Introduction to Numerical Methods in Heat Transfer

Part 1

Introduction

Heat transfer is best understood through theory and application of principles in thermal analysis;

Modern thermal analysis leverages the power of computers and numerical methods to simulate heat transfer in networks representing a physical system;

This lesson is an introduction to numerical methods in heat transfer.

Overview

Students may have experience with numerical methods in college level courses;

Once in the workplace, however, they often use commercially available modeling tools;

Unless graduate study is pursued, students must create opportunities to understand the fundamentals behind these methods;

This course is intended to provide such focus.

Scope of this Lesson

What is heat transfer?

Governing differential equation;

Finite difference;

Finite element;

Thermal radiation in heat transfer analysis.

Lesson Roadmap

- What is Heat Transfer?
- Thermal Modeling Basics
- Fourier's Heat Conduction Law
- Governing Differential Equation for 1-D Heat Transfer
- Solution for 1-D Heat Transfer
- Example 1: Solving the Transient Heat Equation (1-D)
- Numerical Methods
  - Finite Difference
  - Example 2: Transient Thermal Analysis
  - Example 3: Adding Thermal Radiation
  - Implicit Formulation
  - Finite Element
  - Calculus of Variations
  - Establishing the Functional Equation
- Example 4: Steady State Analysis - Distributed Heating
- Example 5: Steady State Analysis
- Example 6: Transient Analysis

Part 1 Roadmap

What is Heat Transfer?

Thermal Modeling Basics

Fourier's Heat Conduction Law

Governing Differential Equation for 1-D Heat Transfer

Solution for 1-D Heat Transfer

Example 1: Solving the Transient Heat Equation (1-D)

Numerical Methods

Finite Difference

Example 2: Transient Thermal Analysis

Example 3: Adding Thermal Radiation

Implicit Formulation

Finite Element

Calculus of Variations

Establishing the Functional Equation

Example 4: Steady State Analysis - Distributed Heating

Example 5: Steady State Analysis

Example 6: Transient Analysis

What is Heat Transfer? (Ref. 1)

Heat transfer is energy transfer due to a temperature difference and can only be measured at the boundary of a system.

<u>**Conduction**</u> - Heat transfer from one substance to another by direct contact.

<u>**Convection**</u> - Heat transfer via movement of fluids.

<u>**Radiation**</u> – Heat transfer via electromagnetic waves.

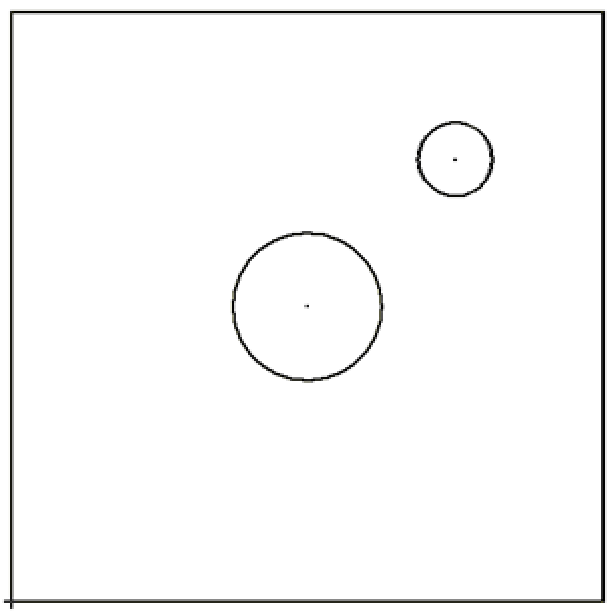
Thermal Modeling Basics

Engineers use thermal models as a tool to aid in design and understand the performance of thermal systems;

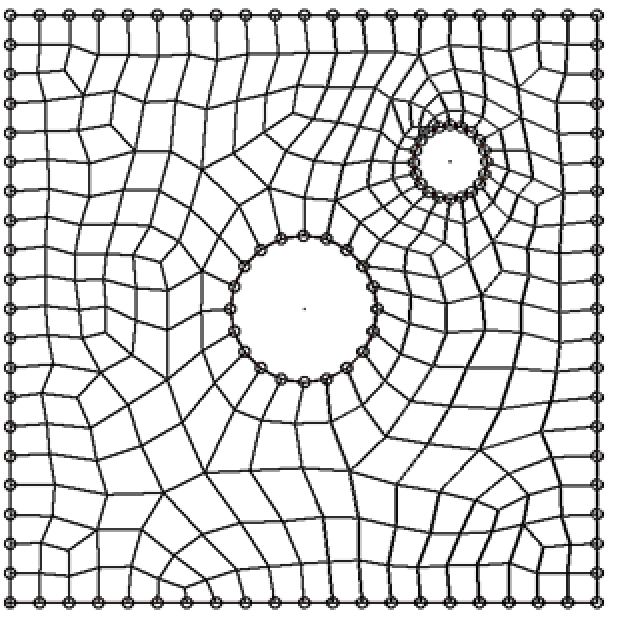
A thermal model is an abstraction of a physical system.

Thermal Modeling Basics

Closed-form solutions do not exist for all physical systems or geometries -- so they are discretized into smaller pieces, called elements.



Geometry



Geometry Discretized into a Finite Element Mesh

Note: Geometry and mesh created using MSC Patran®

Thermal Modeling Basics

Thermal systems can also be represented by representations analogous to electrical resistance-capacitance (RC) networks.

Geometry that is discretized is represented by nodes. A node is isothermal and has a constant temperature throughout the entire volume.

Heat flow between nodes is modeled with conductors (the inverse of thermal resistance) and heat storage is modeled using a capacitor.

Thermal Modeling Basics

Nodes come in the following varieties:

Diffusion -- a node representing a finite mass with finite capacitance;

Arithmetic -- a massless node representing zero capacitance;

Boundary -- a node representing a source or sink with infinite capacitance.

Thermal Modeling Basics

Conductors come in the following varieties:

Conduction -- heat transfer between solid objects (or a solid and a gas/fluid);

Convection -- heat transfer between a solid object and a convecting liquid or gas;

Radiation -- heat transfer via electromagnetic radiation between objects.

Thermal Modeling Basics

Heat transfer via conduction and convection is proportional to ΔT:

$$\dot{Q}_{cond} = G_{cond}\Delta T = \frac{kA}{L}\Delta T$$

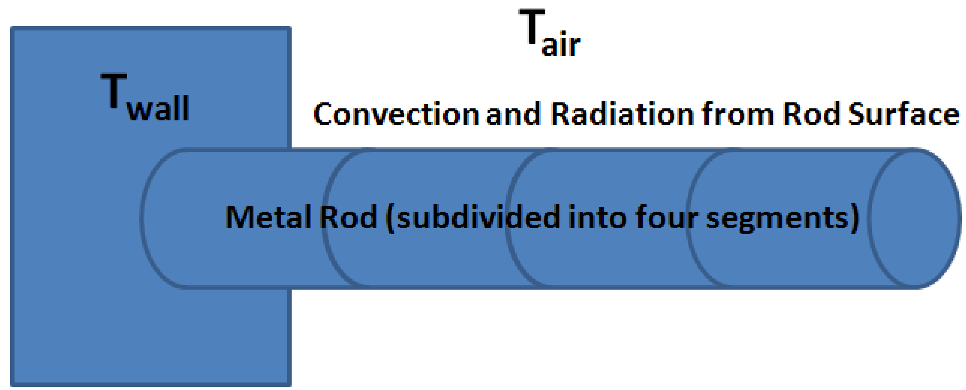
$$\dot{Q}_{conv} = G_{conv}\Delta T = hA\Delta T$$

But, heat transfer via radiation takes this form:

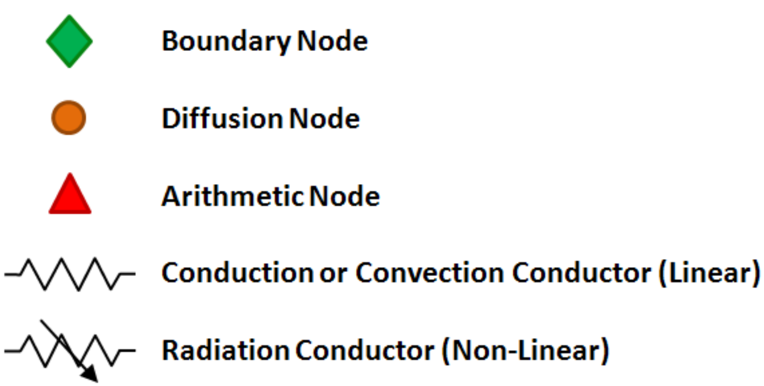
$$\dot{Q}_{rad} = G_{rad}\Delta(T^4)$$

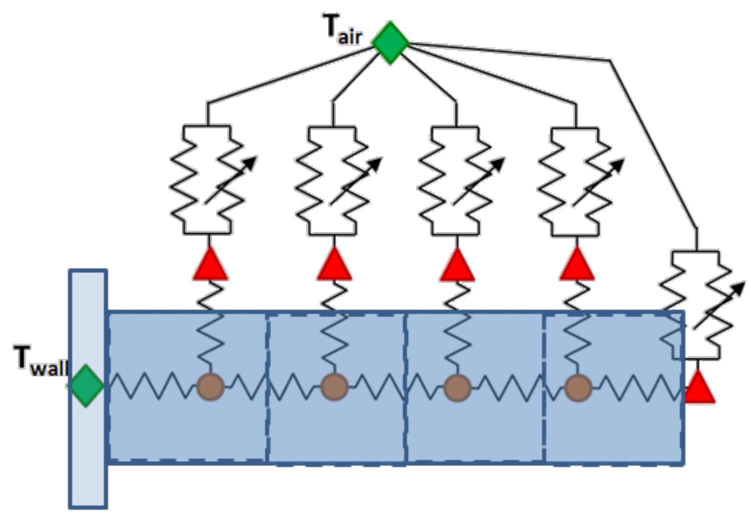
Thermal Modeling Basics

Consider this geometry



The network representation for this configuration might look something like this:





Some Terminology

Heat rate, denoted by $\dot{Q}$, is **[energy/time]**;

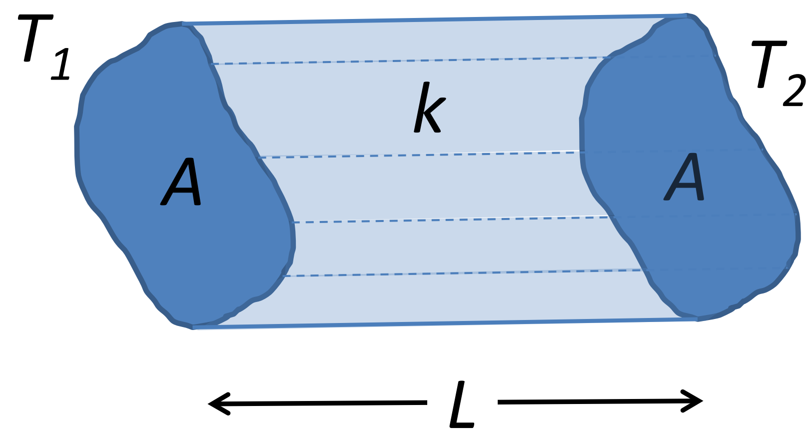
Heat flux, denoted by $\dot{q}$, is **[energy/area/time]**;

Volumetric heat rate, denoted by $\dot{q}_{gen}$, is **[energy/volume/time]**;

Heat per unit length, denoted by $\dot{q}_L$, is **[energy/length/time]**.

Fourier Heat Conduction Law

Fourier *assumed* that conduction heat transfer is directly proportional to the temperature difference across a solid:



$$\dot{Q} \propto \Delta T$$

$$\dot{Q} = \frac{kA}{L}\Delta T = \frac{kA}{L}(T_2 - T_1)$$

Conservation of Energy

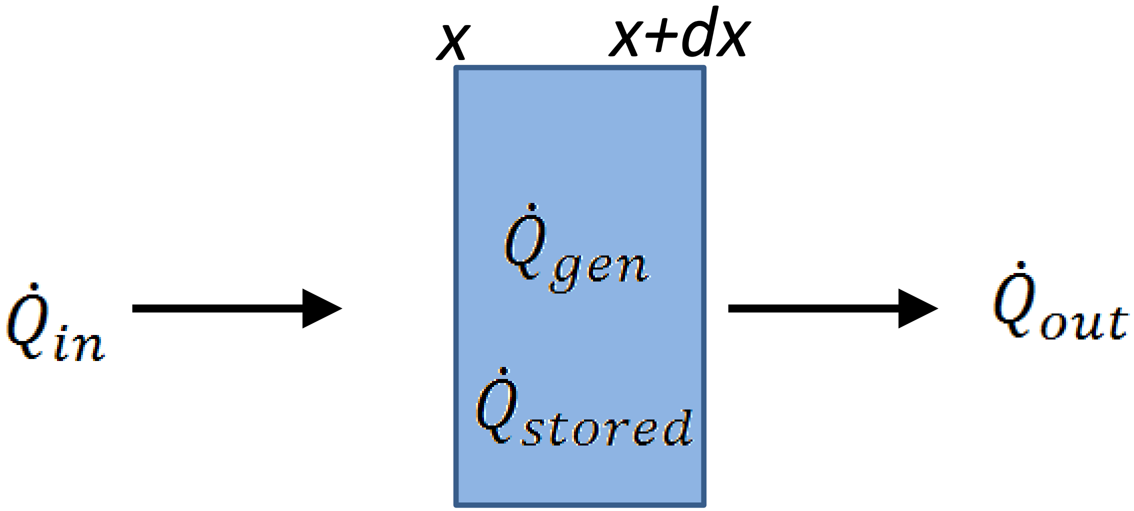
In heat transfer calculations, energy must be conserved:

$$\dot{Q}_{in} + \dot{Q}_{gen} - \dot{Q}_{out} = \dot{Q}_{stored}$$

where...

$\dot{Q}_{in}$	is energy per unit time entering;
$\dot{Q}_{gen}$	is energy per unit time generated;
$\dot{Q}_{out}$	is energy per unit time leaving;
$\dot{Q}_{stored}$	is time rate of change of energy stored.

Deriving the Heat Equation in One Dimension (Ref. 2)



$$\dot{Q}_{in} + \dot{Q}_{gen} - \dot{Q}_{out} = \dot{Q}_{stored}$$

Deriving the Heat Equation in One Dimension (Ref. 2)

The energy per unit time stored in a mass can be expressed as:

$$\dot{Q}_{stored} = mC_p \frac{\partial T}{\partial t} = \rho V C_p \frac{\partial T}{\partial t} = \rho A dx C_p \frac{\partial T}{\partial t}$$

where...

- m is the mass **[mass]**;
- C_p is the specific heat **[energy/mass/temperature]**;
- ρ is the material density **[mass/volume]**;
- V is the material volume **[volume]**;
- A is the differential element cross sectional area **[area]**;
- dx is the differential element length **[length]**;
- $\frac{\partial T}{\partial t}$ is the mass' time rate change of temperature **[temperature/time]**.

Deriving the Heat Equation in One Dimension (Ref. 2)

The heat entering the mass per unit time at location *x* is:

$$\dot{Q}_{in} = -kA \left.\frac{\partial T}{\partial x}\right]_x$$

where...

k is the thermal conductivity **[energy/time/length/temperature]**;

A is the differential element cross sectional area **[area]**;

$\frac{\partial T}{\partial x}$ is temperature gradient at *x* **[temperature/length]**.

Deriving the Heat Equation in One Dimension (Ref. 2)

The heat leaving the mass per unit time at location *x+Δx* is:

$$\dot{Q}_{out} = -kA \left.\frac{\partial T}{\partial x}\right]_{x+dx}$$

where...

- k is the thermal conductivity **[energy/time/length/temperature]**;
- A is the differential element cross sectional area **[area]**;
- $\frac{\partial T}{\partial x}$ is temperature gradient at *x+dx* **[temperature/length]**.

Deriving the Heat Equation in One Dimension (Ref. 2)

But we recognize that the heat transfer at *x+dx* may be expressed as:

$$\dot{Q}_{out} = -kA\frac{\partial T}{\partial x}\Big]_{x+dx} = -\left[kA\frac{\partial T}{\partial x} + \frac{\partial}{\partial x}\left(kA\frac{\partial T}{\partial x}\right)dx\right]$$

where, the first term is simply the heat entering the left face per unit time and the second term is the change in heat transfer rate with respect to *x* over the control volume times the distance to the other end of the control volume.

Deriving the Heat Equation in One Dimension (Ref. 2)

We can now substitute our new expressions for the original terms:

$$\dot{Q}_{in} + \dot{Q}_{gen} - \dot{Q}_{out} = \dot{Q}_{stored}$$

$$-kA\frac{\partial T}{\partial x}\bigg]_x + \dot{q}_{gen}Adx + \left[kA\frac{\partial T}{\partial x} + \frac{\partial}{\partial x}\left(kA\frac{\partial T}{\partial x}\right)dx\right] = \rho A dx C_p\frac{\partial T}{\partial t}$$

This reduces to:

$$\dot{q}_{gen}Adx + \frac{\partial}{\partial x}\left(kA\frac{\partial T}{\partial x}\right)dx = \rho A dx C_p\frac{\partial T}{\partial t}$$

Deriving the Heat Equation in One Dimension (Ref. 2)

Rearranging and simplifying yields:

$$\dot{q}_{gen} + \frac{\partial}{\partial x}\left(k\,\frac{\partial T}{\partial x}\right) = \rho C_p \frac{\partial T}{\partial t}$$

Or, for a constant thermal conductivity:

$$\dot{q}_{gen} + k\,\frac{\partial^2 T}{\partial x^2} = \rho C_p \frac{\partial T}{\partial t}$$

This is known as the ***heat equation in one dimension*** expressed per unit volume.

Extension to 2-D and 3-D

More generally, the heat equation for isotropic thermal conductivity can be expanded into additional dimensions:

$$\dot{q}_{gen} + k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) = \rho C_p \frac{\partial T}{\partial t}$$

or, using the Laplacian:

$$\dot{q}_{gen} + k\nabla^2 T = \rho C_p \frac{\partial T}{\partial t}$$

One Dimensional Heat Equation:
Steady State

Assuming the heat generation term is constant, the only time dependency appears on the right hand side of the equation:

$$\dot{q}_{gen} + k\frac{\partial^2 T}{\partial x^2} = \rho C_p \frac{\partial T}{\partial t} \rightarrow 0$$

At steady state, the time rate of change of temperature, $\partial T/\partial t = 0$;

We see that steady state behavior is *independent of density,* ρ*, and specific heat,* C_p.

One Dimensional Heat Equation: Transient

For a case with no internal heat generation, we have:

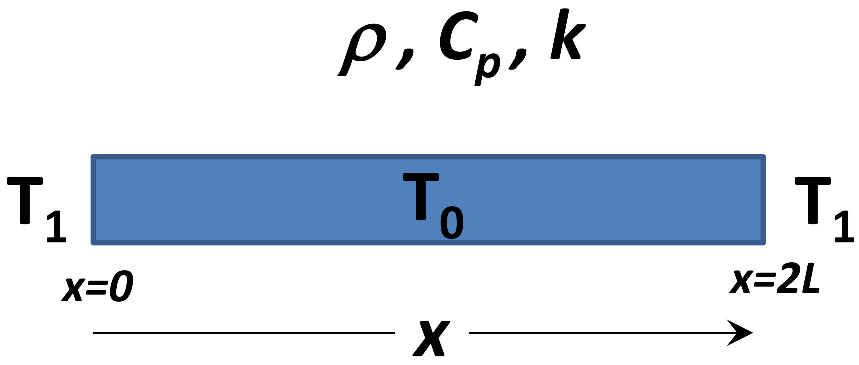
$$\dot{q}_{gen}^{\,0} + k\frac{\partial^2 T}{\partial x^2} = \rho C_p \frac{\partial T}{\partial t}$$

We recognize this to be a linear differential equation of second order in the spatial dimension, *x* and first order in time, *t*.

Fortunately, there exists a solution.

Example 1: Solving the Transient Heat Equation (Adapted from Ref. 2)

Consider the following example:



The rod is initially at a uniform temperature, T_0. At time, $t = 0$, the temperature at both ends ($x=0$ and $x=2L$) is raised to T_1;

How does the temperature profile in the rod change with time?

Example 1: Solving the Transient Heat Equation (Adapted from Ref. 2)

The governing differential equation is the previously derived heat equation:

$$k\frac{\partial^2 T}{\partial x^2} = \rho C_p \frac{\partial T}{\partial t}$$

We recast the equation noting that $\alpha = k/\rho C_p$ and $\theta = T - T_1$:

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{\alpha}\frac{\partial \theta}{\partial t}$$

Example 1: Solving the Transient Heat Equation (Adapted from Ref. 2)

Using this new form, we examine our boundary conditions for $\theta(x,t)$:

For $0 \leq x \leq 2L$ and $t = 0$: $\theta(x,0) = T_0 - T_1$

For $x = 0$ and $t > 0$: $\theta(0,t) = 0$

For $x = 2L$ and $t > 0$: $\theta(2L,t) = 0$

Example 1: Solving the Transient Heat Equation (Adapted from Ref. 2)

To solve this differential equation, we can use separation of variables by assuming:

$$\theta(x,t) = X(x)Y(t)$$

where...

X(x) is a function of *only x*, and;
Y(t) is a function of *only t*.

Example 1: Solving the Transient Heat Equation (Adapted from Ref. 2)

Forming the required derivatives of $\theta(x, t)$:

$$\frac{\partial^2 \theta(x,t)}{\partial x^2} = \frac{\partial^2 X(x)}{\partial x^2} Y(t)$$

$$\frac{\partial \theta(x,t)}{\partial t} = X(x) \frac{\partial Y(t)}{\partial t}$$

So our transformed differential equation becomes:

$$\frac{\partial^2 X(x)}{\partial x^2} Y(t) = \frac{1}{\alpha} X(x) \frac{\partial Y(t)}{\partial t}$$

Example 1: Solving the Transient Heat Equation (Adapted from Ref. 2)

We can rearrange the previous equation to read:

$$\left(\frac{\frac{\partial^2 X(x)}{\partial x^2}}{X(x)}\right) = \frac{1}{\alpha}\left(\frac{\frac{\partial Y(t)}{\partial t}}{Y(t)}\right)$$

To facilitate solution, we can set both equations equal to a constant, where λ is called the *separation constant*:

$$\left(\frac{\frac{\partial^2 X(x)}{\partial x^2}}{X(x)}\right) = \frac{1}{\alpha}\left(\frac{\frac{\partial Y(t)}{\partial t}}{Y(t)}\right) = -\lambda^2$$

Example 1: Solving the Transient Heat Equation (Adapted from Ref. 2)

We can now write two, independent linear differential equations:

$$\frac{\partial^2 X(x)}{\partial x^2} + \lambda^2 X(x) = 0$$

$$\frac{\partial Y(t)}{\partial t} + \lambda^2 \alpha Y(t) = 0$$

Example 1: Solving the Transient Heat Equation (Adapted from Ref. 2)

We recognize that the first equation:

$$\frac{\partial^2 X(x)}{\partial x^2} + \lambda^2 X(x) = 0$$

has a solution of the form:

$$X(x) = c_1 cos(\lambda x) + c_2 sin(\lambda x)$$

Example 1: Solving the Transient Heat Equation (Adapted from Ref. 2)

The second equation:

$$\frac{\partial Y(t)}{\partial t} + \lambda^2 \alpha Y(t) = 0$$

has a solution of the form:

$$Y(t) = c_3 e^{-\lambda^2 \alpha t}$$

Example 1: Solving the Transient Heat Equation (Adapted from Ref. 2)

Our expression for $\theta(x, t)$, then, becomes:

$$\theta(x,t) = X(x)Y(t)$$
$$= [c_1 cos(\lambda x) + c_2 sin(\lambda x)]c_3 e^{-\lambda^2 \alpha t}$$

We can simplify this expression by letting:

$$C_1 = c_1 c_3$$
$$C_2 = c_2 c_3$$

Example 1: Solving the Transient Heat Equation (Adapted from Ref. 2)

The expression becomes:

$$\theta(x,t) = X(x)Y(t)$$
$$= [C_1 cos(\lambda x) + C_2 sin(\lambda x)]e^{-\lambda^2 \alpha t}$$

The previously defined boundary conditions are now used to solve for constants, C_1 and C_2:

For $0 \leq x \leq 2L$ and $t = 0$: $\theta(x,0) = T_0 - T_1$
For $x = 0$ and $t > 0$: $\theta(0,t) = 0$
For $x = 2L$ and $t > 0$: $\theta(2L,t) = 0$

Example 1: Solving the Transient Heat Equation (Adapted from Ref. 2)

When we apply the second boundary condition (i.e., for *x = 0* and *t > 0)*:

$$\theta(0,t) = [C_1 cos(\lambda 0) + C_2 sin(\lambda 0)]e^{-\lambda^2 \alpha t} = 0$$

From this, we see that C_1 must be zero.

$$\theta(x,t) = C_2 e^{-\lambda^2 \alpha t} sin(\lambda x)$$

We note that C_2 cannot be zero -- if it was, $\theta(x, t) = 0$ everywhere.

Example 1: Solving the Transient Heat Equation (Adapted from Ref. 2)

When we apply the third boundary condition (i.e., $x = 2L$ and $t > 0$):

$$\theta(2L, t) = C_2 e^{-\lambda^2 \alpha t} sin(\lambda 2L) = 0$$

Since $C_2 \neq 0$, we see that the only way for this to happen is when:

$$sin(\lambda 2L) = 0$$

Example 1: Solving the Transient Heat Equation (Adapted from Ref. 2)

This happens when:

$$\lambda = \frac{n\pi}{2L}$$

So the solution may be expressed in the form of a series given by:

$$\theta(L,t) = \sum_{n=1}^{\infty} C_n e^{-\left(\frac{n\pi}{2L}\right)^2 \alpha t} sin\left(\frac{n\pi x}{2L}\right)$$

$$C_n = \frac{1}{L}\int_0^{2L} (T_0 - T_1)\, sin\left(\frac{n\pi x}{2L}\right) dx \qquad n = 1,3,5,\ldots$$

Example 1: Solving the Transient Heat Equation (Adapted from Ref. 2)

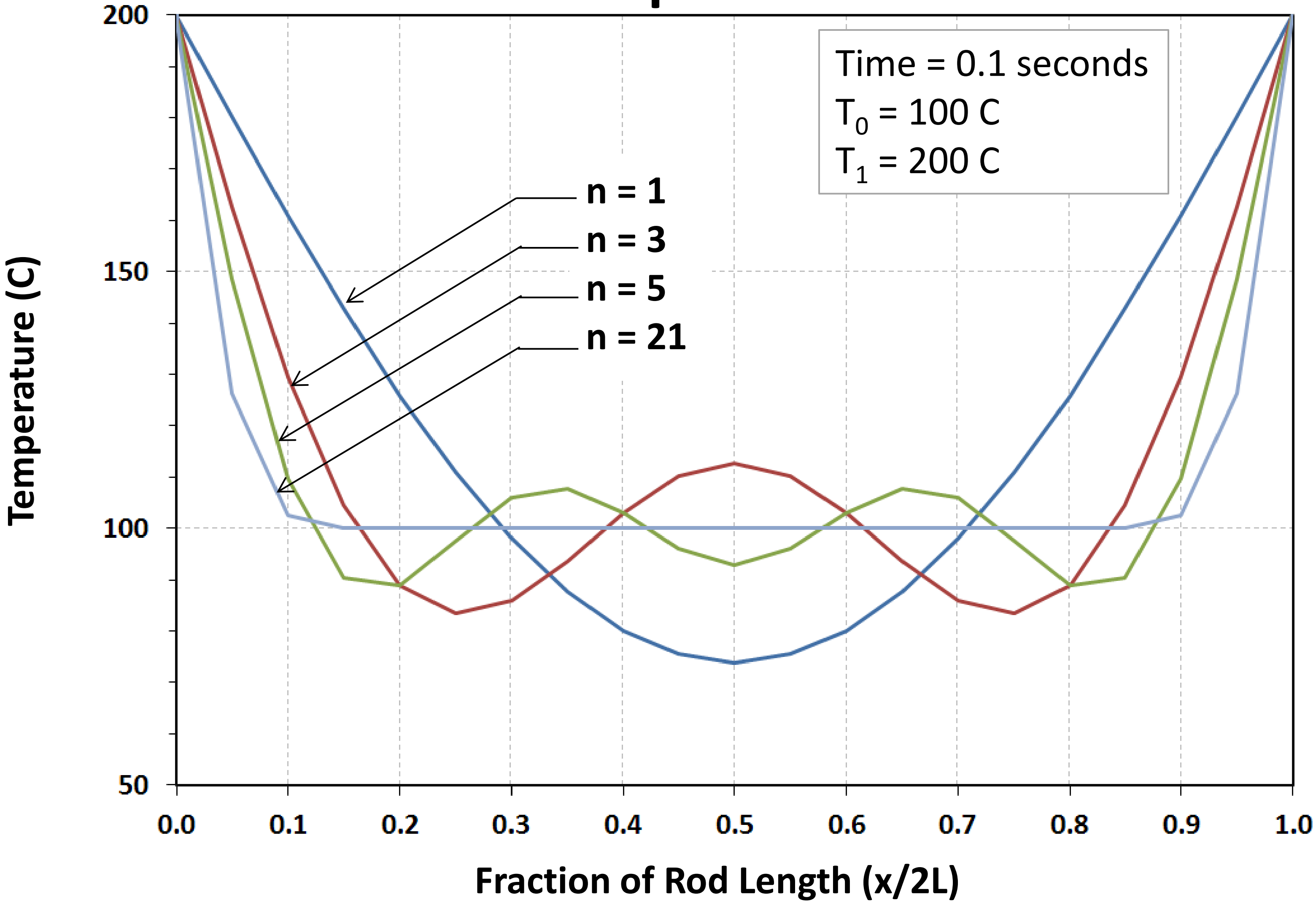
C_n may be determined by integrating considering the initial conditions (for $n = 1, 3, 5, \ldots$):

$$C_n = \frac{1}{L}\int_0^{2L} (T_0 - T_1)\, sin\left(\frac{n\pi x}{2L}\right) dx = \frac{4}{n\pi}(T_0 - T_1)$$

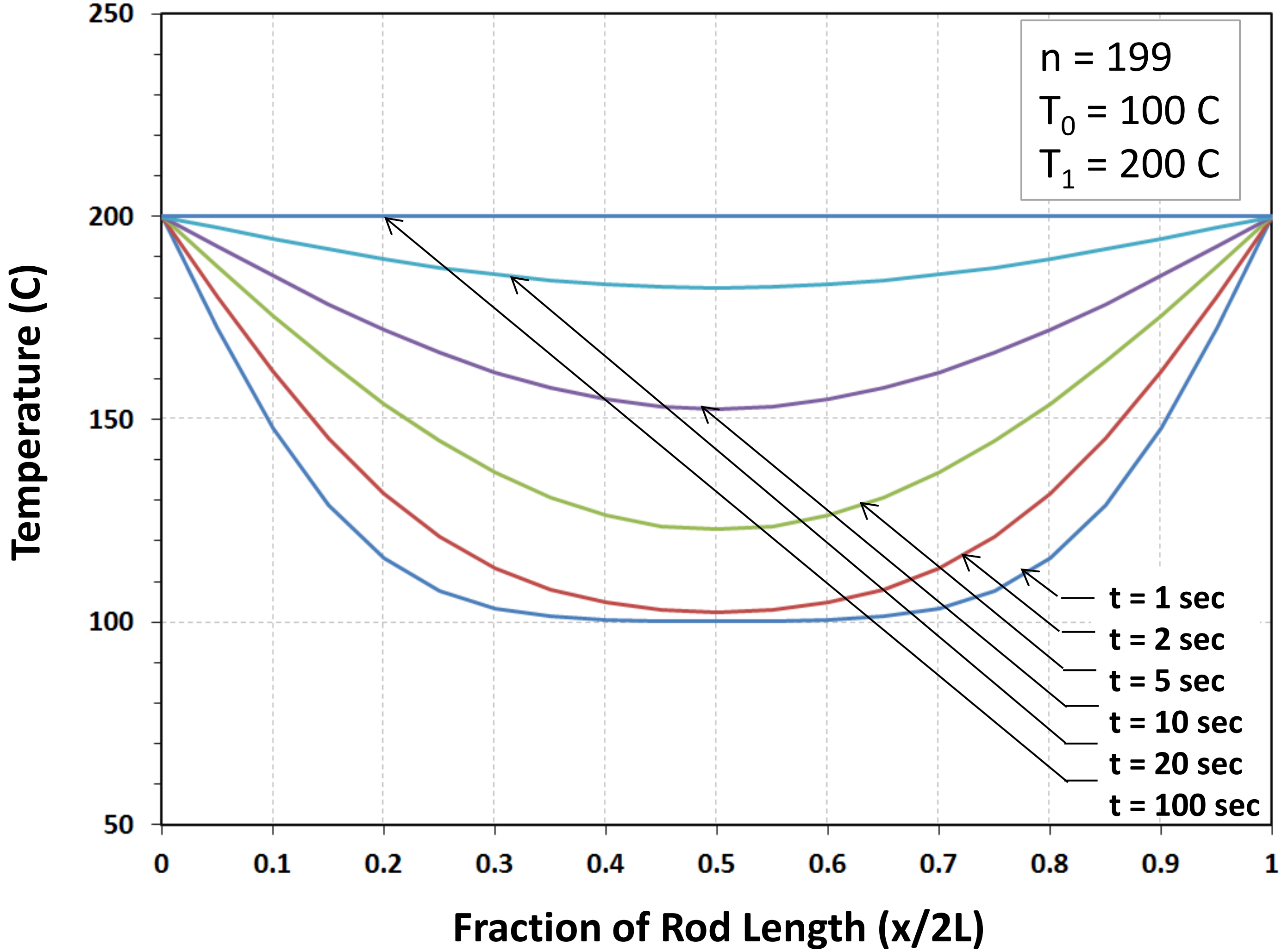
The overall solution becomes (for $n = 1, 3, 5, \ldots$):

$$\frac{T - T_1}{T_0 - T_1} = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{n} e^{-\left(\frac{n\pi}{2L}\right)^2 \alpha t} sin\left(\frac{n\pi x}{2L}\right)$$

Example 1: Solving the Transient Heat Equation



Example 1: Solving the Transient Heat Equation



Wrap-Up for Part 1

Heat transfer basics;

Thermal modeling;

Derived the heat equation one dimension;

Solved the heat equation for a specified initial condition and boundary conditions.

Part 2

Part 2 Roadmap

What is Heat Transfer?

Thermal Modeling Basics

Fourier's Heat Conduction Law

Governing Differential Equation for 1-D Heat Transfer

Solution for 1-D Heat Transfer

Example 1: Solving the Transient Heat Equation (1-D)

Numerical Methods

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Example 3: Adding Thermal Radiation

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Example 4: Steady State Analysis - Distributed Heating

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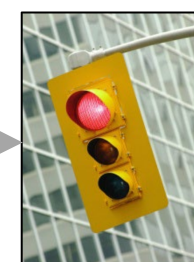


Numerical Methods

We'll focus on two different numerical methods:

Finite Difference -- uses the differential formulation -- i.e., equations are formulated using the governing differential equation -- where we replace the partial derivatives by approximations obtained by Taylor expansions near the point of interest;

Finite Element -- uses a variational formulation -- i.e., equations are formulated from an integral formulation arising from the Calculus of Variations.

Formulation of the Finite Difference for 1-D Heat Transfer

Finite difference relies on a differential formulation -- that is, a description of the heat transfer using derivatives;

For our one-dimensional heat transfer case, recall the governing differential equation is:

$$\dot{q}_{gen} + k\frac{\partial^2 T}{\partial x^2} = \rho C_p \frac{\partial T}{\partial t}$$

Formulation of the Finite Difference for 1-D Heat Transfer

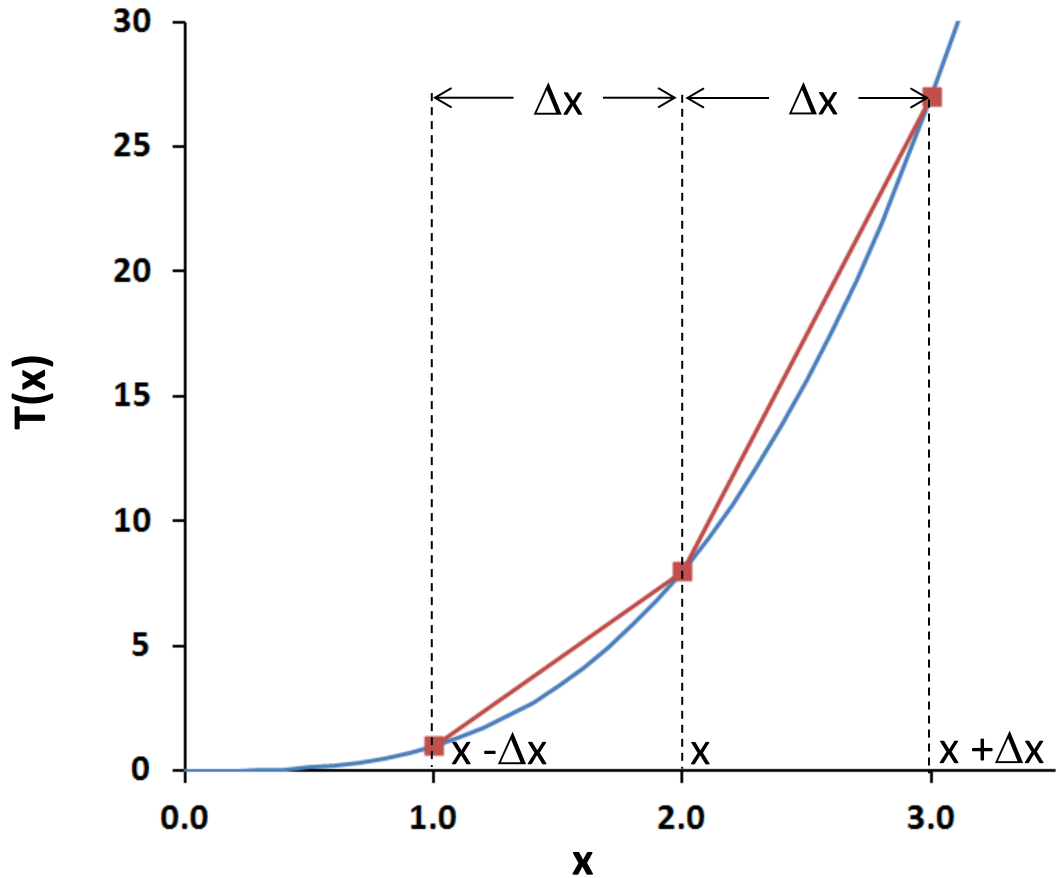
This equation involves the second derivative of temperature with respect to a spatial dimension (i.e., *x*) and the first derivative of temperature with respect to time;

Let's take a closer look at how these derivatives are formed for a numerical solution.

$$\dot{q}_{gen} + k\frac{\partial^2 T}{\partial x^2} = \rho C_p \frac{\partial T}{\partial t}$$

Formulation of the Finite Difference for 1-D Heat Transfer

Consider this temperature distribution about *x*:



Formulation of the Finite Difference for 1-D Heat Transfer (Ref. 3)

At a given instant in time, the first derivative, taken to the "right" of *x*, is given by:

$$\left.\frac{\partial T}{\partial x}\right|_{+} \approx \frac{T_{x+\Delta x} - T_x}{\Delta x}$$

Similarly, the first derivative, taken to the "left" of *x*, is given by:

$$\left.\frac{\partial T}{\partial x}\right|_{-} \approx \frac{T_x - T_{x-\Delta x}}{\Delta x}$$

Each expression yields an approximation of the slope in the specified region about *x*.

Formulation of the Finite Difference for 1-D Heat Transfer (Ref. 3)

The second derivative is just the slope of the slope over the region, or, the slope of the first derivatives:

$$\frac{\partial^2 T}{\partial x^2} \approx \frac{\left.\frac{\partial T}{\partial x}\right|_{+} - \left.\frac{\partial T}{\partial x}\right|_{-}}{\Delta x}$$

which becomes...

$$\frac{\partial^2 T}{\partial x^2} \approx \frac{(T_{x+\Delta x} - T_x) - (T_x - T_{x-\Delta x})}{\Delta x^2}$$

Formulation of the Finite Difference for 1-D Heat Transfer (Ref. 3)

Further simplification and rearranging yields:

$$\frac{\partial^2 T}{\partial x^2} \approx \frac{T_{x-\Delta x} - 2T_x + T_{x+\Delta x}}{\Delta x^2}$$

Hence we have an expression for the second derivative *in terms of temperatures at specific locations*.

But this derivation assumed the distance between nodes was Δx on either side of the node. What if it isn't?

Formulation of the Finite Difference for 1-D Heat Transfer (Adapted from Ref. 3)

Consider the case where the distance between the last node and the boundary is only $\Delta x/2$:

If we go through the same process, our expression for the second derivative at the left end and right end, respectively becomes:

$$\frac{\partial^2 T}{\partial x^2} \approx \frac{0.5T_{x-\Delta x} - 0.75T_x + 0.25T_{x+\Delta x}}{\Delta x^2}$$

$$\frac{\partial^2 T}{\partial x^2} \approx \frac{0.25T_{x-\Delta x} - 0.75T_x + 0.5T_{x+\Delta x}}{\Delta x^2}$$

Formulation of the Finite Difference for 1-D Heat Transfer (Ref. 3)

Forming the first derivative of temperature, *T* with respect to time, *t for a specific node* is considerably easier:

$$\frac{\partial T}{\partial t} \approx \frac{T^{t+\Delta t} - T^{t}}{\Delta t}$$

Again, *we have an expression for a derivative in terms of temperature and time*.

Subscripting and Superscripting

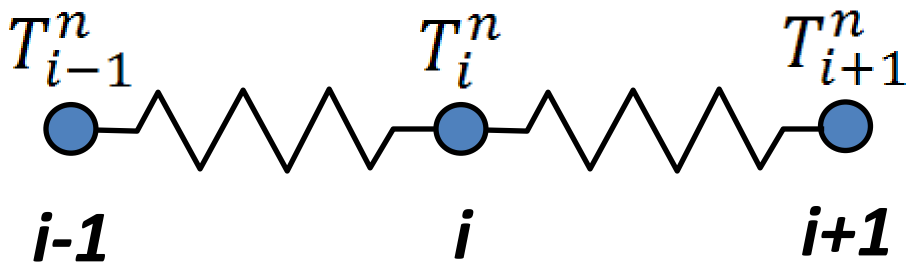
We're going to be working with both time and spatial dimension;

We'll need to do some bookkeeping;

A subscripting and superscripting scheme is shown here and will be used in the example.

Subscripting and Superscripting

For the node of interest, *i*, at time, *n*, we have:



Time	T of Node *i-1*	T of Node *i*	T of Node *i+1*
n	T_{i-1}^{n}	T_{i}^{n}	T_{i+1}^{n}
n+1	T_{i-1}^{n+1}	T_{i}^{n+1}	T_{i+1}^{n+1}

Review

We have...

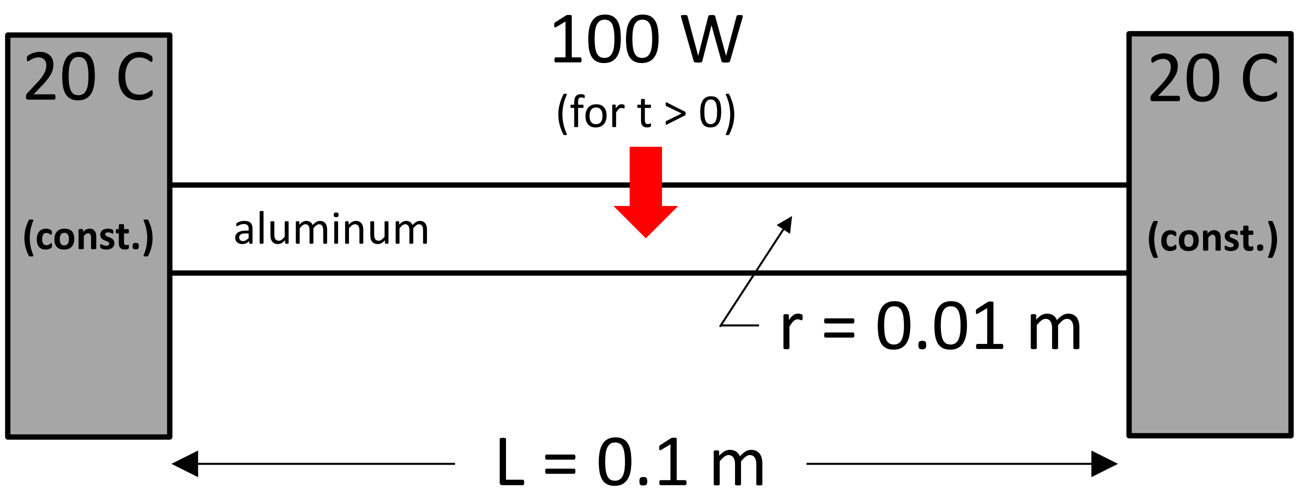
derived expressions for the second derivative of temperature with respect to distance and for the derivative of temperature with respect to time, and;

established a subscript and superscript convention to aid our analysis;

We're now ready to present an example problem.

Example 2: Explicit Finite Difference

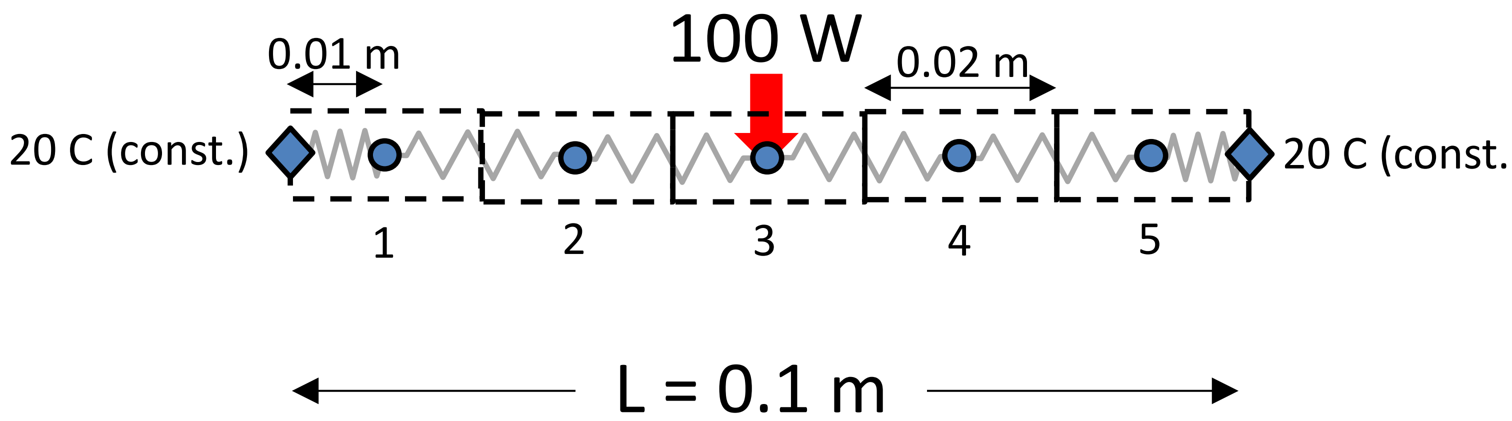
Consider the following configuration:



For an initial rod temperature of 20 C, determine the system transient response.

Example 2: Explicit Finite Difference

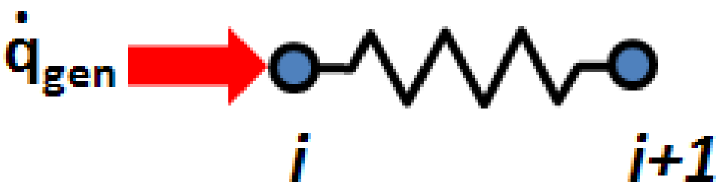
We'll model the system with five diffusion (●) nodes:



We also establish two boundary (◆) nodes at either end of the rod.

Example 2: The Explicit Finite Difference Solution

Consider the following simplified geometry;



We wish to write the difference equation for node *i* accounting for heat transfer via conduction to node *i+1* and the addition of heat to node *i*.

Example 2: The Explicit Finite Difference Solution

The overall energy balance for a segment of the rod is:

$$\dot{q}_{cond} + \dot{q}_{gen} = \dot{q}_{stored}$$

where each term represents *energy per unit time per unit volume*;

This leads to the governing differential equation:

$$k\frac{\partial^2 T}{\partial x^2} + \dot{q}_{gen} = \rho C_p \frac{\partial T}{\partial t}$$

Example 2: The Explicit Finite Difference Solution

We arrive at the difference equation describing our system:

$$T_i^n + \Delta t \frac{k}{\rho C_p} \frac{(T_i^n - T_{i+1}^n)}{(\Delta x)^2} + \frac{\Delta t}{m C_p} \dot{q}_{gen,i} = T_i^{n+1}$$

which becomes...

$$T_i^n + \Delta t \alpha \frac{(T_i^n - T_{i+1}^n)}{(\Delta x)^2} + \frac{\Delta t}{\rho C_p} \dot{Q}_i = T_i^{n+1}$$

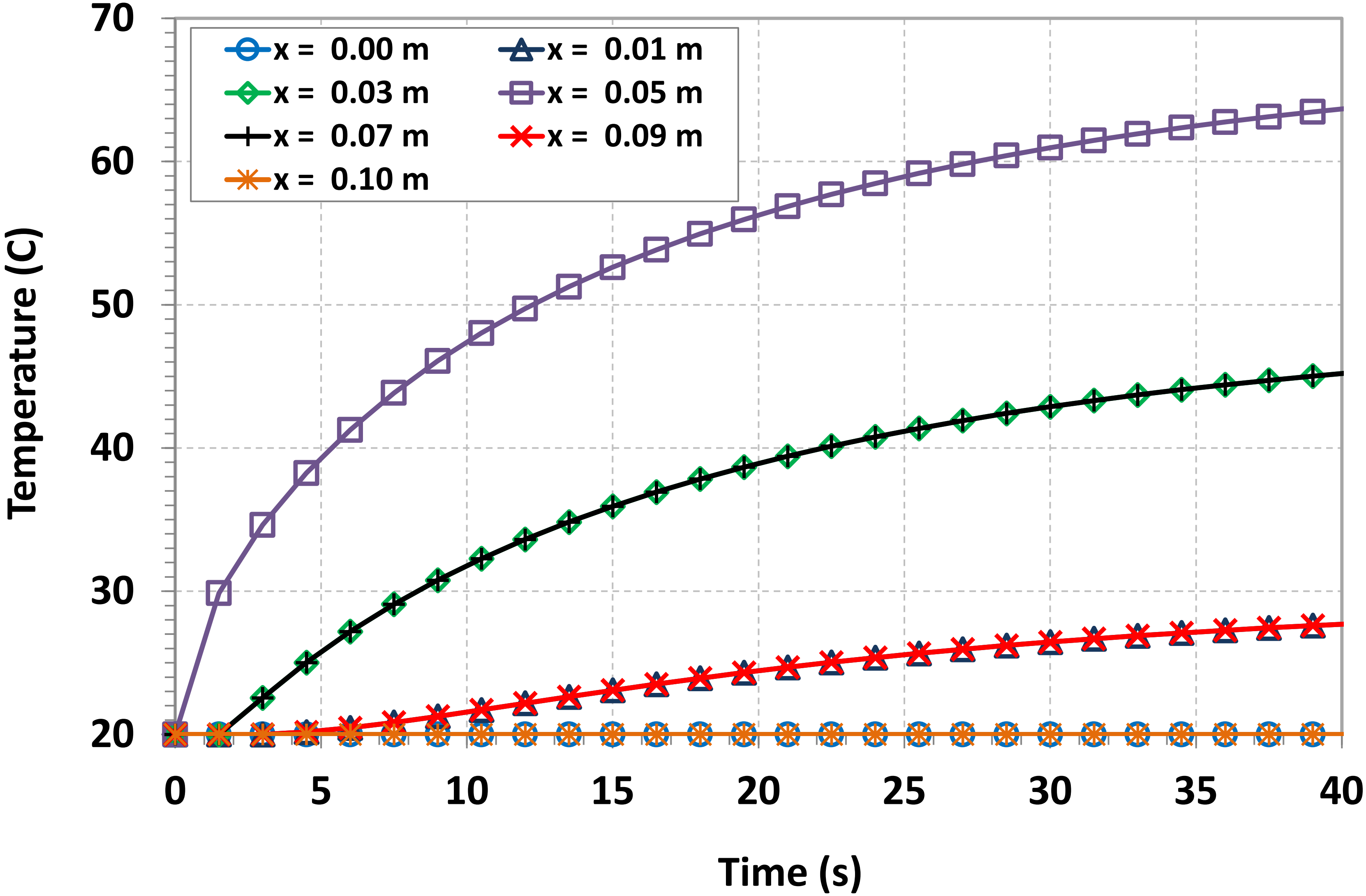
For an explicit scheme, note that the temperature at time, *n+1* is *completely* determined by parameters known at time *n*.

Example 2: The Explicit Finite Difference Solution

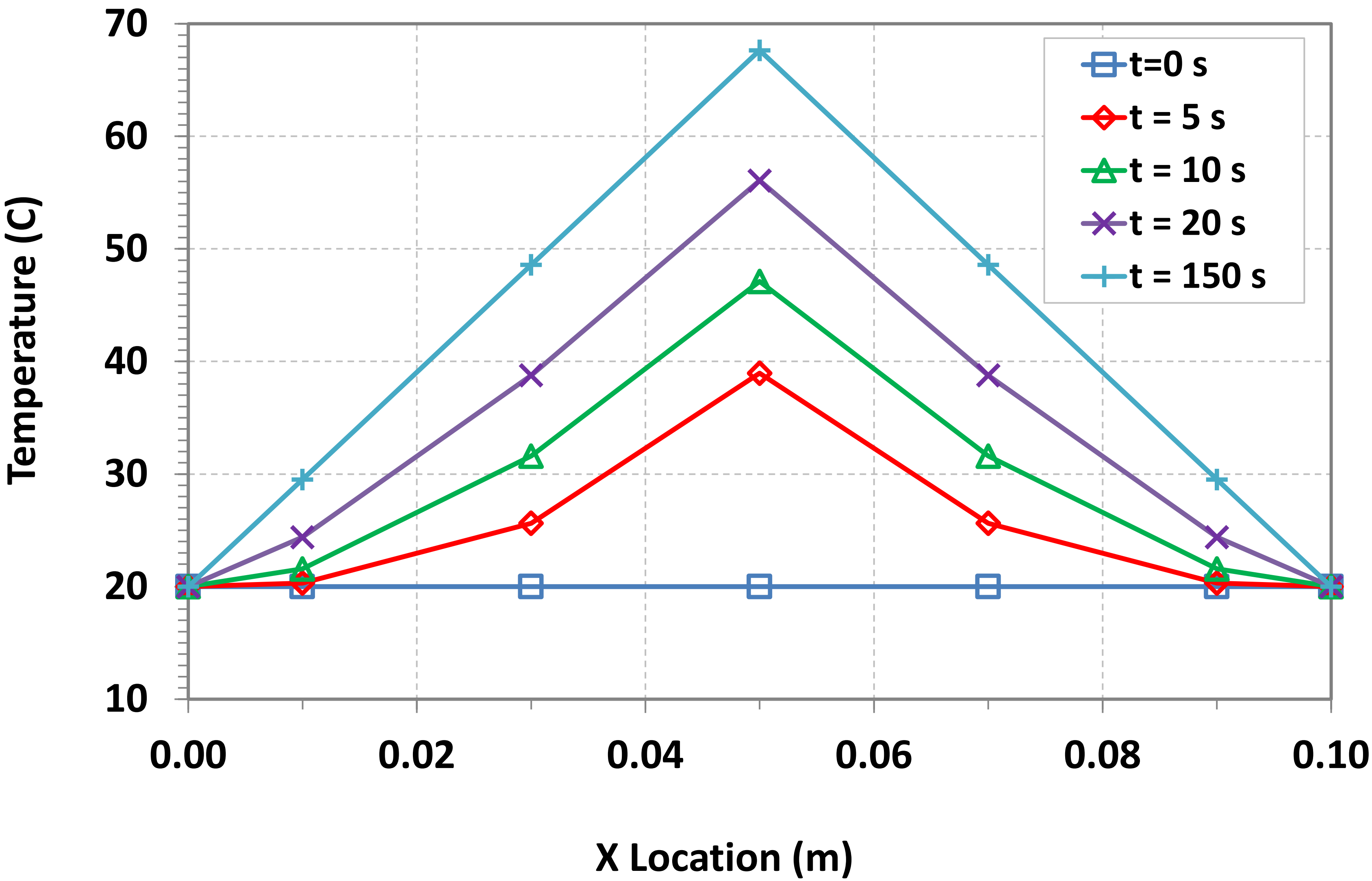
Temperatures at time *n* are known and are used to calculate temperatures at time *n+1* -- the *n+1* solution becomes the *n* solution for the next iteration.

$$\begin{bmatrix} T_{b1}^{n} \\ T_1^{n} \\ T_2^{n} \\ T_3^{n} \\ T_4^{n} \\ T_5^{n} \\ T_{b5}^{n} \end{bmatrix} + \frac{\alpha \Delta t}{(\Delta x)^2} \begin{bmatrix} - & - & - & - & - & - & - \\ 0.5 & -0.75 & 0.25 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0.25 & -0.75 & 0.5 \\ - & - & - & - & - & - & - \end{bmatrix} \begin{bmatrix} T_{b1}^{n} \\ T_1^{n} \\ T_2^{n} \\ T_3^{n} \\ T_4^{n} \\ T_5^{n} \\ T_{b5}^{n} \end{bmatrix} + \frac{\Delta t}{\rho C_p} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \dot{Q} \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} T_{b1}^{n+1} \\ T_1^{n+1} \\ T_2^{n+1} \\ T_3^{n+1} \\ T_4^{n+1} \\ T_5^{n+1} \\ T_{b5}^{n+1} \end{bmatrix}$$

Example: Explicit Finite Difference



Example: Explicit Finite Difference



Time Step and Time Constant

Finite difference temperature solution is calculated at discrete time steps;

Accuracy and stability of the solution is influenced by the time step;

Commercial software will auto calculate the time step;

Upper limit to the time step required for solution stability- function of the systems time constant, τ.

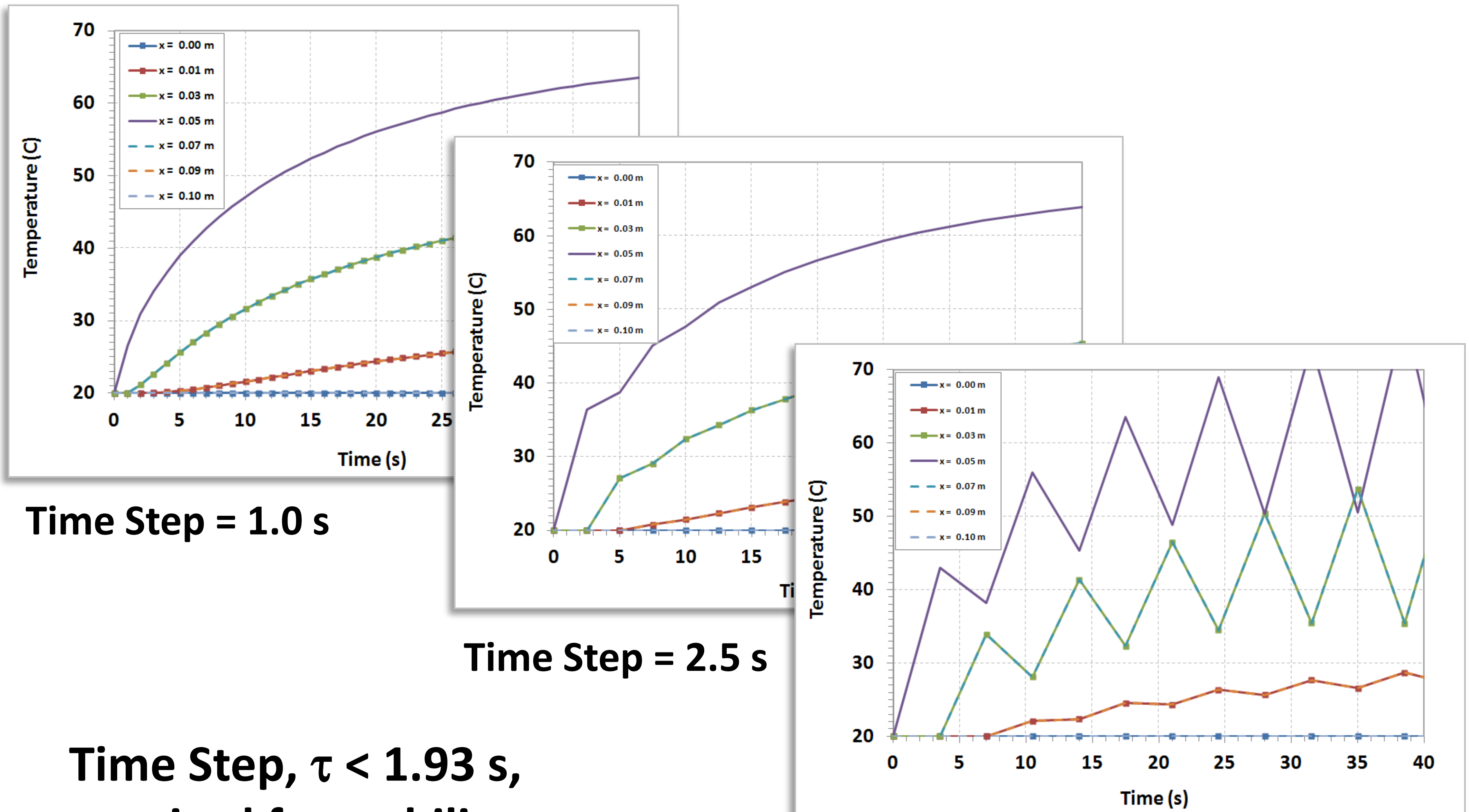
Time Step and Time Constant

For transient solutions, the time constant, τ, is determined by:

$$\tau = min\left(\frac{mC_p}{\sum G}\right)$$

In other words, the minimum of the quotient of node capacitance divided by the sum of the conductances to that node is the limiting rate at which marching in time can occur.

Example 2: Effect of the Time Step



Time Step = 1.0 s

Time Step = 2.5 s

Time Step = 3.5 s

Time Step, τ < 1.93 s, required for stability

Adding Radiation

Conduction and convection heat transfer are linear functions of the temperature difference;

$$\dot{Q}_{cond} = \frac{kA}{L}\Delta T \qquad \dot{Q}_{conv} = hA\Delta T$$

Radiation heat transfer is nonlinear and is proportional to difference of the fourth power of absolute temperatures.

$$\dot{Q}_{rad} = G_{rad}\Delta(T^4)$$

Adding Radiation (Ref. 5)

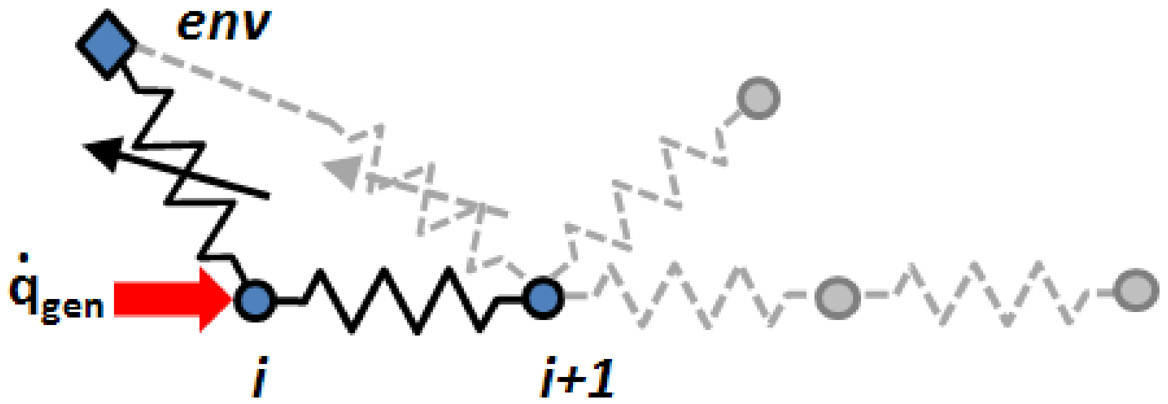
We seek to express the radiation heat transfer between the two nodes of interest in terms of *ΔT*.

$$\left(\dot{Q}_{rad}\right)_{12} = G_{rad}(T_1^4 - T_2^4)$$
$$= G_{rad}(T_1^2 + T_2^2)(T_1^2 - T_2^2)$$
$$= G_{rad}(T_1^2 + T_2^2)(T_1 + T_2)(T_1 - T_2)$$
$$= \boxed{G_{rad}(T_1^2 + T_2^2)(T_1 + T_2)}\Delta T$$

Linearized Radiation Conductor

Adding Radiation

Consider the following simplified geometry;



We wish to write the difference equation for node i including radiation to the environment.

Adding Radiation

We can perform an energy balance on node *i* (adapted from Ref. 6):

$$\dot{q}_{cond} + \dot{q}_{gen} + \dot{q}_{rad} = \dot{q}_{stored}$$

This is an expression of energy transfer per unit time *per unit volume*;

$$k\frac{\partial^2 T}{\partial x^2} + \dot{q}_{gen} + \dot{q}_{rad} = \rho C_p \frac{\partial T}{\partial t}$$

Adding Radiation

Expanding and assuming nodes are of equal length, Δx:

$$k\frac{\partial^2 T}{\partial x^2} + \frac{\dot{q}_{gen}}{A\Delta x} + \frac{\varepsilon A_s \sigma (T^4 - T_{env}^4)}{A\Delta x} = \rho C_p \frac{\partial T}{\partial t}$$

Add nodal subscripting and linearization:

$$k\frac{\partial^2 T}{\partial x^2} + \frac{\dot{q}_{gen}}{A\Delta x} + \frac{\varepsilon A_s \sigma \boxed{\left((\boldsymbol{T_i^n})^2 + (\boldsymbol{T_{env}^n})^2\right)(\boldsymbol{T_i^n} + \boldsymbol{T_{env}^n})}(T_i^n - T_{env}^n)}{A\Delta x} = \rho C_p \frac{\partial T}{\partial t}$$

Adding Radiation

Next, we add accommodation for node-specific heating and a means of indexing the equation in time (i.e., *n*, *n+1*):

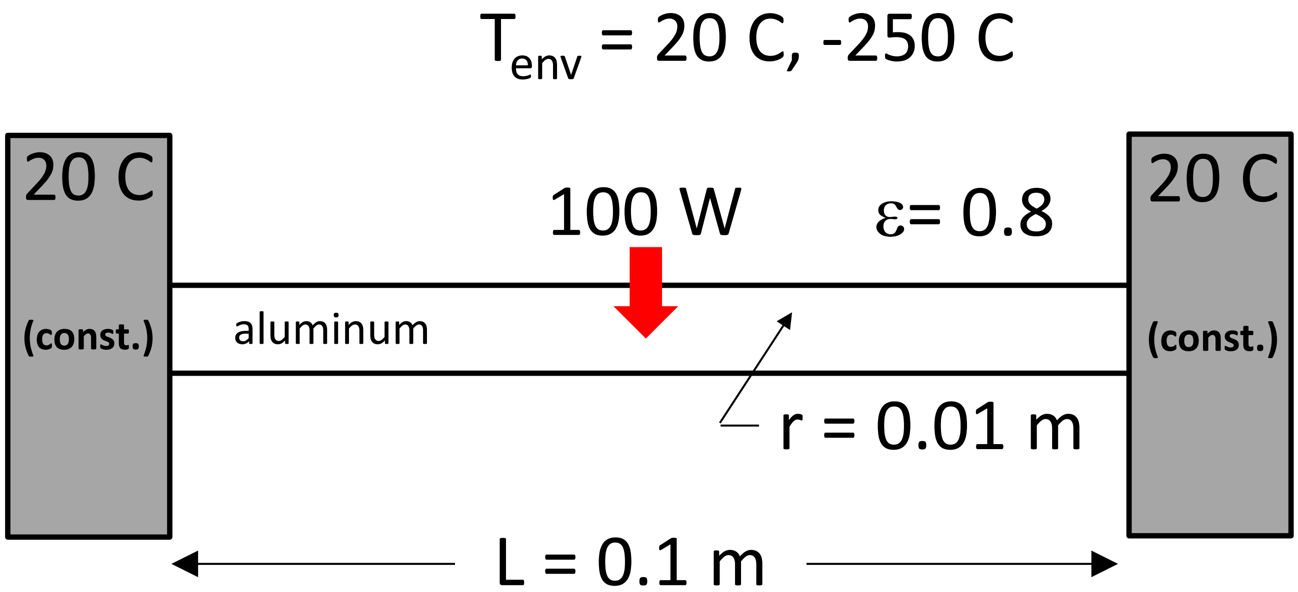
$$k\frac{(T_i^n - T_{i+1}^n)}{(\Delta x)^2} + \frac{\dot{q}_{gen,i}}{A\Delta x} + \frac{\varepsilon A_s \sigma((T_i^n)^2 + (T_{env}^n)^2)(T_i^n + T_{env}^n)(T_i^n - T_{env}^n)}{A\Delta x} = \rho C_p \frac{T_i^{n+1} - T_i^n}{\Delta t}$$

With some additional manipulation, we arrive at our desired result:

$$T_i^n + \Delta t\alpha\frac{(T_i^n - T_{i+1}^n)}{(\Delta x)^2} + \frac{\Delta t}{mC_p}\dot{q}_{gen,i} + \frac{\Delta t\varepsilon A_s \sigma\left((\boldsymbol{T_i^n})^2 + (\boldsymbol{T_{env}^n})^2\right)(\boldsymbol{T_i^n} + \boldsymbol{T_{env}^n})}{mC_p}(T_i^n - T_{env}^n) = T_i^{n+1}$$

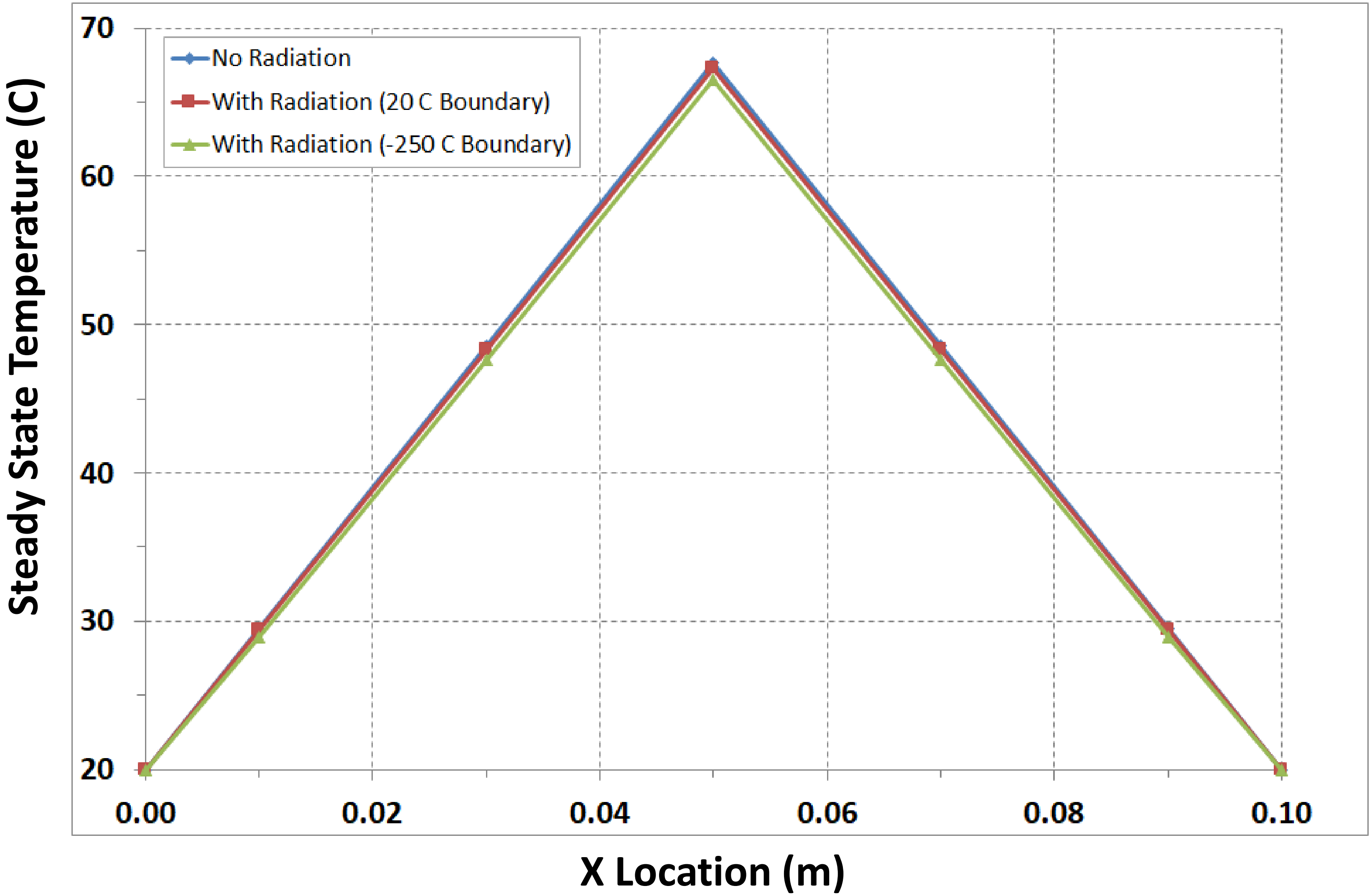
Example 3: Explicit Finite Difference with Radiation Added

Consider the following configuration:



Assume a surface $\varepsilon = 0.8$ and solve for two different T_{env}: 20 C and -250 C.

Example 3: Explicit Finite Difference with Radiation Added



The Implicit Finite Difference Solution

The explicit technique shown allows determination of temperatures at time *n+1* in terms of known temperatures at time *n*;

However, different formulations of the differencing equation exist;

Common Implicit formulations include the Backward difference and the Central difference (Crank-Nicolson) methods.

The Implicit Finite Difference Solution (Adapted from Ref. 3)

Crank-Nicolson takes the form:

$$\frac{T_x^{n+1} - T_x^n}{\Delta t} = \alpha \frac{\frac{1}{2}(T_{x+\Delta x}^{n+1} + T_{x+\Delta x}^n) + \frac{1}{2}(2T_x^{n+1} - 2T_x^n) + \frac{1}{2}(T_{x-\Delta x}^{n+1} + T_{x-\Delta x}^n)}{(\Delta x)^2}$$

Note that since the *n+1* superscript appears on both sides of the equation, we can't express temperatures at that time as explicit functions of *n*;

Hence, this is referred to as an implicit technique.

Solution Accuracy (Adapted from Ref. 4)

An assessment of the solution accuracy can be made by observing the order of the terms truncated in the Taylor series approximation;

For a first derivative formed as:

$$\frac{\partial T}{\partial x} \approx \frac{T(x+\Delta x) - T(x)}{\Delta x} + \cdots$$

The truncated terms are: $\mathcal{O}(\Delta x)$

Solution Accuracy (Adapted from Ref. 4)

A second order derivative with respect to x, though, is inherently first order accurate in *x* because it is formed using first derivatives that are accurate to the first order in *x*.

$$\frac{\partial^2 T}{\partial x^2} \approx \frac{\left.\frac{\partial T}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial T}{\partial x}\right|_{x}}{\Delta x} + \cdots$$

Techniques exist to boost Forward and Backward differences to higher accuracy.

Solution Accuracy (Adapted from Ref. 4)

Both Forward and Backward differencing are inherently first order accurate in *x*;

A Central difference, such as Crank-Nicolson,, however, is inherently: $\mathcal{O}(\Delta x^2)$

$$\frac{\partial T}{\partial x} \approx \frac{T(x+\Delta x) - T(x-\Delta x)}{2\Delta x} + \cdots$$

Wrap-Up for Part 2

Formulated the finite difference;

Demonstrated explicit solution technique;

Brief discussion of time step and time constant;

Added effects of radiation;

Implicit techniques;

Solution accuracy.

Part 3

Part 3 Roadmap

What is Heat Transfer?

Thermal Modeling Basics

Fourier's Heat Conduction Law

Governing Differential Equation for 1-D Heat Transfer

Solution for 1-D Heat Transfer

Example 1: Solving the Transient Heat Equation (1-D)

Numerical Methods

Finite Difference

Example 2: Transient Thermal Analysis

Example 3: Adding Thermal Radiation

Implicit Formulation

Finite Element

Calculus of Variations

Establishing the Functional Equation

Example 4: Steady State Analysis - Distributed Heating

Example 5: Steady State Analysis

Example 6: Transient Analysis

The Finite Element Method

The finite element formulation centers around minimization of a functional;

But what is a functional?

To understand this, we need some background on the Calculus of Variations.

Calculus of Variations

Finite differencing relies on a *differential* formulation of the heat equation;

Finite element relies on a *variational* formulation;

For steady state, one-dimensional heat transfer, we seek to minimize the following integral, called a functional:

$$I = \int_{x=0}^{L} \frac{1}{2} k \left(\frac{dT}{dx}\right)^2 dx$$

Calculus of Variations (Adapted from Ref. 7)

But where does the functional come from and what is the theoretical basis for establishing the functional?

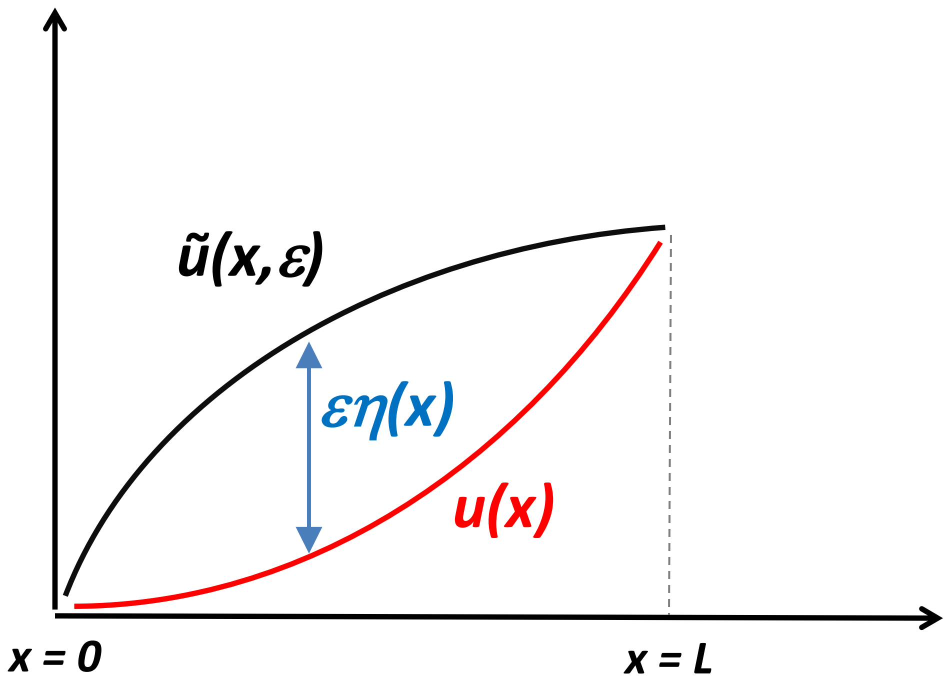
The functional is established using the Calculus of Variations;

In general, we seek a function to *minimize* the integral:

$$I = \int_{x=0}^{L} F\big(x, u(x), u'(x)\big)dx$$

Calculus of Variations (Adapted from Ref. 7)

u(x) is the function we want to minimize.
ũ(x,ε) is the set of all functions that can minimize the integral, *I*.



Calculus of Variations (Adapted from Ref. 7)

The set of all functions to minimize *I* can be expressed as:

$$\tilde{u}(x, \varepsilon) = u(x) + \varepsilon\eta(x)$$

where...

u(x) is the function we seek to minimize;

η*(x)* is an arbitrary function constrained by:

$$\eta(0) = 0$$
$$\eta(L) = 0$$

Calculus of Variations (Adapted from Ref. 7)

This ensures that $\tilde{u}(x,\varepsilon)$ is correct at the boundaries:

$$\tilde{u}(0,\varepsilon) = u(0)$$
$$\tilde{u}(L,\varepsilon) = u(L)$$

This works if $\varepsilon = 0$ so that:

$$\tilde{u}(0,0) = u(0)$$
$$\tilde{u}(L,0) = u(L)$$

or...

$$\tilde{u}(x,0) = u(x) + \varepsilon\eta(x)$$

0

Calculus of Variations (Adapted from Ref. 7)

Let's substitute our expression for $\tilde{u}(x,\varepsilon)$ into the original equation but note we've added the variable, ε:

$$I = \int_{x=0}^{L} F\big(x, \tilde{u}(x,\varepsilon), \tilde{u}'(x,\varepsilon)\big)dx$$

and note that when $\varepsilon = 0$, we return to the original expression.

Calculus of Variations (Adapted from Ref. 7)

So, we want our function *I(ε)* to be a minimum when $\varepsilon = 0$;

We can accomplish this by differentiating the expression with respect to ε;

$$\frac{\partial I}{\partial \varepsilon} = \int_{x=0}^{L} \frac{\partial F}{\partial \varepsilon}\Big(x, \tilde{u}(x,\varepsilon), \tilde{u}'(x,\varepsilon)\Big)dx$$

Calculus of Variations (Adapted from Ref. 7)

By chain rule, the expression becomes:

$$\frac{\partial I}{\partial \varepsilon} = \int_{x=0}^{L} \left[\frac{\partial F}{\partial \tilde{u}} \frac{\partial \tilde{u}}{\partial \varepsilon} + \frac{\partial F}{\partial \tilde{u}'} \frac{\partial \tilde{u}'}{\partial \varepsilon} \right] dx$$

This looks messy, but we can clean it up.

Calculus of Variations (Adapted from Ref. 7)

Remember:

$$\tilde{u}(x,\varepsilon) = u(x) + \varepsilon\eta(x)$$

So that...

$$\frac{\partial\tilde{u}}{\partial x} = \tilde{u}' = u' + \varepsilon\eta'(x)$$

and...

$$\frac{\partial\tilde{u}}{\partial\varepsilon} = \eta(x) \qquad \frac{\partial\tilde{u}'}{\partial\varepsilon} = \eta'(x)$$

Calculus of Variations (Adapted from Ref. 7)

The integral expression becomes:

$$\frac{\partial I}{\partial \varepsilon} = \int_{x=0}^{L} \left[\frac{\partial F}{\partial \tilde{u}} \eta(x) + \frac{\partial F}{\partial \tilde{u}'} \eta'(x) \right] dx$$

But note that:

$$\frac{\partial F}{\partial \tilde{u}'} \eta'(x)$$

w **dv**

can be integrated by parts.

Calculus of Variations (Adapted from Ref. 7)

Recall integration by parts:

$$\int w\,dv = wv - \int v\,dw$$

So the entire integral expression becomes:

$$\frac{dI}{d\varepsilon} = \int_0^L \frac{\partial F}{\partial \tilde{u}} \eta(x) dx + \left[\frac{\partial F}{\partial \tilde{u}'} \eta(x)\right]_0^L - \int_0^L \eta(x) \frac{d}{dx}\left(\frac{\partial F}{\partial \tilde{u}'}\right) dx$$

Calculus of Variations (Adapted from Ref. 7)

But, recall that $\eta(0) = 0$ and $\eta(L) = 0$, so:

$$\frac{dI}{d\varepsilon} = \int_0^L \frac{\partial F}{\partial \tilde{u}} \eta(x) dx + \cancelto{0}{\left[\frac{\partial F}{\partial \tilde{u}'} \eta(x)\right]_0^L} - \int_0^L \eta(x) \frac{d}{dx}\left(\frac{\partial F}{\partial \tilde{u}'}\right) dx$$

So the expression simplifies to:

$$\frac{dI}{d\varepsilon} = \int_0^L \eta(x) \left[\frac{\partial F}{\partial \tilde{u}} - \frac{d}{dx}\left(\frac{\partial F}{\partial \tilde{u}'}\right)\right] dx$$

Calculus of Variations (Adapted from Ref. 7)

Consider the quantity inside the brackets:

$$\frac{dI}{d\varepsilon} = \int_0^L \eta(x) \left[\frac{\partial F}{\partial \tilde{u}} - \frac{d}{dx}\left(\frac{\partial F}{\partial \tilde{u}'}\right)\right] dx$$

We force ε to be zero so that:

$$\frac{dI}{d\varepsilon} = 0$$

Calculus of Variations (Adapted from Ref. 7)

Since $\eta(x)$ is arbitrary, this forces us to conclude that the bracketed quantity:

$$\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right) = 0$$

Note that since $\varepsilon = 0$, the $\tilde{u}$ and $\tilde{u}'$ become u and u'.

This is called the Euler-Lagrange equation and we will use it to establish the functional for heat transfer.

Forming the Functional

Recall the governing differential equation for steady state, one-dimensional conduction:

$$A\frac{d}{dx}\left(k\frac{dT}{dx}\right) = 0$$

Area, *A*, cancels out and we can re-write this as:

$$\frac{d}{dx}(kT') = 0$$

where

$$T' = \frac{dT}{dx}$$

Forming the Functional

Next, we note that *T'* replaces *u'* and which means that *T* replaces *u* in the Euler-Lagrange equation:

$$\frac{d}{dx}(kT') = 0 \qquad\qquad \frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right) = 0$$

So the equation becomes:

$$\frac{\partial F}{\partial T} - \frac{d}{dx}\left(\frac{\partial F}{\partial T'}\right) = 0$$

Forming the Functional

The heat equation be rewritten as:

$$0 - \frac{d}{dx}(kT') = 0$$

In comparing it with our expression:

$$\frac{\partial F}{\partial T} - \frac{d}{dx}\left(\frac{\partial F}{\partial T'}\right) = 0$$

We see that:

$$\frac{\partial F}{\partial T} = 0$$

Forming the Functional

And since:

$$\frac{\partial F}{\partial T'} = kT'$$

We can integrate this expression with respect to *T'* to get:

0

$$F = \frac{1}{2}k(T')^2 + g(T)$$

Forming the Functional

Finally, we arrive at the desired expression:

$$I = \frac{1}{2}\int_{x=0}^{L} k\left(\frac{dT}{dx}\right)^2 dx$$

Wrap-Up for Part 3

Differentiated between the finite difference and finite element techniques;

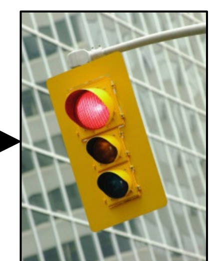
Established the functional using the Euler-Lagrange equation;

Formed the functional for steady state heat transfer.

Part 4

Part 4 Roadmap

- What is Heat Transfer?
- Thermal Modeling Basics
- Fourier's Heat Conduction Law
- Governing Differential Equation for 1-D Heat Transfer
- Solution for 1-D Heat Transfer
- Example 1: Solving the Transient Heat Equation (1-D)
- **Numerical Methods**
  - Finite Difference
  - Example 2: Transient Thermal Analysis
  - Example 3: Adding Thermal Radiation
  - Implicit Formulation
  - Finite Element
  - Calculus of Variations
  - Establishing the Functional Equation
- **Example 4: Steady State Analysis - Distributed Heating**
- **Example 5: Steady State Analysis**
- **Example 6: Transient Analysis**



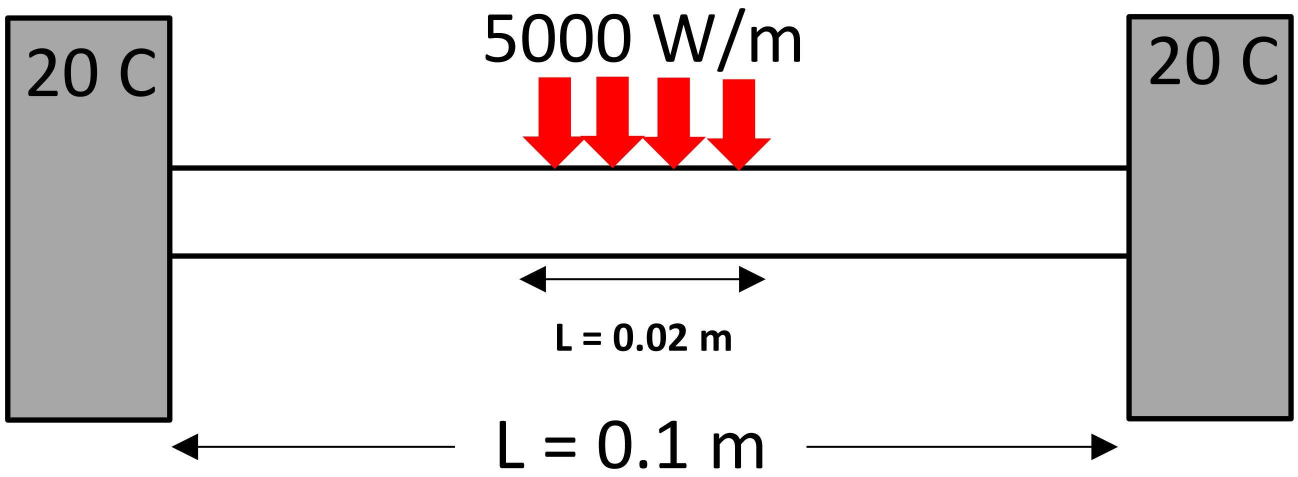
Finite Element Method Strategy

We will employ the following strategy to solve a sample problem:

Form the variational statement;
Formulate relations at the element level;
Minimize the integral at the element level;
Assemble equations into a system for solution;
Apply boundary conditions;
Solve for nodal temperatures.

Example 4: One-Dimensional Steady State Finite Element

Consider, the following configuration:



What is the steady state temperature distribution in the rod?

Example 4: One-Dimensional Steady State Finite Element

This is a steady state conduction problem with heat application/generation and fixed boundary temperatures;

The governing differential equation is:

$$\dot{q}_{gen} + k\frac{\partial^2 T}{\partial x^2} = \rho C_p \frac{\partial T}{\partial t}$$

0

but for steady state, the right hand side is equal to zero.

Example 4: One-Dimensional Steady State Finite Element

First, we form the variational relationship;

From the Euler-Lagrange equation, we have:

$$\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right) = 0$$

And note that:

$$T = u \qquad\qquad T' = \frac{dT}{dx} = u'$$

Example 4: One-Dimensional Steady State Finite Element

So the Euler-Lagrange equation becomes:

$$\frac{\partial F}{\partial T} - \frac{d}{dx}\left(\frac{\partial F}{\partial T'}\right) = 0$$

We can rewrite our governing differential equation in this form:

$$-\dot{q}_{gen} - \frac{d}{dx}(kT') = 0$$

Example 4: One-Dimensional Steady State Finite Element

By comparing terms, we see that:

$$\frac{\partial F}{\partial T} = -\dot{q}_{gen}$$

and...

$$\frac{\partial F}{\partial T'} = kT'$$

Example 4: One-Dimensional Steady State Finite Element

We can solve for *F* by integrating the first expression:

$$F = \int \frac{\partial F}{\partial T} dT = -\int \dot{q}_{gen} dT = -\,\dot{q}_{gen} T + g_1(T')$$

But, by integrating the second expression (i.e., the one with the *T'* term), we get *another* expression for *F*.

$$F = \int \frac{\partial F}{\partial T'} dT' = \int (kT')\, dT' = \frac{1}{2} k (T')^2 + g_2(T)$$

Example 4: One-Dimensional Steady State Finite Element

But both of these expressions *must be equal to one another because they both represent F*;

So we conclude that...

$$F = \frac{1}{2}k(T')^2 - \dot{q}_{gen}T$$

And the integral we seek to minimize is:

$$I = \int_{x=0}^{L} \left[\frac{1}{2}k(T')^2 - \dot{q}_{gen}T\right] dx$$

Example 4: One-Dimensional Steady State Finite Element (Adapted from Ref. 7)

Now that we have our variational statement, we need to apply it to our problem;

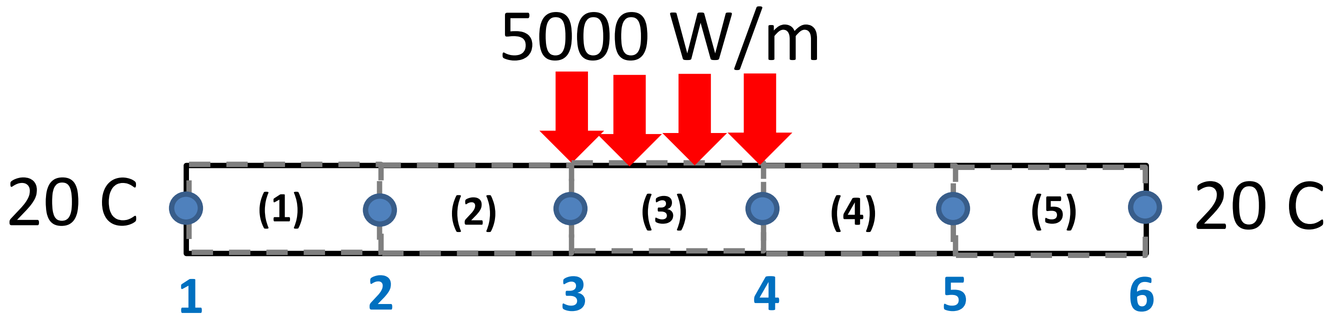
Most geometries are quite complex and we cannot apply this expression directly;

So, we discretize the system into finite elements and apply the expression to each element:

$$I = \sum_{e=1}^{E} I^{(e)} = I^{(1)} + I^{(2)} + \cdots + I^{(E)}$$

Example 4: One-Dimensional Steady State Finite Element

Let's consider the following element breakdown;



Note that for finite elements, the node points are at the boundary of each element;

For finite difference, the nodes were centered in the element.

Example 4: One-Dimensional Transient Finite Element (Adapted from Ref. 7)

In this case, we seek to minimize an integral expression considering the effects of conduction (I_k), and applied (or internally generated) heating (I_q):

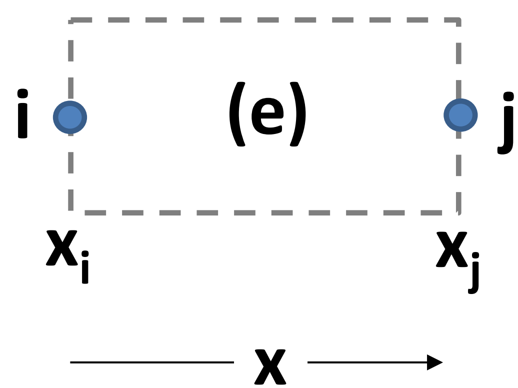
$$I = I_k + I_q$$

Example 4: One-Dimensional Steady State Finite Element (Adapted from Ref. 7)

For an individual element, our integral becomes:

$$I^{(e)} = \int_{x_i}^{x_j} \left[\frac{1}{2} k^{(e)} A^{(e)} \left(T'^{(e)} \right)^2 - \dot{q}_L^{(e)} T^{(e)} \right] dx$$

where x_i and x_j represent the bounds of the element, $A^{(e)}$ is the element cross sectional area and $\dot{q}_L$ is the heating per unit length per unit time.



Example 4: One-Dimensional Steady State Finite Element (Adapted from Ref. 7)

Let's assume a linear temperature profile across each element

$$T^{(e)}(x) = c_1^{(e)} + c_2^{(e)} x$$

At the element boundaries, we have:

$$T_i = c_1^{(e)} + c_2^{(e)} x_i$$
$$T_j = c_1^{(e)} + c_2^{(e)} x_j$$

But we are left with unknown constants, c_1 and c_2.

Example 4: One-Dimensional Steady State Finite Element (Adapted from Ref. 7)

But, with two equations and two unknowns, we can solve for constants c_1 and c_2.

$$c_1 = \frac{x_j T_i - x_i T_j}{x_j - x_i}$$

$$c_2 = \frac{T_j - T_i}{x_j - x_i}$$

Example 4: One-Dimensional Steady State Finite Element (Adapted from Ref. 7)

But, we can express the unknown constants in terms of known variables from the T_i and T_j expressions to give us an expression for $T^{(e)}$ in terms of, either, known or to-be-determined quantities:

$$T^{(e)}(x) = \frac{1}{(x_j - x_i)}\left[(x_j T_i - x_i T_j) + (T_j - T_i)x\right]$$

Example 4: One-Dimensional Steady State Finite Element (Adapted from Ref. 7)

We'll also need an expression for the derivative of *T* with respect to *x*;

Differentiating the previous expression yields:

$$\frac{dT^{(e)}}{dx} = \frac{(T_j - T_i)}{(x_j - x_i)}$$

Example 4: One-Dimensional Steady State Finite Element (Adapted from Ref. 8)

Similarly, we must establish an expression for the distribution of heating across the element:

$$\dot{q}_L^{(e)}(x) = c_3^{(e)} + c_4^{(e)} x$$

As with our temperature calculation, this leads to:

$$\dot{q}_L^{(e)}(x) = \frac{1}{(x_j - x_i)} \left[(x_j \dot{q}_{L,i} - x_i \dot{q}_{L,j}) + (\dot{q}_{L,j} - \dot{q}_{L,i}) x \right]$$

Example 4: One-Dimensional Steady State Finite Element (Adapted from Ref. 7)

So, starting with the integral expression for an element, we have:

$$I^{(e)} = \int_{x_i}^{x_j} \left[\frac{1}{2} k^{(e)} A^{(e)} \left(T'^{(e)} \right)^2 - \dot{q}_L^{(e)} T^{(e)} \right] dx$$

Example 4: One-Dimensional Steady State Finite Element (Adapted from Ref. 7)

And substituting our expressions for the element temperature and gradient, we arrive at:

$$I^{(e)} = \frac{1}{2}\int_{x_i}^{x_j}\Bigg[k^{(e)}A^{(e)}\left(\frac{(T_j - T_i)}{(x_j - x_i)}\right)^2 - 2\left\{\frac{1}{(x_j - x_i)}\left[(x_j\dot{q}_{L,i} - x_i\dot{q}_{L,j}) + (\dot{q}_{L,j} - \dot{q}_{L,i})x\right]\right\}\left\{\frac{1}{(x_j - x_i)}\left[(x_jT_i - x_iT_j) + (T_j - T_i)x\right]\right\}\Bigg]dx$$

Example 4: One-Dimensional Steady State Finite Element (Adapted from Ref. 7)

Before we integrate, we differentiate this previous expression with respect to T_i:

$$I^{(e)} = \frac{1}{2}\int_{x_i}^{x_j}\Bigg[k^{(e)}A^{(e)}\left(\frac{(T_j - T_i)}{(x_j - x_i)}\right)^2 - 2\left\{\frac{1}{(x_j - x_i)}\left[\left(x_j\dot{q}_{L,i} - x_i\dot{q}_{L,j}\right) + \left(\dot{q}_{L,j} - \dot{q}_{L,i}\right)x\right]\right\}\left\{\frac{1}{(x_j - x_i)}\left[\left(x_jT_i - x_iT_j\right) + \left(T_j - T_i\right)x\right]\right\}\Bigg]dx$$

Example 4: One-Dimensional Steady State Finite Element (Adapted from Ref. 7)

And we arrive at...

$$\frac{\partial I^{(e)}}{\partial T_i} = \int_{x_i}^{x_j} \left[\frac{k^{(e)} A^{(e)}}{\left(x_j - x_i\right)^2} \left(T_i - T_j\right) - \frac{x_j \dot{q}_{L,i}^{(e)} - x_i \dot{q}_{L,j}^{(e)} + \left(\dot{q}_{L,i}^{(e)} - \dot{q}_{L,j}^{(e)}\right) x}{\left(x_j - x_i\right)} \right] dx$$

Example 4: One-Dimensional Steady State Finite Element (Adapted from Ref. 7)

We can now integrate the expression with respect to *x*:

$$\frac{\partial I^{(e)}}{\partial T_i} = \left[\frac{k^{(e)}A^{(e)}}{\left(x_j - x_i\right)^2}\left(T_i - T_j\right)\right] x \Bigg|_{x_i}^{x_j} - \frac{1}{\left(x_j - x_i\right)}\left[\left(x_j \dot{q}_{L,i}^{(e)} - x_i \dot{q}_{L,j}^{(e)}\right) x + \frac{\left(\dot{q}_{L,i}^{(e)} - \dot{q}_{L,j}^{(e)}\right) x^2}{2}\right]\Bigg|_{x_i}^{x_j}$$

Example 4: One-Dimensional Steady State Finite Element (Adapted from Ref. 7)

This simplifies to:

$$\frac{\partial I^{(e)}}{\partial T_i} = \frac{k^{(e)}A^{(e)}}{(x_j - x_i)}(T_i - T_j) - \frac{\left(2\dot{q}_{L,i}^{(e)} + \dot{q}_{L,j}^{(e)}\right)}{6}(x_j - x_i)$$

But, for constant heating across an element:

$$\dot{q}_L^{(e)} = \dot{q}_{L,i}^{(e)} = \dot{q}_{L,j}^{(e)}$$

So the expression becomes:

$$\frac{\partial I^{(e)}}{\partial T_i} = \frac{k^{(e)}A^{(e)}}{(x_j - x_i)}(T_i - T_j) - \frac{\dot{q}_L^{(e)}}{2}(x_j - x_i)$$

Example 4: One-Dimensional Steady State Finite Element (Adapted from Ref. 7)

Remember, the overall integral, *I*, is a function of all of the *m* temperatures in the network:

$$I = I(T_1, T_2, T_3, \cdots, T_m)$$

To minimize the overall integral, we'll need to find the derivative of each element integral with respect to every temperature and set the sum equal to zero:

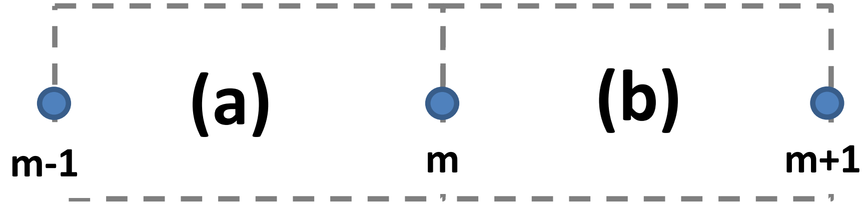
$$\frac{\partial I}{\partial T_m} = \frac{\partial I^{(1)}}{\partial T_m} + \frac{\partial I^{(2)}}{\partial T_m} + \frac{\partial I^{(a)}}{\partial T_m} + \frac{\partial I^{(b)}}{\partial T_m} \cdots + \frac{\partial I^{(E)}}{\partial T_m}$$

Example 4: One-Dimensional Steady State Finite Element (Adapted from Ref. 7)

For our linear, one-dimensional elements, though, $I^{(e)}$ is a function of only two temperatures;

The expression reduces to:

$$\frac{\partial I}{\partial T_m} = \frac{\partial I^{(a)}}{\partial T_m} + \frac{\partial I^{(b)}}{\partial T_m}$$



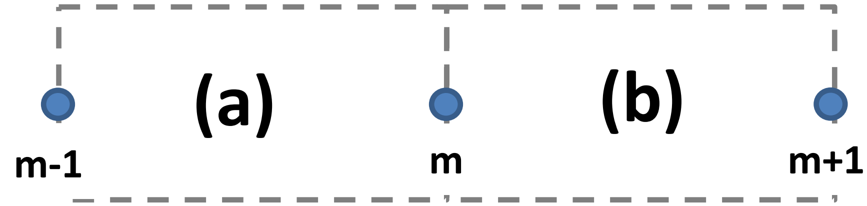
Example 4: One-Dimensional Steady State Finite Element (Adapted from Ref. 7)

To simplify matters for our example, let's assume all elements are the same size, so:

$$\Delta x = \Delta x^{(a)} = \Delta x^{(b)} = etc.$$

First, *for conduction only*, the expression reduces to:

$$\frac{\partial I}{\partial T_m} = \frac{\partial I^{(a)}}{\partial T_m} + \frac{\partial I^{(b)}}{\partial T_m} = \frac{kA}{\Delta x}\left(-T_{m-1} + 2T_m - T_{m+1}\right)$$



Example 4: One-Dimensional Steady State Finite Element (Adapted from Ref. 7)

The expression is different when a boundary is considered;

For our example, both the left and right boundary temperatures are the same, T_{bound} :

$$\left.\frac{\partial I}{\partial T_m}\right|_{left} = \frac{kA}{\Delta x}(-T_{bound} + 2T_m - T_{m+1})$$

$$\left.\frac{\partial I}{\partial T_m}\right|_{right} = \frac{kA}{\Delta x}(-T_{m-1} + 2T_m - T_{bound})$$

Example 4: One-Dimensional Steady State Finite Element (Adapted from Ref. 7)

We can perform a similar operation for heating and we arrive at:

$$\left.\frac{\partial I}{\partial T_m}\right|_{left} = \frac{\dot{q}_L^{(e)} \Delta x}{2}$$

$$\left.\frac{\partial I}{\partial T_m}\right|_{right} = \frac{\dot{q}_L^{(e)} \Delta x}{2}$$

As a result, half of the heating is applied to one node and the remaining half to the other.

Example 4: One-Dimensional Steady State Finite Element (Adapted from Ref. 7)

The system of equations in matrix form is given by:

$$\frac{kA}{\Delta x}\begin{bmatrix} \text{Not used} & & 0 & 0 & 0 & 0 \\ -1 & +2 & -1 & 0 & 0 & 0 \\ 0 & -1 & +2 & -1 & 0 & 0 \\ 0 & 0 & -1 & +2 & -1 & 0 \\ 0 & 0 & 0 & -1 & +2 & -1 \\ 0 & 0 & 0 & 0 & \text{Not used} & \end{bmatrix}\begin{Bmatrix} T_{b1} \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_{b6} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \dot{q}_L \Delta x/2 \\ \dot{q}_L \Delta x/2 \\ 0 \\ 0 \end{Bmatrix}$$

Example 4: One-Dimensional Steady State Finite Element (Adapted from Ref. 7)

But, before solving, we must apply the boundary conditions

$$\frac{kA}{\Delta x}\begin{bmatrix} & & 0 & 0 & 0 & 0 \\ -1 & +2 & -1 & 0 & 0 & 0 \\ 0 & -1 & +2 & -1 & 0 & 0 \\ 0 & 0 & -1 & +2 & -1 & 0 \\ 0 & 0 & 0 & -1 & +2 & -1 \\ 0 & 0 & 0 & 0 & & \end{bmatrix}\begin{Bmatrix} T_{b1} \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_{b6} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \dot{q}_L \Delta x/2 \\ \dot{q}_L \Delta x/2 \\ 0 \\ 0 \end{Bmatrix}$$

Example 4: One-Dimensional Steady State Finite Element (Adapted from Ref. 7)

Moving the terms associated with the boundary temperatures over to the right hand side yields a reduced matrix:

$$\frac{kA}{\Delta x}\begin{bmatrix} +2 & -1 & 0 & 0 \\ -1 & +2 & -1 & 0 \\ 0 & -1 & +2 & -1 \\ 0 & 0 & -1 & +2 \end{bmatrix}\begin{Bmatrix} T_2 \\ T_3 \\ T_4 \\ T_5 \end{Bmatrix} = \begin{Bmatrix} (kA/\Delta x)T_{b1} \\ \dot{q}_L \Delta x/2 \\ \dot{q}_L \Delta x/2 \\ (kA/\Delta x)T_{b6} \end{Bmatrix}$$

Example 4: One-Dimensional Steady State Finite Element

We see this is of the form:

$$[\boldsymbol{K}]\{\boldsymbol{T}\} = \{\boldsymbol{Q}\}$$

where...

$$[\boldsymbol{K}] = \frac{kA}{\Delta x}\begin{bmatrix} +2 & -1 & 0 & 0 \\ -1 & +2 & -1 & 0 \\ 0 & -1 & +2 & -1 \\ 0 & 0 & -1 & +2 \end{bmatrix} \quad \{\boldsymbol{T}\} = \begin{Bmatrix} T_2 \\ T_3 \\ T_4 \\ T_5 \end{Bmatrix} \quad \{\boldsymbol{Q}\} = \begin{Bmatrix} (kA/\Delta x)T_{b1} \\ \dot{q}_L \Delta x/2 \\ \dot{q}_L \Delta x/2 \\ (kA/\Delta x)T_{b6} \end{Bmatrix}$$

Example 4: One-Dimensional Steady State Finite Element

For our sample problem, consider 5 equally sized elements with 6 equally spaced nodes:

$L = 0.1$ m

$r = 0.01$ m

$k = 167$ W/mC

$T_{boundary} = 20$ C

$\dot{q}_L = 5000$ W/m

Example 4: One-Dimensional Steady State Finite Element

This leads to the following derived quantities:

$\Delta x = 0.02$ m

$A = 3.14159 \times 10^{-4}$ m^2

$\dot{q}_L \Delta x = 100$ W

$kA/\Delta x = 2.623$ W/C

$T_{b1} = T_{b6} = 20$ C

Example 4: One-Dimensional Steady State Finite Element

Filling in numbers:

$$[\boldsymbol{K}]\{\boldsymbol{T}\} = \{\boldsymbol{Q}\}$$

we get...

$$\begin{bmatrix} 5.246 & -2.623 & 0 & 0 \\ -2.623 & 5.246 & -2.623 & 0 \\ 0 & -2.623 & 5.246 & -2.623 \\ 0 & 0 & -2.623 & 5.246 \end{bmatrix} \begin{Bmatrix} T_2 \\ T_3 \\ T_4 \\ T_5 \end{Bmatrix} = \begin{Bmatrix} 52.465 \\ 50.000 \\ 50.000 \\ 52.465 \end{Bmatrix}$$

Example 4: One-Dimensional Steady State Finite Element

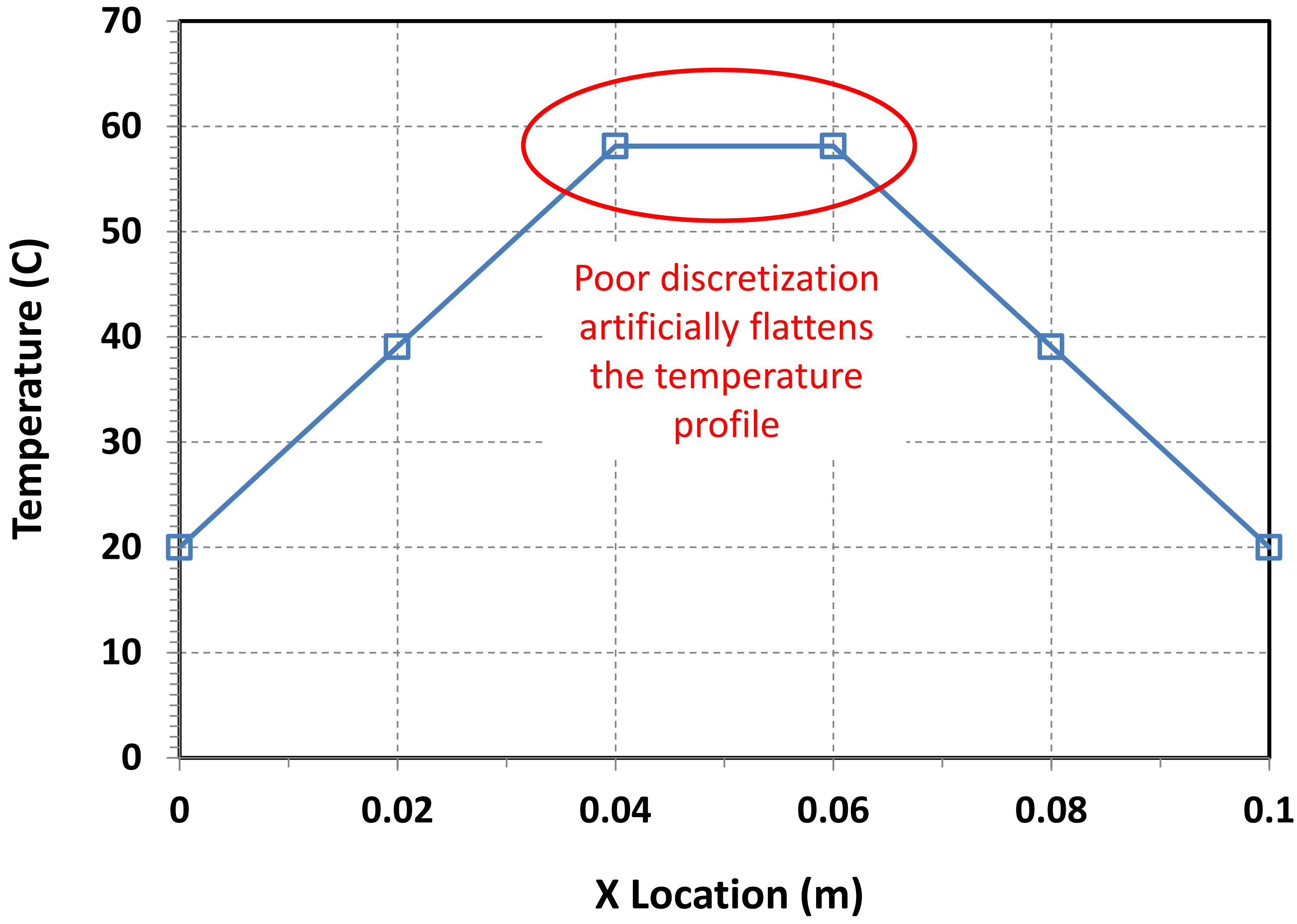
To solve for temperatures:

$$[\boldsymbol{K}]^{-1}\{\boldsymbol{Q}\} = \{\boldsymbol{T}\}$$

$$\begin{bmatrix} 0.305 & 0.229 & 0.152 & 0.076 \\ 0.229 & 0.457 & 0.305 & 0.152 \\ 0.152 & 0.305 & 0.457 & 0.229 \\ 0.076 & 0.152 & 0.229 & 0.305 \end{bmatrix} \begin{Bmatrix} 52.465 \\ 50.000 \\ 50.000 \\ 52.465 \end{Bmatrix} = \begin{Bmatrix} T_2 \\ T_3 \\ T_4 \\ T_5 \end{Bmatrix}$$

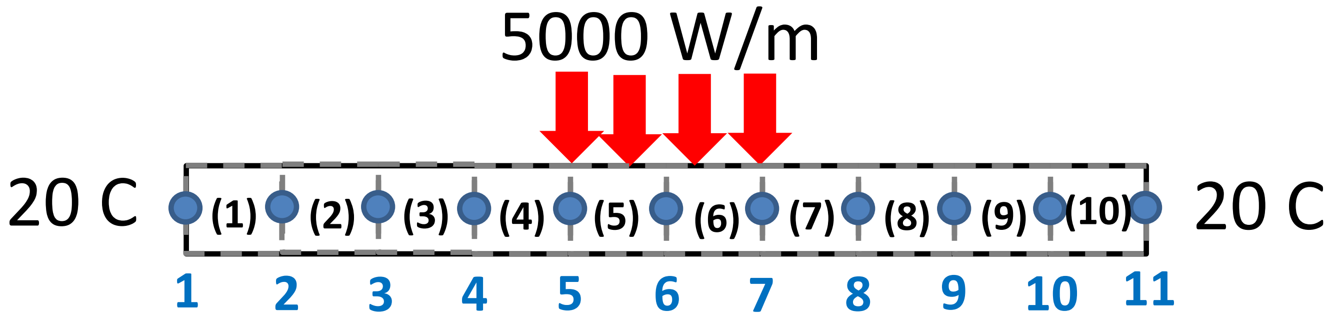
$$\begin{Bmatrix} T_{b1} \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_{b6} \end{Bmatrix} = \begin{Bmatrix} 20.000 \\ 39.060 \\ 58.121 \\ 58.121 \\ 39.060 \\ 20.000 \end{Bmatrix}$$

Example 4: Finite Element Solution



Example 4: Finite Element Solution Revisited

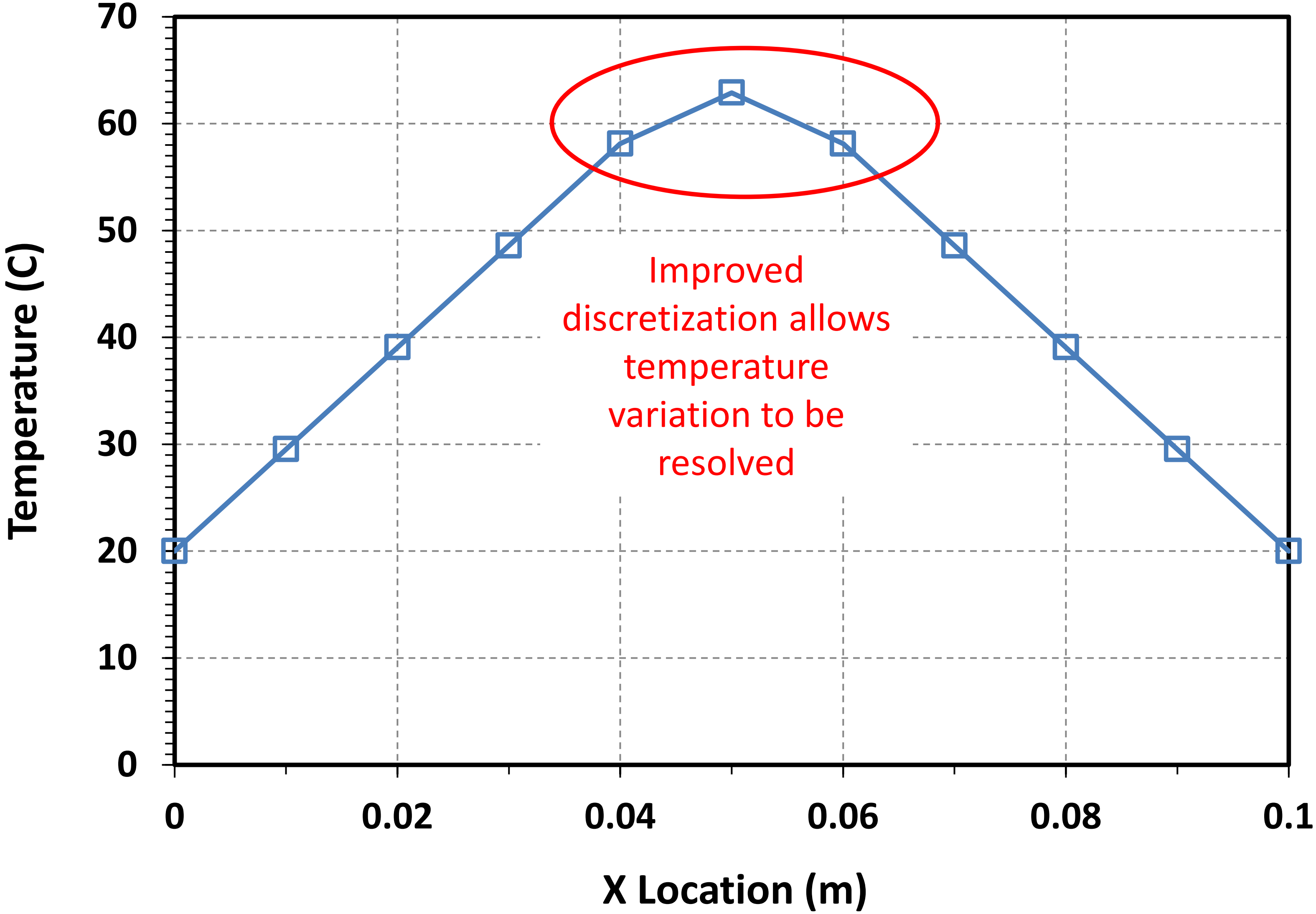
To fix the problem, adjust discretization*;



We seek a calculation point somewhere within the heated region to resolve local temperature differences;

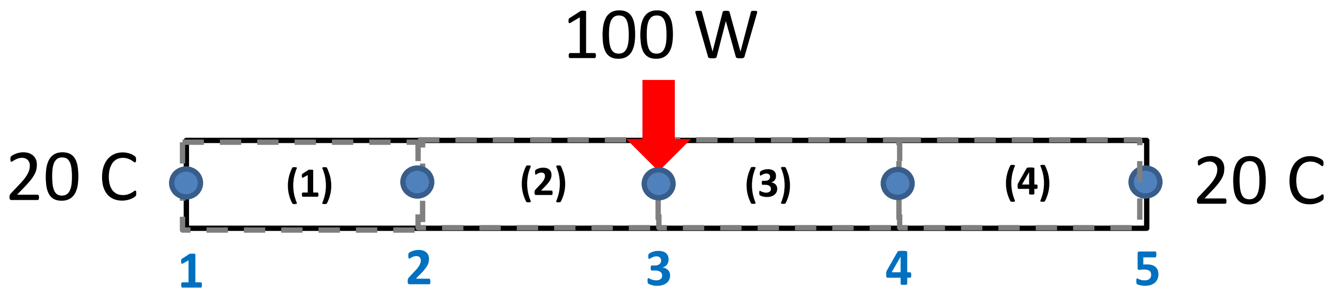
*Note: Discretization was increased across the entire rod to maintain a constant element size.

Example 4: Finite Element Solution Revisited



Example 5: One-Dimensional Steady State Finite Element

Let's solve the Example 2 geometry using the finite element method;



Instead of a distributed heat load (as was the case in Example 4), all heating is applied at the center (i.e., Node 3).

Example 5: One-Dimensional Steady State Finite Element

The system of equations in matrix form is given by:

$$\frac{kA}{\Delta x}\begin{bmatrix} \text{Not used} & & 0 & 0 & 0 \\ -1 & +2 & -1 & 0 & 0 \\ 0 & -1 & +2 & -1 & 0 \\ 0 & 0 & -1 & +2 & -1 \\ 0 & 0 & 0 & \text{Not used} & \end{bmatrix}\begin{Bmatrix} T_{b1} \\ T_2 \\ T_3 \\ T_4 \\ T_{b5} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \dot{Q} \\ 0 \\ 0 \end{Bmatrix}$$

Example 5: One-Dimensional Steady State Finite Element

But, before solving, we must apply the boundary conditions

$$\frac{kA}{\Delta x}\begin{bmatrix} \multicolumn{2}{c}{\text{Not used}} & 0 & 0 & 0 \\ -1 & +2 & -1 & 0 & 0 \\ 0 & -1 & +2 & -1 & 0 \\ 0 & 0 & -1 & +2 & -1 \\ 0 & 0 & 0 & \multicolumn{2}{c}{\text{Not used}} \end{bmatrix}\begin{Bmatrix} T_{b1} \\ T_2 \\ T_3 \\ T_4 \\ T_{b5} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \dot{Q} \\ 0 \\ 0 \end{Bmatrix}$$

Example 5: One-Dimensional Steady State Finite Element

Moving the terms associated with the boundary temperatures over to the right hand side yields the reduced matrix:

$$\frac{kA}{\Delta x}\begin{bmatrix} +2 & -1 & 0 \\ -1 & +2 & -1 \\ 0 & -1 & +2 \end{bmatrix}\begin{Bmatrix} T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} (kA/\Delta x)T_{b1} \\ \dot{Q} \\ (kA/\Delta x)T_{b5} \end{Bmatrix}$$

Example 5: One-Dimensional Steady State Finite Element

Consider 4 equally spaced elements with 5 nodes:

$L = 0.1$ m

$r = 0.01$ m

$k = 167$ W/mC

$T_{boundary} = 20$ C

$\dot{Q} = 100$ W

Example 5: One-Dimensional Steady State Finite Element

This leads to the following derived quantities:

$\Delta x = 0.025$ m

$A = 3.14159 \times 10^{-4}$ m^2

$\dot{Q} = 100$ W

$kA/\Delta x = 2.099$ W/C

$T_{b1} = T_{b5} = 20$ C

Example 5: One-Dimensional Steady State Finite Element

Filling in numbers:

$$[\boldsymbol{K}]\{\boldsymbol{T}\} = \{\boldsymbol{Q}\}$$

we get...

$$\begin{bmatrix} 4.197 & -2.099 & 0 \\ -2.099 & 4.197 & -2.099 \\ 0 & -2.099 & 4.197 \end{bmatrix} \begin{Bmatrix} T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} 41.972 \\ 100.000 \\ 41.972 \end{Bmatrix}$$

Example 5: One-Dimensional Steady State Finite Element

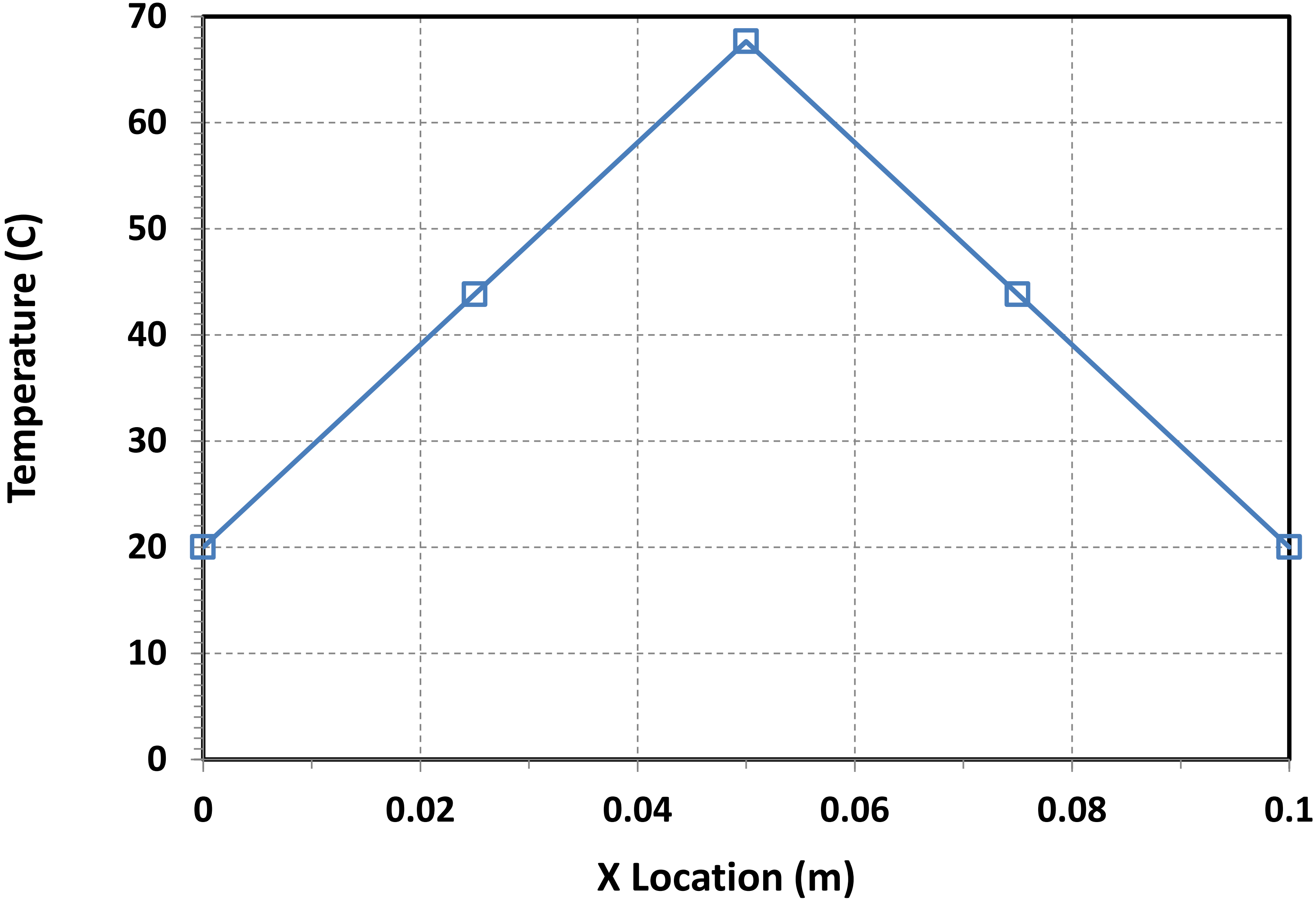
To solve for temperatures:

$$[\boldsymbol{K}]^{-1}\{\boldsymbol{Q}\} = \{\boldsymbol{T}\}$$

$$\begin{bmatrix} 0.357 & 0.238 & 0.119 \\ 0.238 & 0.477 & 0.238 \\ 0.119 & 0.238 & 0.357 \end{bmatrix} \begin{Bmatrix} 41.972 \\ 100.000 \\ 41.972 \end{Bmatrix} = \begin{Bmatrix} T_2 \\ T_3 \\ T_4 \end{Bmatrix}$$

$$\begin{Bmatrix} T_{b1} \\ T_2 \\ T_3 \\ T_4 \\ T_{b5} \end{Bmatrix} = \begin{Bmatrix} 20.000 \\ 43.826 \\ 67.651 \\ 43.826 \\ 20.000 \end{Bmatrix}$$

Example 5: Finite Element Solution



Example 6: One-Dimensional Transient Finite Element (Adapted from Ref. 7)

Let's look at the same problem geometry but, this time, we'll model the transient response.

In this case, we seek to minimize an integral expression considering the effects of conduction (I_k), applied heating (I_q) and capacitance (I_c):

$$I = I_k + I_c + I_q$$

Example 6: One-Dimensional Transient Finite Element (Adapted from Ref. 7)

The individual components are:

$$I_k = \int_{x=0}^{L} \frac{1}{2} kA(T')^2 dx$$

$$I_q = \int_{x=0}^{L} \dot{Q} T dx$$

$$I_c = \frac{\rho C_p}{2} \int_{x=0}^{L} \frac{\partial (T^2)}{\partial \theta} dx$$

where $\dot{Q}$ is heating rate and θ is time.

Example 6: One-Dimensional Transient Finite Element (Adapted from Ref. 7)

And the overall integral to be minimized is:

$$I = \int_{x=0}^{L} \left[\frac{1}{2} kA(T')^2 + \dot{Q}T + \frac{\rho C_p}{2} \frac{\partial (T^2)}{\partial \theta} \right] dx$$

We already have expressions for the conduction and heating so let's focus on the capacitance term.

$$I_c = \frac{\rho C_p}{2} \int_{x=0}^{L} \frac{\partial (T^2)}{\partial \theta} dx$$

Example 6: One-Dimensional Transient Finite Element (Adapted from Ref. 7)

Recall, from Example 4, our expression for temperature as a function of location in an element:

$$T^{(e)}(x) = c_1^{(e)} + c_2^{(e)} x$$

And because we know the temperatures at the ends, we solved for constants c_1 and c_2:

$$c_1 = \frac{x_j T_i - x_i T_j}{x_j - x_i} \qquad c_2 = \frac{T_j - T_i}{x_j - x_i}$$

Example 6: One-Dimensional Transient Finite Element (Adapted from Ref. 7)

So, for an individual element, the integral becomes:

$$I_c^{(e)} = \frac{\rho C_p}{2} \frac{d}{d\theta} \int_{x_i}^{x_j} \left(c_1^{(e)} + c_2^{(e)} x \right)^2 dx$$

After substituting in expressions for c_1 and c_2, the expression inside the integral is a function of, only, T_i, T_j, x_i, x_j and x.

Example 6: One-Dimensional Transient Finite Element (Adapted from Ref. 7)

We differentiate the expression with respect to T_i and then integrate over x to obtain:

$$\frac{dI_c^{(e)}}{dT_i} = \frac{\rho C_p}{6} \frac{d}{d\theta} \left(x_j - x_i\right)\left(2T_i + T_j\right)$$

Similarly, we perform the differentiation with respect to T_j and integrate, we obtain:

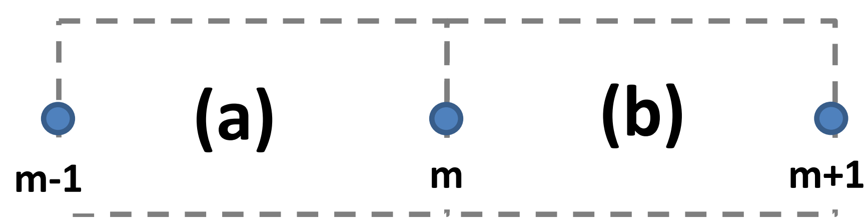
$$\frac{dI_c^{(e)}}{dT_j} = \frac{\rho C_p}{6} \frac{d}{d\theta} \left(x_j - x_i\right)\left(T_i + 2T_j\right)$$

Example 6: One-Dimensional Transient Finite Element (Adapted from Ref. 7)

So, for an element, we see that...

$$\frac{dI_c^{(a)}}{dT_m} = \frac{\rho C_p}{6}\frac{d}{d\theta}(x_m - x_{m-1})(T_{m-1} + 2T_m)$$

$$\frac{dI_c^{(b)}}{dT_m} = \frac{\rho C_p}{6}\frac{d}{d\theta}(x_{m+1} - x_m)(2T_m + T_{m+1})$$



Example 6: One-Dimensional Transient Finite Element (Adapted from Ref. 7)

But...

$$\Delta x = (x_m - x_{m-1}) = (x_{m+1} - x_m)$$

and over a specified time interval, $\Delta\theta$, gives us...

$$\frac{dI_c^{(a)}}{dT_m} = \frac{\rho C_p}{6\Delta\theta}\Delta x(T_{m-1} + 2T_m)$$

$$\frac{dI_c^{(b)}}{dT_m} = \frac{\rho C_p}{6\Delta\theta}\Delta x(2T_m + T_{m+1})$$

Example 6: One-Dimensional Transient Finite Element (Adapted from Ref. 7)

The previous expressions form the basis for our element capacitance matrix:

$$[C]^{(e)} = \frac{\rho C_p \Delta x}{6\Delta\theta}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Example 6: One-Dimensional Transient Finite Element

But we can assemble the element matrices into a global capacitance matrix through superposition;

For similarly sized elements, the matrix becomes:

$$[C] = \frac{\rho C_p \Delta x}{6\Delta\theta}\begin{bmatrix} 2 & 1 & & & \\ 1 & 2+2 & 1 & & \\ & 1 & 2+2 & 1 & \\ & & 1 & 2+2 & 1 \\ & & & 1 & 2 \end{bmatrix}$$

Example 6: One-Dimensional Transient Finite Element

With the previously derived conduction matrix, we can now assemble our system of equations:

$$-kA\frac{\partial^2 T}{\partial x^2} + \dot{Q} = \rho C_p \frac{\partial T}{\partial t}$$

Becomes...

$$-[\boldsymbol{K}]\{\boldsymbol{T}\} + \{\dot{\boldsymbol{Q}}\} = [\boldsymbol{C}]\{\dot{\boldsymbol{T}}\}$$

Example 6: One-Dimensional Transient Finite Element (Ref. 9)

This equation can be expressed in a more general form :

$$
\begin{aligned}
&([C] + \phi \Delta\theta [K])\{T^{n+1}\} \\
&\qquad = ([C] - (1-\phi)\Delta\theta[K])\{T^{n}\} \\
&\qquad + \Delta\theta\big((1-\phi)\{Q^{n}\} + \phi\{Q^{n+1}\}\big)
\end{aligned}
$$

so when...

ϕ = 0, we have the Forward difference

ϕ = 1/2, we have Crank-Nicolson

ϕ = 2/3, we have Galerkin

ϕ = 1, we have the Backward difference

Example 6: One-Dimensional Transient Finite Element (Ref. 9)

Let's use Backward differencing to solve this problem, i.e., $\phi = 1$;

When we substitute into the general equation, we arrive at:

$$([C] + \Delta\theta[K])\{T^{n+1}\} = [C]\{T^{n}\} + \Delta\theta\{Q^{n+1}\}$$

Our unknowns are the temperatures at time *n+1*, shown in blue.

Example 6: One-Dimensional Transient Finite Element (Ref. 9)

We recognize this equation is of the form:

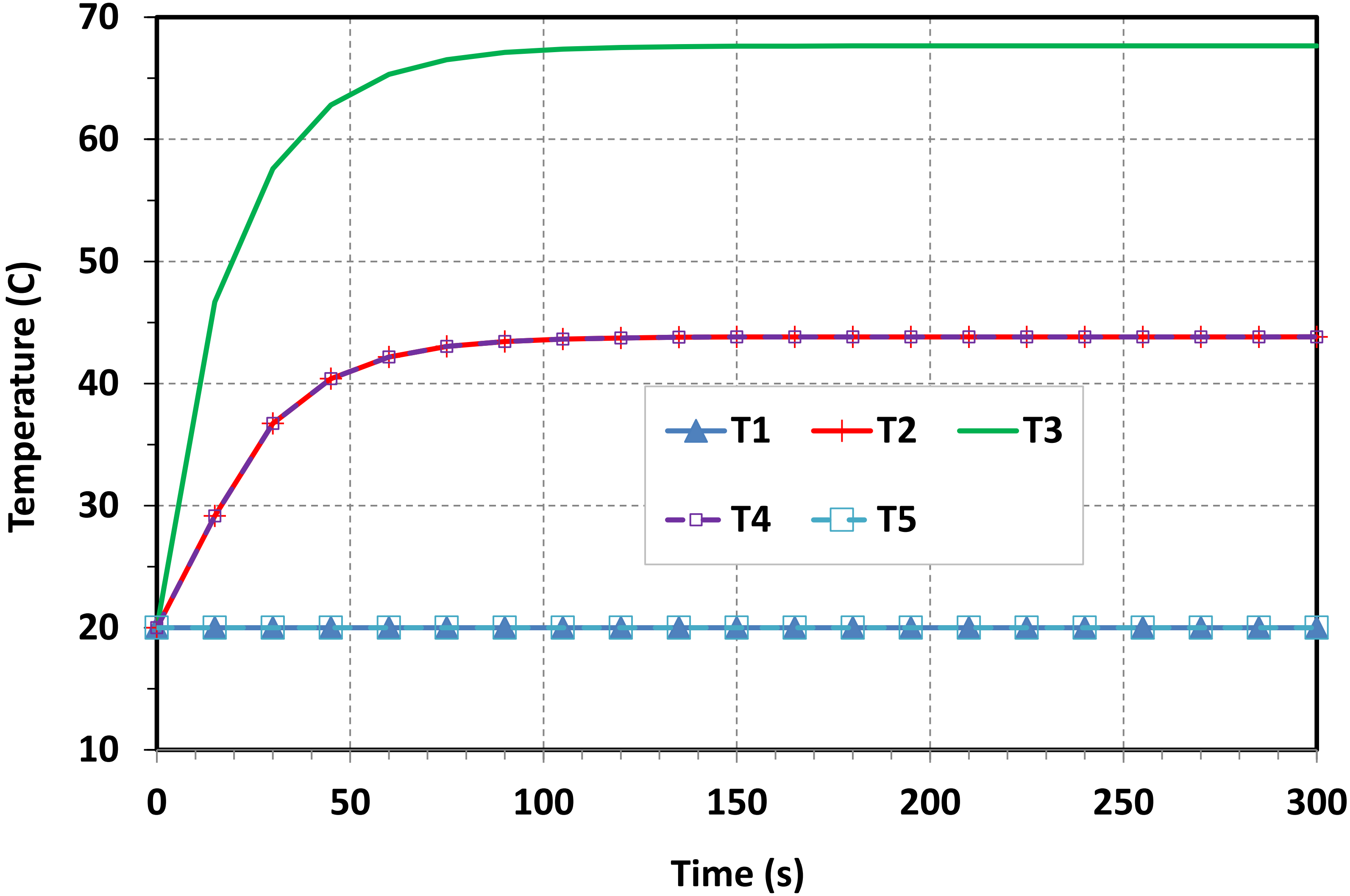
$$[\boldsymbol{A}]\{\boldsymbol{T}^{n+1}\} = \{\boldsymbol{B}\}$$

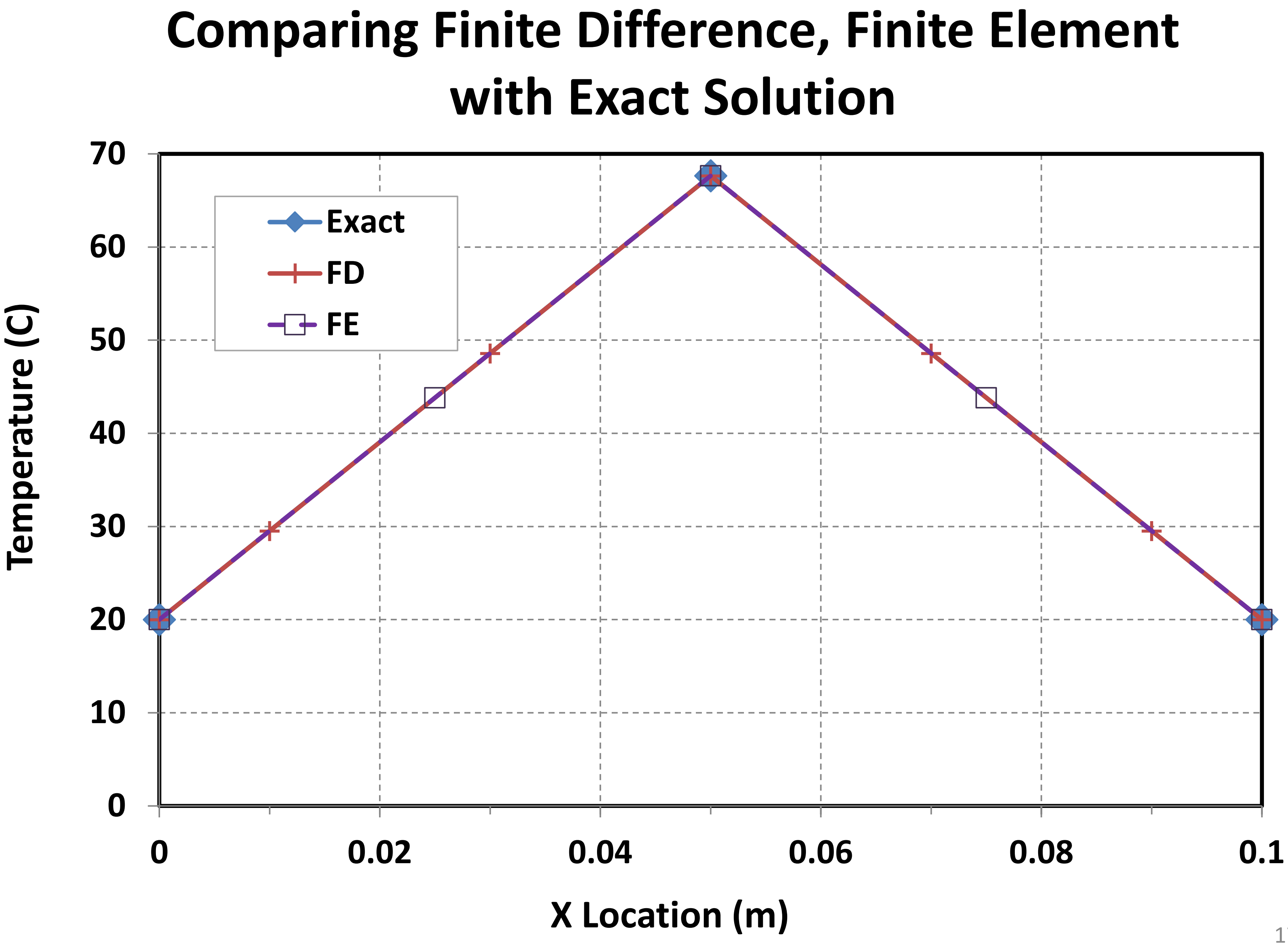
Inverting **[*A*]** and multiplying it with **{*B*}** yields:

$$\{\boldsymbol{T}^{n+1}\} = [\boldsymbol{A}]^{-1}\{\boldsymbol{B}\}$$

But we must remember to apply the boundary conditions as we did before.

Example 6: One-Dimensional Transient Finite Element




Comparing Finite Difference, Finite Element
with Exact Solution
Exact
FD
FE
Temperature (C)
X Location (m)
0
10
20
30
40
50
60
70
0
0.02
0.04
0.06
0.08
0.1

Conclusion

An overview of numerical methods in heat transfer has been presented;

The governing differential equation was formulated from first principles;

The finite difference was developed and demonstrated through examples;

Finite element was developed and demonstrated through examples.

Acknowledgements

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Microsoft® Clip Art was used in this presentation.

Wolfram Mathematica® was used for some calculations in this presentation.

For Additional Information

Variational Principles

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