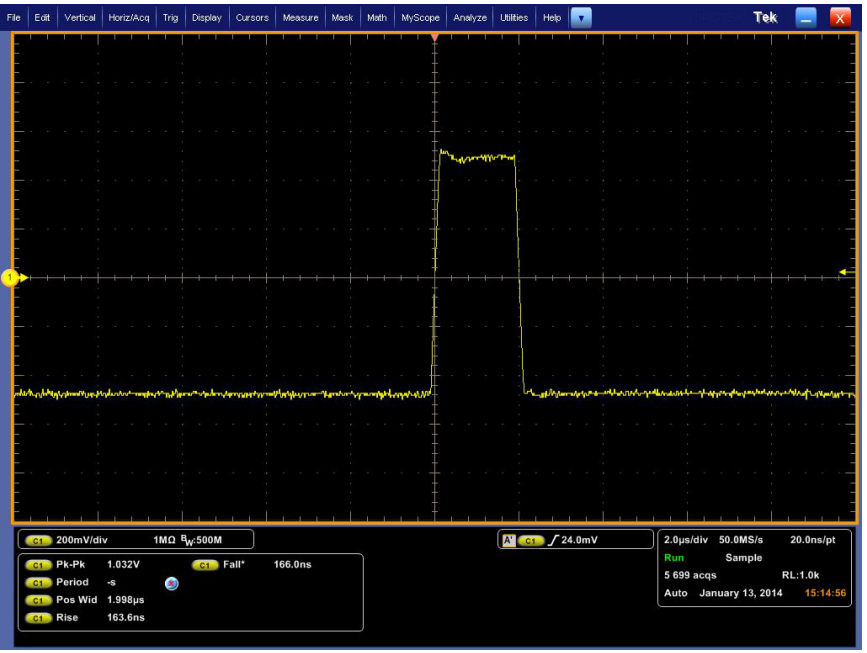


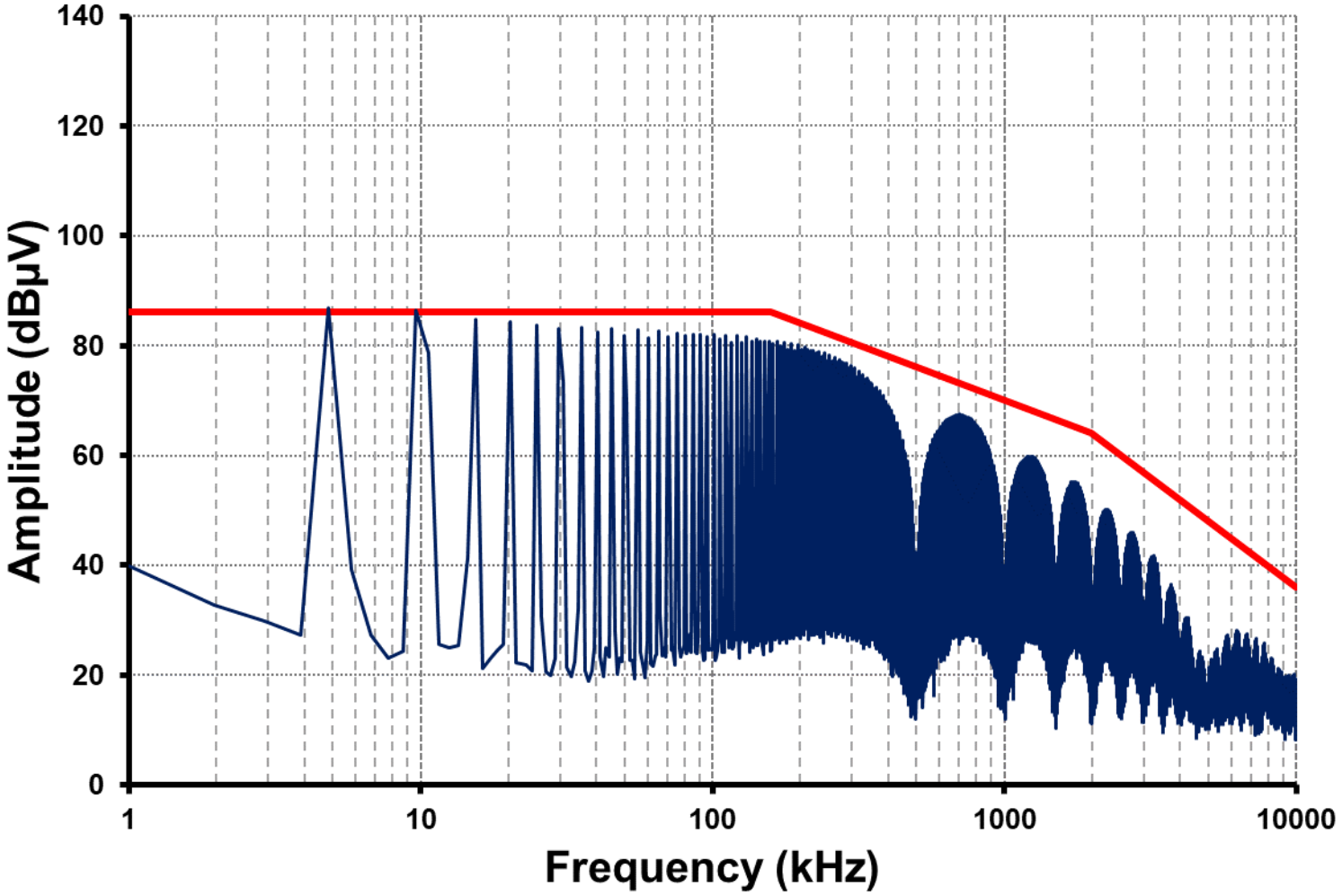
τ = 2 µs (τ/T = 1%); $\tau_r = \tau_f$ = 20 ns



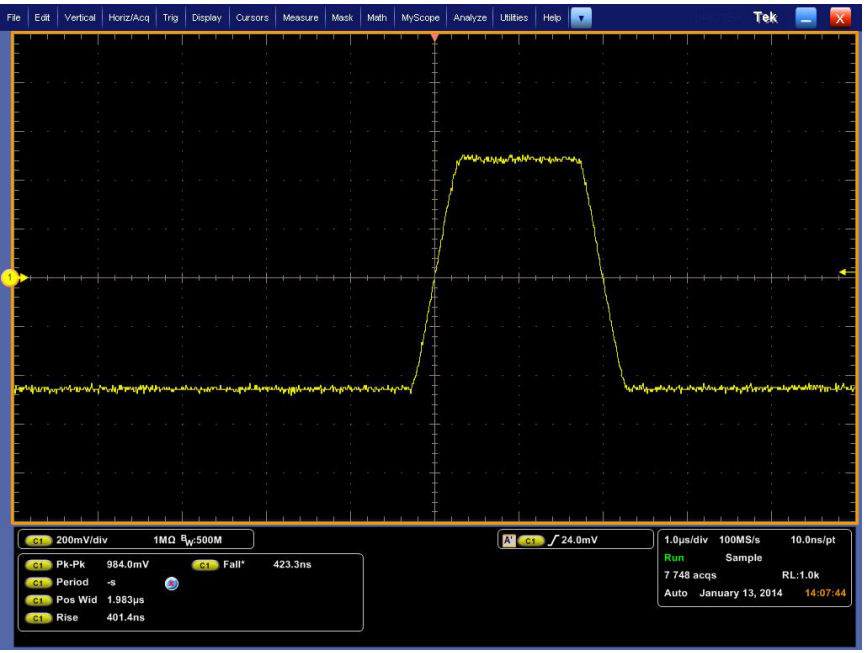


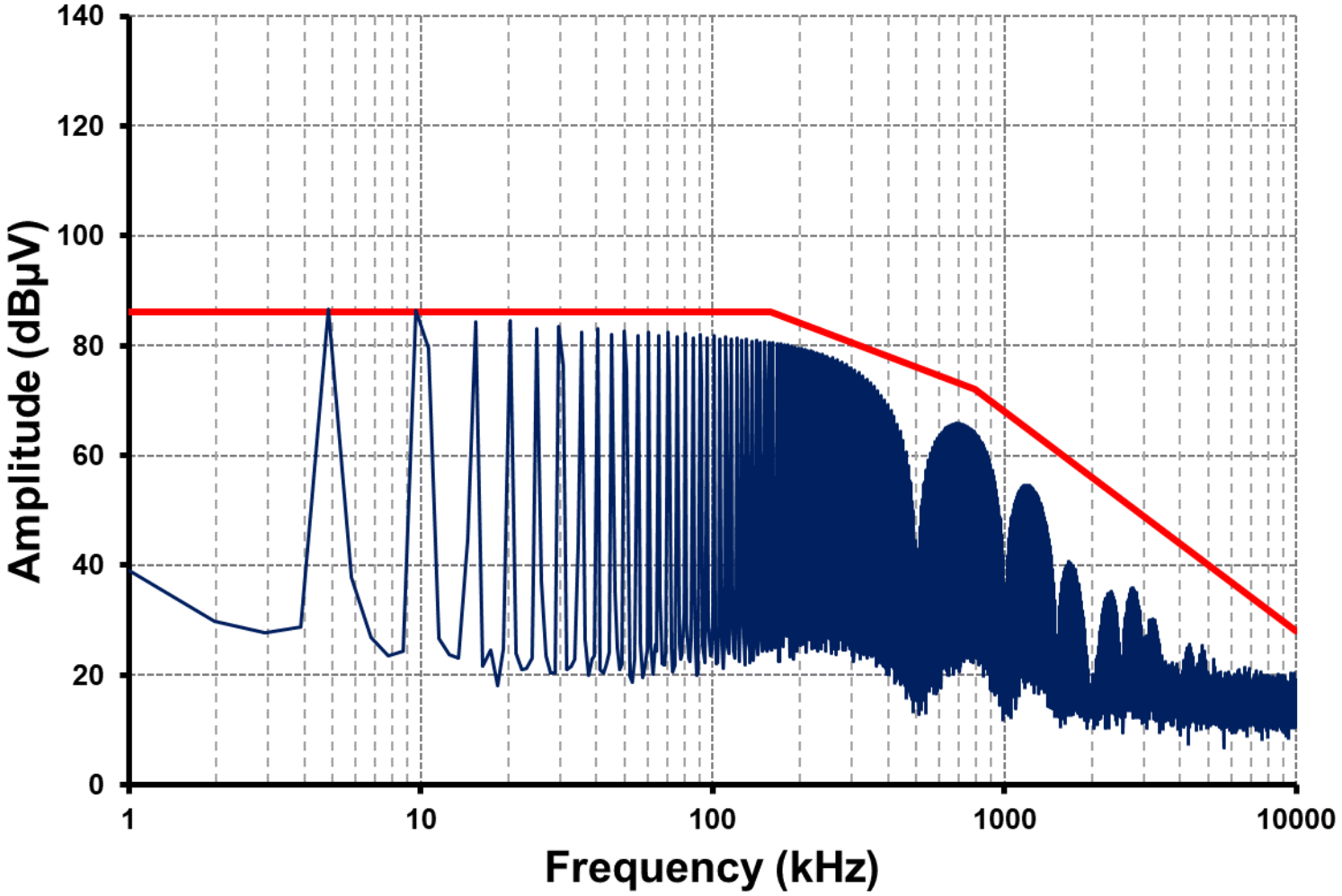
$\tau = 2\ \mu s\ (\tau/T = 1\%);\ \tau_r = \tau_f = 160\ ns$



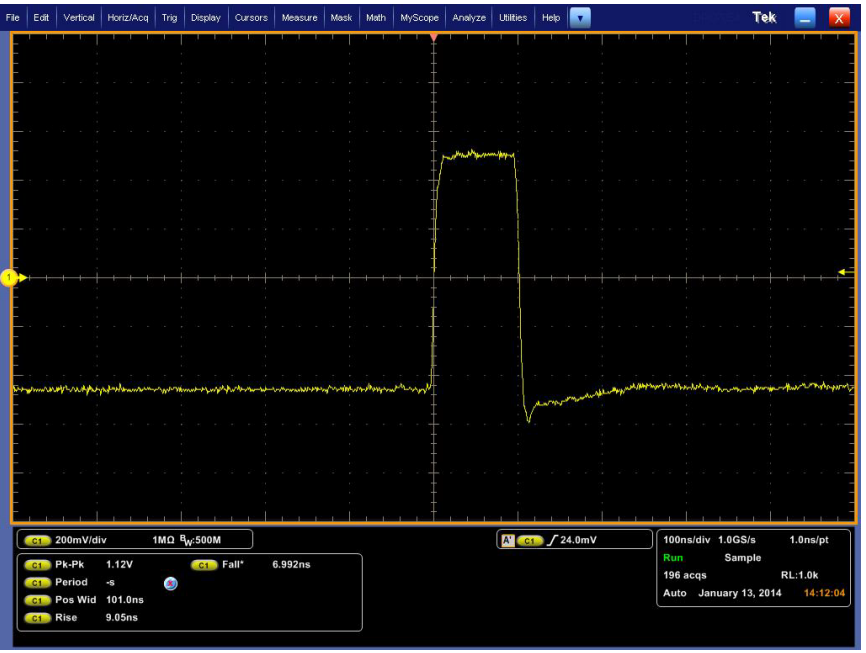


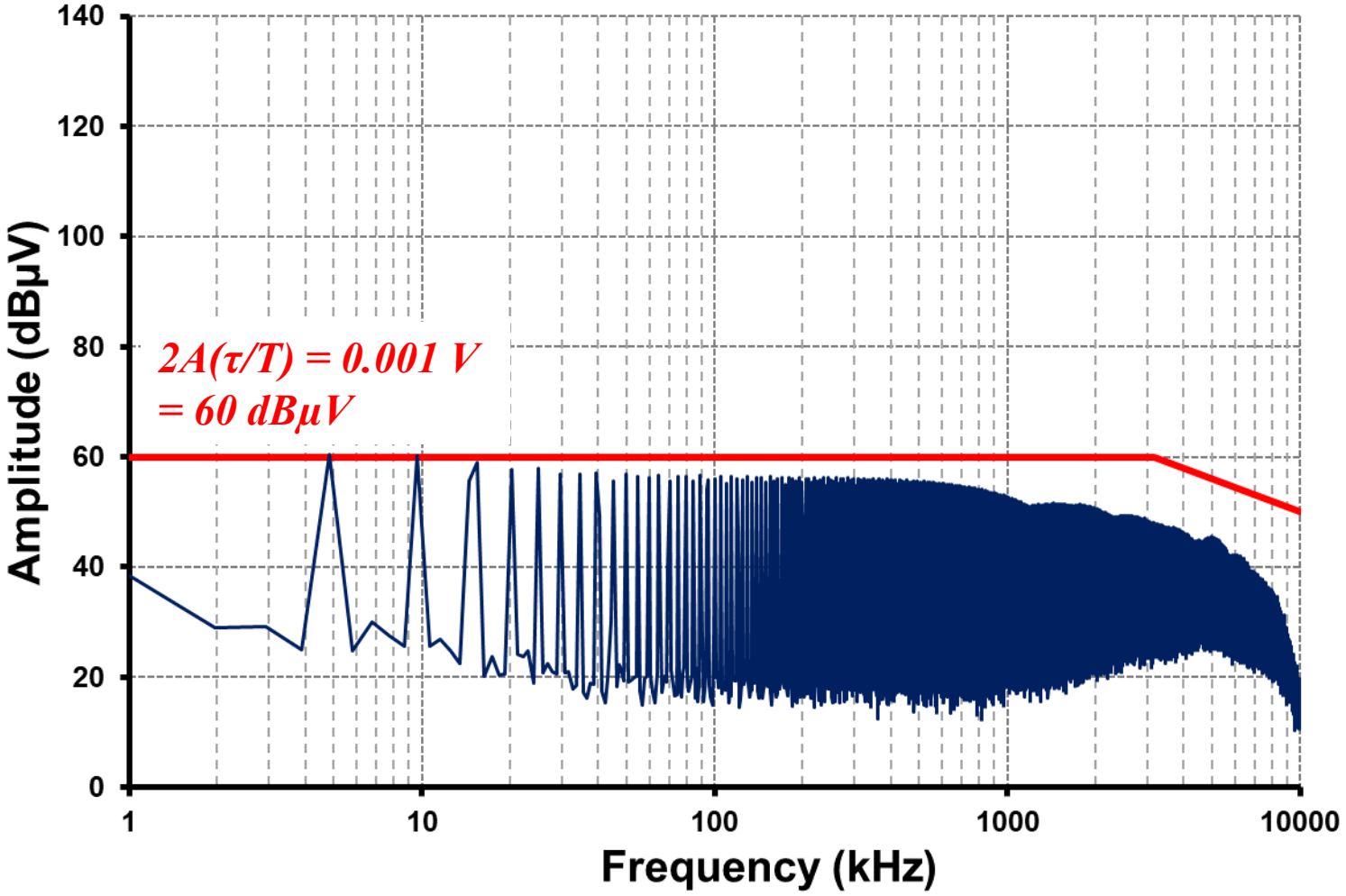
$\tau = 2\ \mu s\ (\tau/T = 1\%);\ \tau_r = \tau_f = 400\ ns$



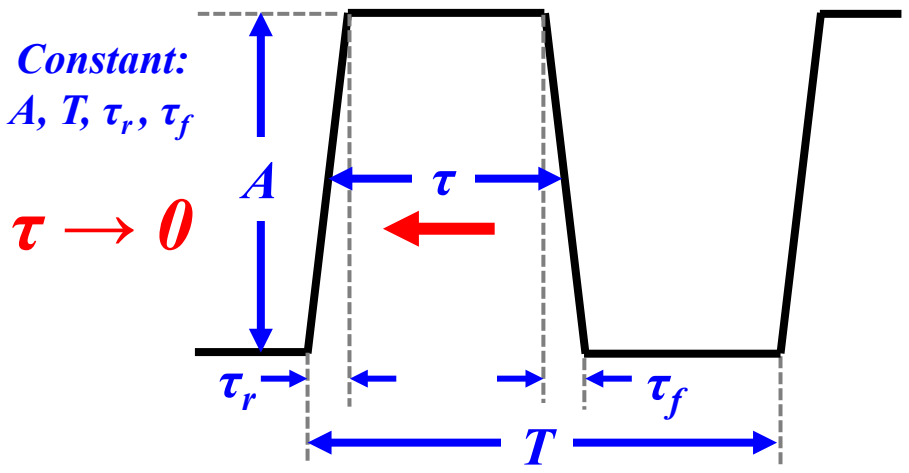


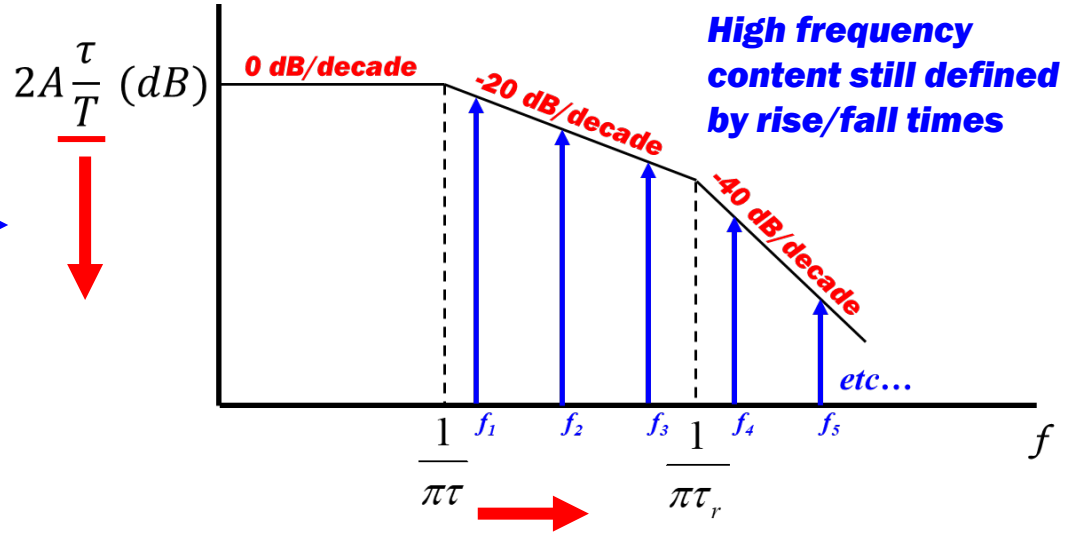
$\tau = 100$ ns ($\tau/T = 0.05\%$); $\tau_r = \tau_f = 8$ ns

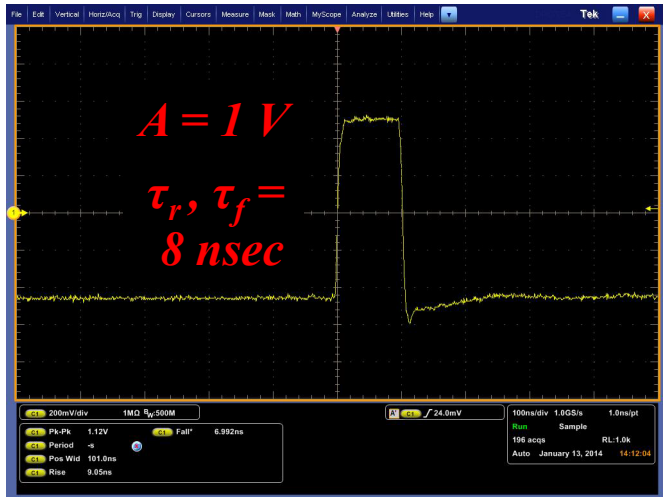




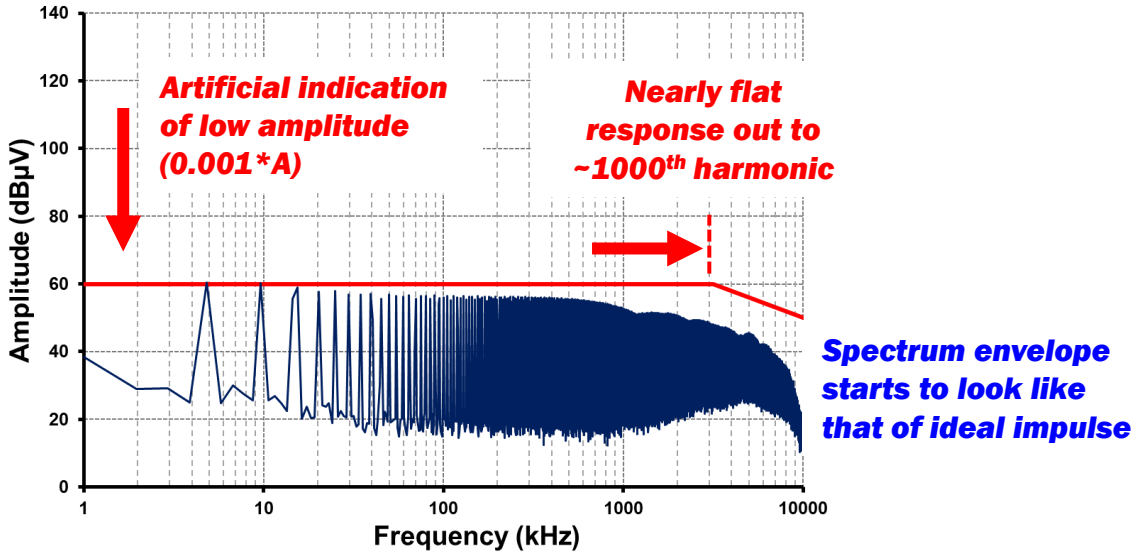
Low Duty Cycle





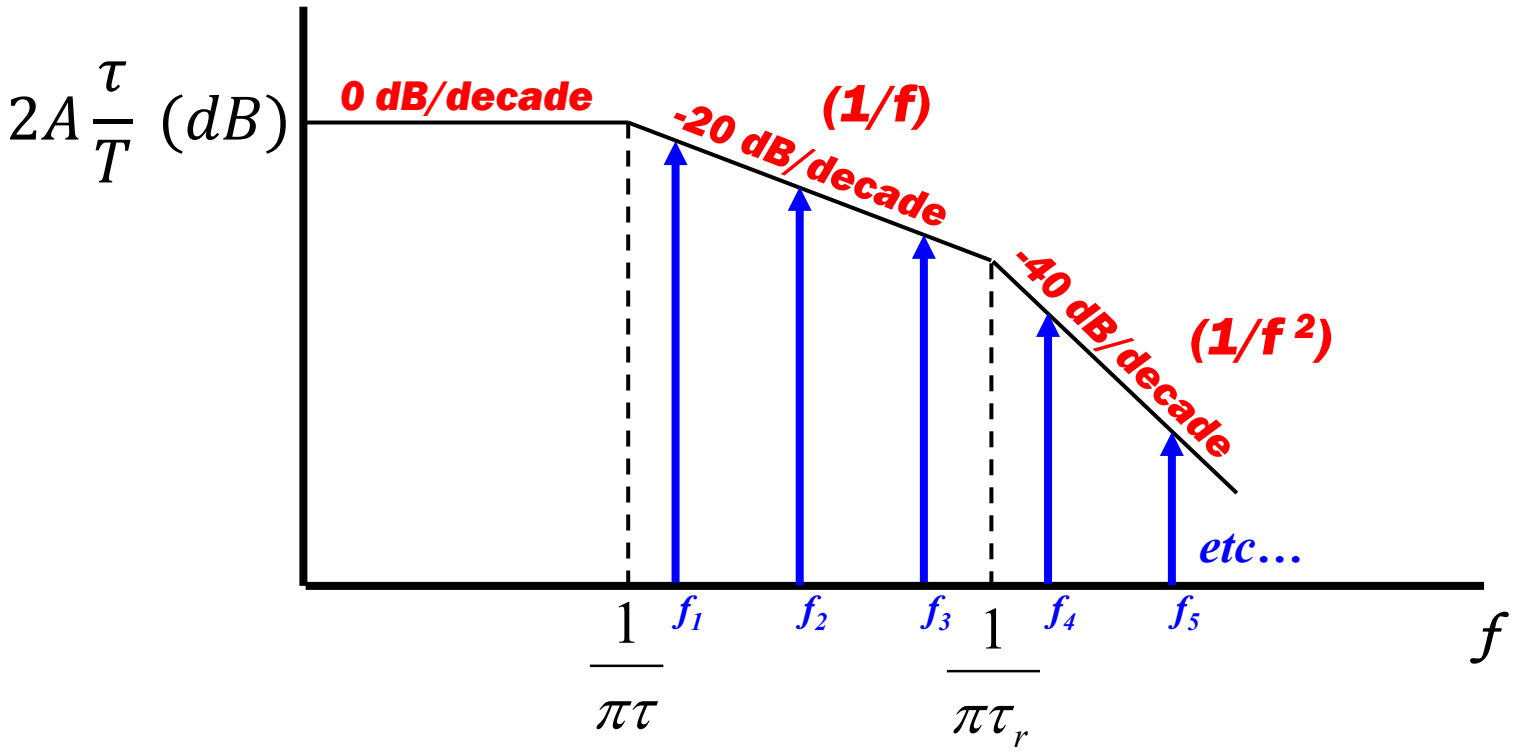


τ/T = 0.05%



Summary (1 of 3)

- "3-line" spectral envelope provides simple, accurate, and powerful analytical tool for correlating signal spectra to digital/clock type trapezoidal waveforms



Summary (2 of 3)

- **Time domain and frequency domain**
  - 2 sides of same coin
  - Both are useful, and it is essential to be comfortable translating from one to the other
  - Fourier transform and Fourier series are your friends
- **Even harmonics**
  - Reduced, but not zero, amplitude for “50%” duty cycle
  - Varying amplitude for other than 50% duty cycle
  - Must be considered as potentially significant contributors to spectrum
- **Low frequency plateau scales with duty cycle**
  - Spectrum for low duty cycle waveforms gives artificial indication of low amplitude
  - First “knee” frequency may be quite high, producing relatively flat spectrum for many harmonics
    - 100s or 1000s in these examples
  - Low duty cycle waveforms should be observed in time domain (oscilloscope) as well as frequency domain (spectrum analyzer)

Summary (3 of 3)

- **Signal spectra significantly more determined by rise/fall times than by fundamental frequency**
- **Uncontrolled rise/fall times can produce significant frequency content out to > 1000th harmonic**
- **Common cause of radiated emissions**
  - 5 MHz clock can easily produce harmonics out to 5 GHz and beyond
  - Potential interference to S-band receiver (~2 GHz), GPS (~1.5 GHz), etc.
- **Controlling rise/fall times can significantly limit high frequency content at source**

LIMIT THOSE RISE/FALL TIMES!!!



References

- Paul, Clayton: “Introduction to Electromagnetic Compatibility,” chapter 3
- U.S. Army Electronics Laboratories Interference Reduction Guide for Design Engineers, 1 August 1964