Robust Control Design and Analysis

NASA Workshop: Winter 2025

Lecture 4: Robust Stability With Unstructured Uncertainty

Key Takeaways

This lecture begins with a review of the Nyquist stability theorem.

We use the Nyquist condition to prove the SISO small gain theorem.

- This provides a necessary and sufficient condition to prove robust stability in terms of the gain (H_∞ norm) of the nominal system.
- The proof constructs a destabilizing uncertainty that can be studied further in nonlinear simulations.

The small gain theorem is stated for a generic feedback loop.

It can be adapted to derive robust stability conditions for many classes of uncertainty.

We then generalize the small gain theorem to MIMO systems.

Nyquist Stability Theorem

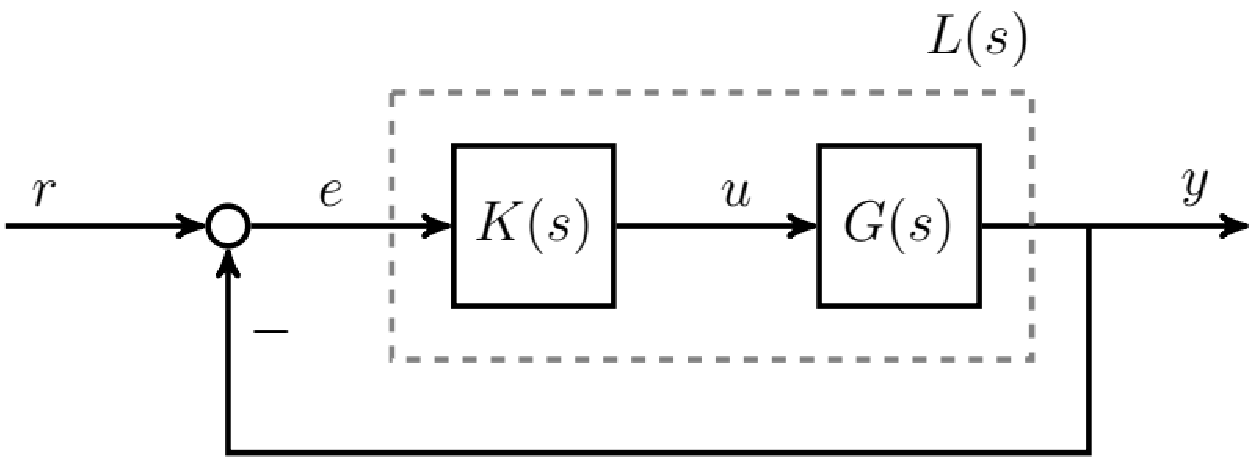
Course Notes: Section 7.6

Critical -1 Point

The transfer function *L(s) = G(s)K(s)* is called the (open) loop transfer function.

If the Nyquist curve of $L(s)$ passes through the critical point $s = -1$ then the closed-loop is unstable.

- Suppose $L(j\omega_0)=-1$ at some frequency ω_0. Hence $1+L(j\omega_0)=0$.
- The sensitivity $S(s) = \frac{1}{1+L(s)}$ has a pole on the imaginary axis at $s=j\omega_0$.



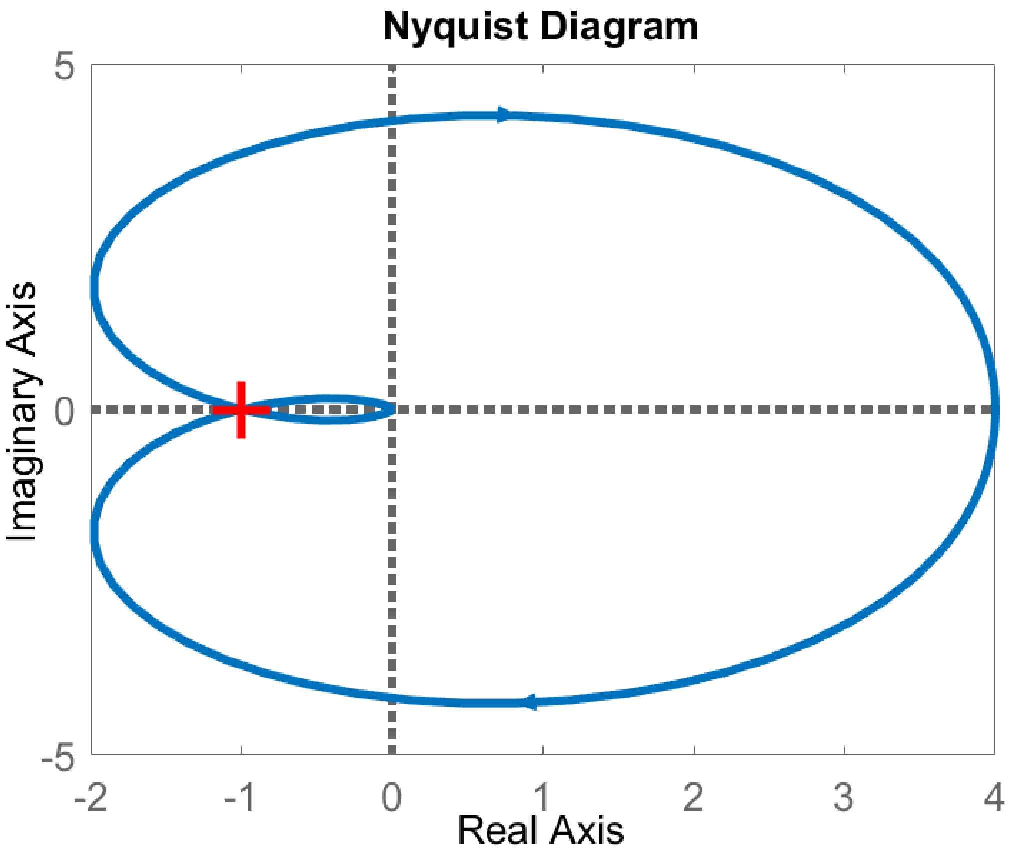
Example

```
>> G = tf(4,[1 2.0407 4]);
>> K = tf(20,[1 5]);
>> L = G*K;
>> nyquist(L);

>> S=feedback(1,L);
>> pole(S)
ans =
   -7.0407 + 0.0000i
   -0.0000 + 3.7687i
   -0.0000 - 3.7687i
>> evalfr(L,1j*3.7687)
ans =
   -1.0000 - 0.0000i
```

$$G(s) = \frac{4}{s^2+2.0407s+4}$$

$$K(s) = \frac{20}{s+5}$$



Nyquist Theorem

Notation:

- P_{CL}: Number of poles of the closed-loop in the CRHP.
- P_{OL}: Number of poles of the open-loop $L(s)$ in the CRHP.
- N_{CCW}: This denotes the number of times the Nyquist curve of L(s) encircles the critical −1 point. $N_{CCW}>0$ for counterclockwise (CCW) encirclements and $N_{CCW}<0$ for clockwise (CW) encirclements.

Nyquist Theorem: Assume $L(s)=G(s)K(s)$ has no pole/zero cancellations in the CRHP and no poles on the imaginary axis. Then

$$P_{CL} = P_{OL} - N_{CCW}.$$

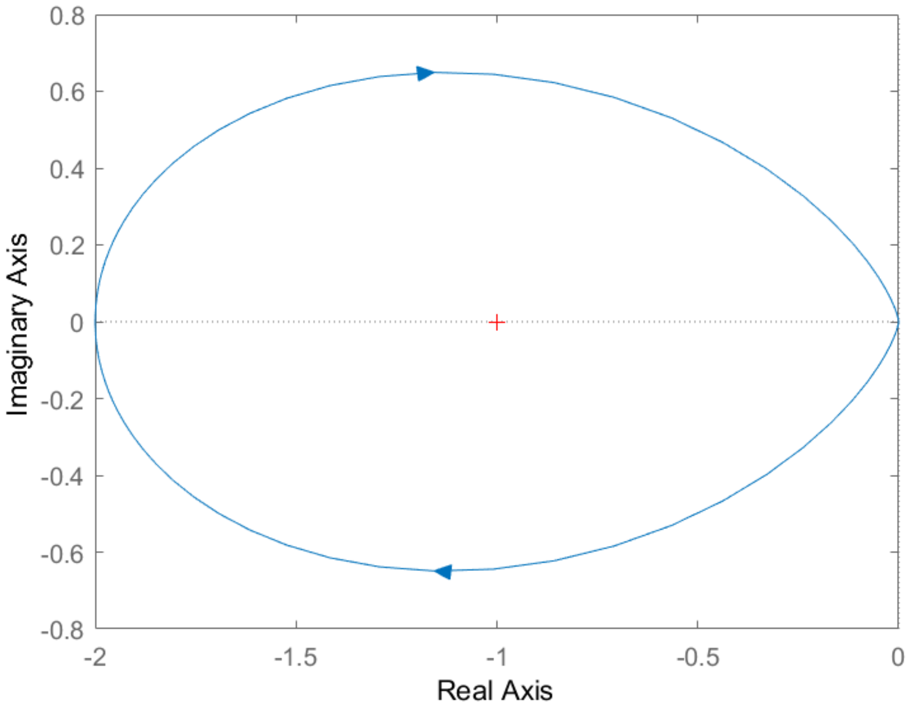
The closed-loop is stable ($P_{CL} = 0$) if and only if $N_{CCW} = P_{OL}$.

Benefit: Closed-loop stability can be determined from a Nyquist plot of the open loop transfer function L(s).

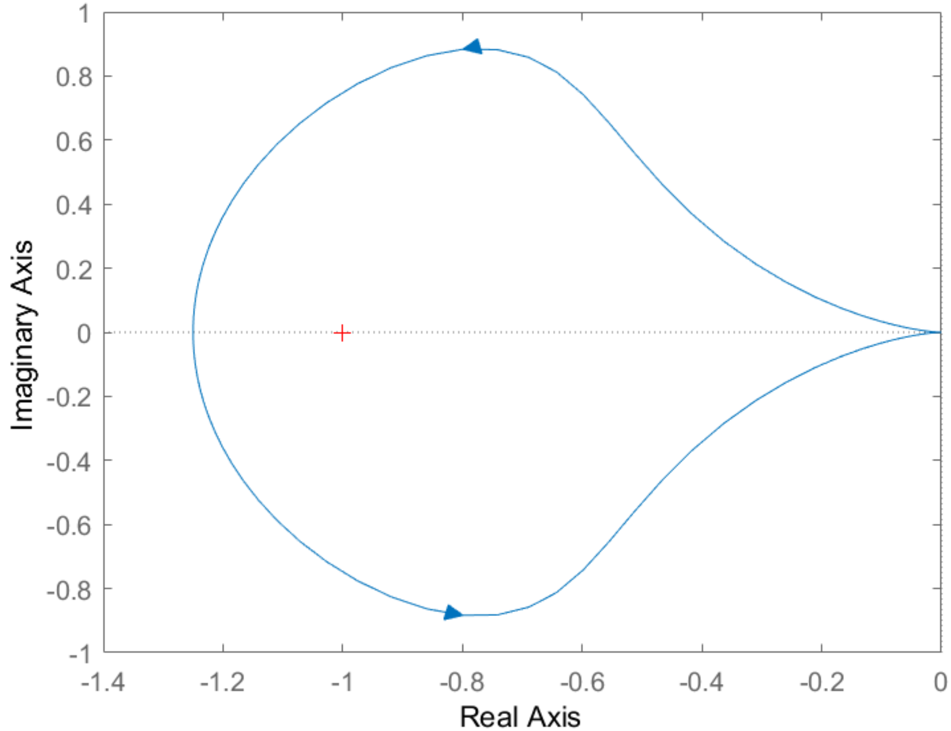
Problem

Consider a standard closed-loop system with the loop transfer function L(s). Apply the Nyquist stability theorem to predict the number of closed loop poles of the feedback system in the right-half plane. Is the closed-loop system stable or unstable?

A) $L(s) = \frac{20}{s-10}\frac{100}{s^2+20s+100}$



B) $L(s) = \frac{50}{s+5}\frac{s+2}{s^2+4s-16}$



Solution A

Apply the Nyquist stability theorem to predict the number of closed loop poles of the feedback system in the right-half plane. Is the closed-loop system stable or unstable?

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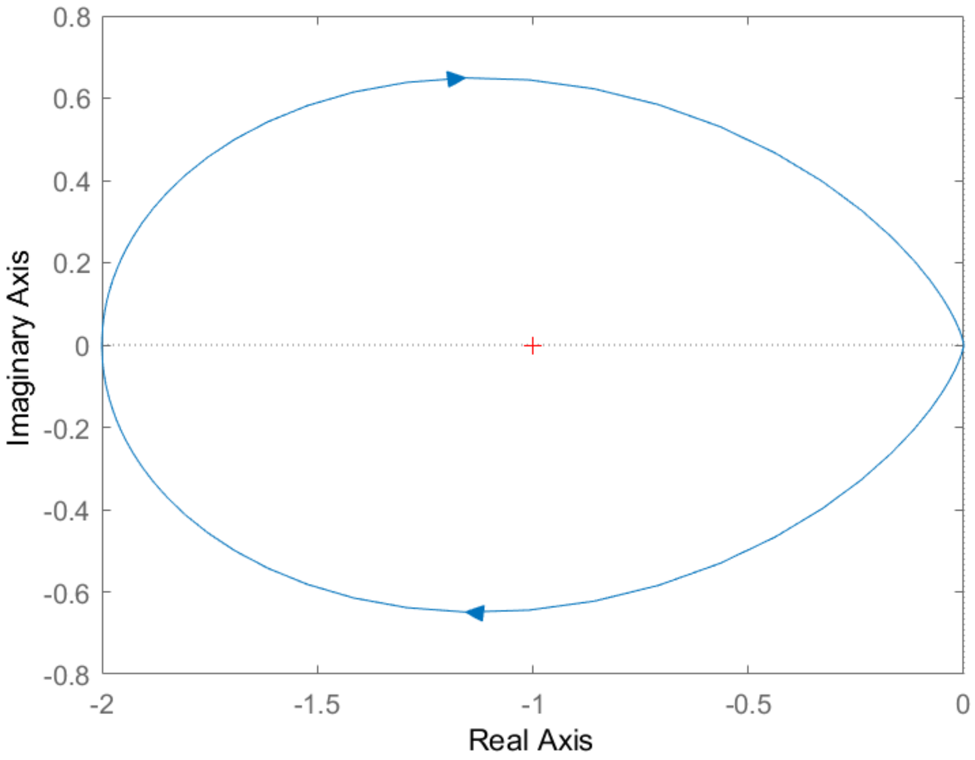
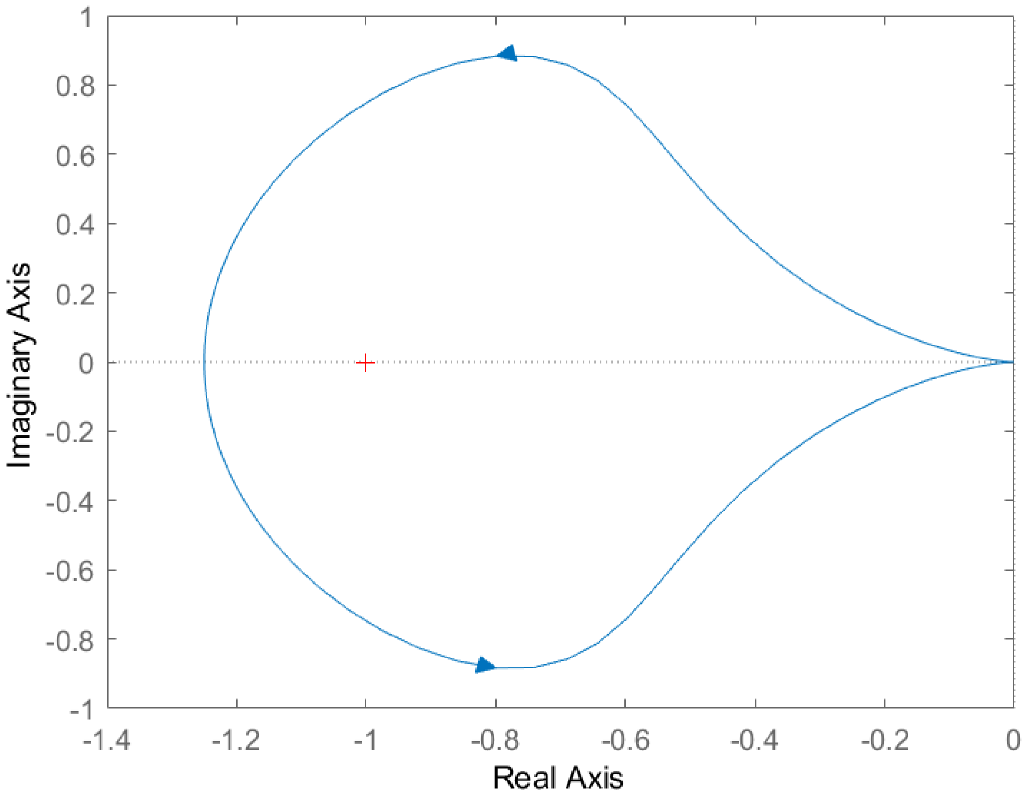


Solution B

Apply the Nyquist stability theorem to predict the number of closed loop poles of the feedback system in the right-half plane. Is the closed-loop system stable or unstable?

$$\text{B)}\ L(s) = \frac{50}{s+5}\frac{s+2}{s^2+4s-16}$$



Solution-Extra Space

SISO Small Gain Condition

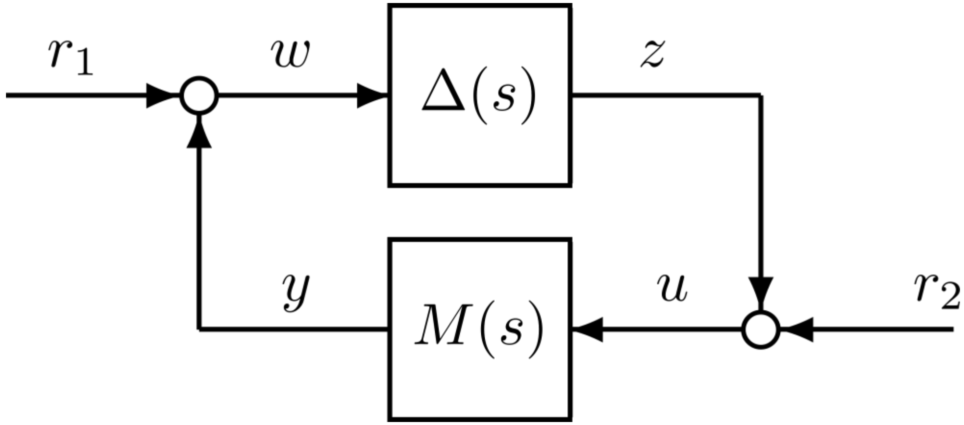
SISO Small Gain Condition

Consider the *positive* feedback system below where:

- $M(s)$ is a known stable, SISO, LTI system.
- $\Delta(s)$ is an unknown SISO, LTI system, i.e. it is uncertain. Also assume that $\Delta(s)$ is stable and is norm-bounded:

$$||\Delta||_\infty := \max_{\omega \in \mathbb{R} \cup \{+\infty\}} |\Delta(j\omega)| < 1$$

We will develop a simple condition to prove stability of the feedback system.



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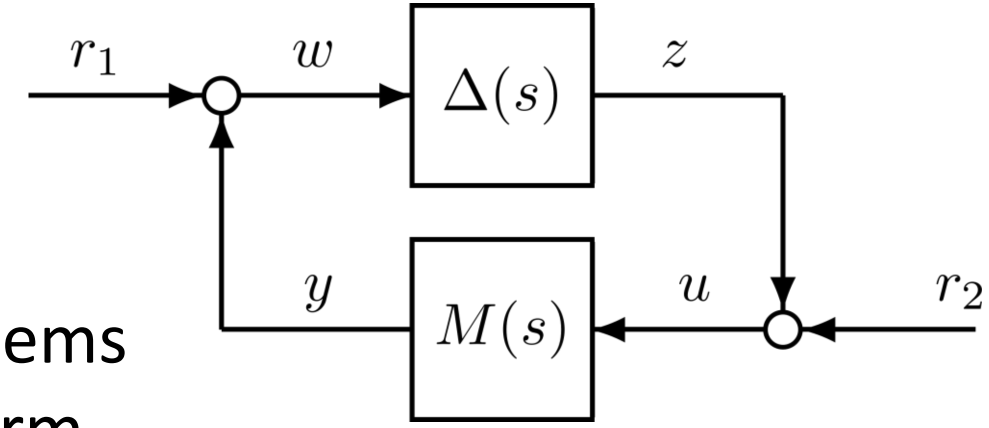
We will develop a simple condition to prove stability of the feedback system.

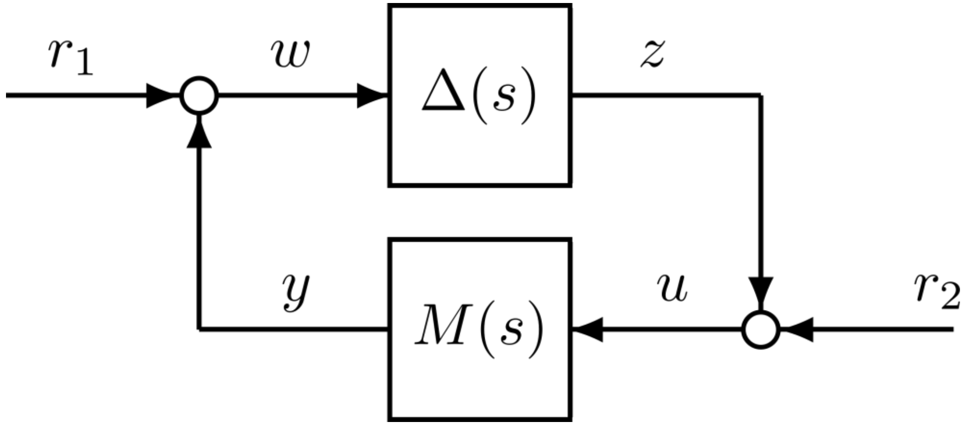
The feedback diagram is in a general, abstracted form. We will show that many robust stability problems can be converted to this form.



SISO Small Gain Condition

Theorem: Consider the positive feedback system below where $M(s)$ is stable.

A) If $||\mathrm{M}||_{\infty} \leq 1$ then the feedback system is stable for all $\Delta(s)$ that are stable and norm-bounded $||\Delta||_{\infty} < 1$.



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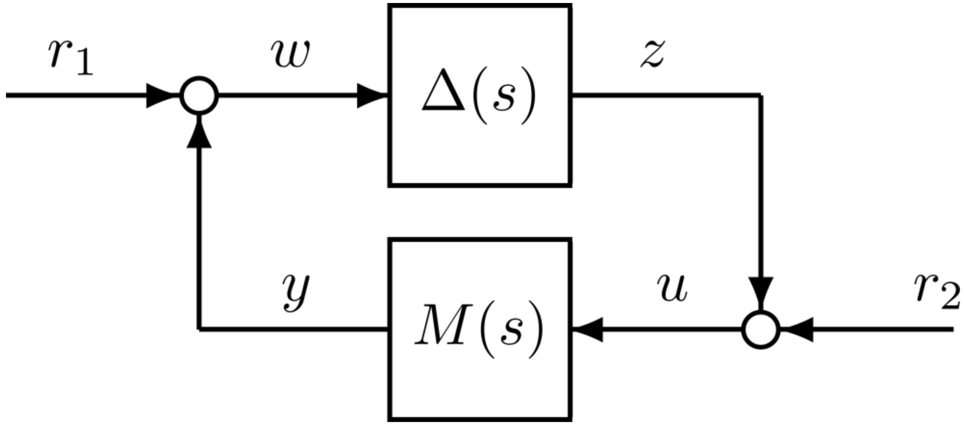
A) If $||\mathrm{M}||_{\infty} \leq 1$ then the feedback system is stable for all $\Delta(s)$ that are stable and norm-bounded $||\Delta||_{\infty} < 1$.

B) If $||\mathrm{M}||_{\infty} > 1$ then there is a stable $\Delta(s)$ with $||\Delta||_{\infty} < 1$ such that the feedback system is unstable.



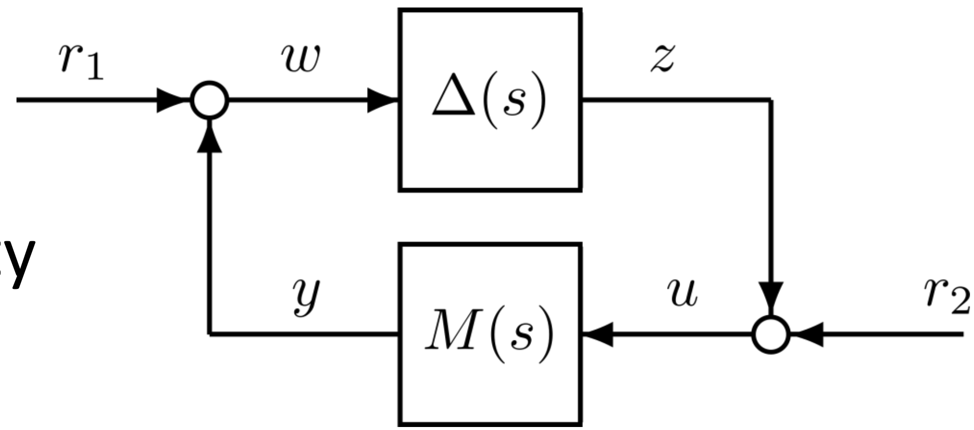
SISO Small Gain Condition

Theorem: Consider the positive feedback system below where $M(s)$ is stable.

A) If $||M||_\infty \leq 1$ then the feedback system is stable for all $\Delta(s)$ that are stable and norm-bounded $||\Delta||_\infty < 1$.

B) If $||M||_\infty > 1$ then there is a stable $\Delta(s)$ with $||\Delta||_\infty < 1$ such that the feedback system is unstable.

Thus $||M||_\infty \leq 1$ is a necessary and sufficient condition for robust stability with respect to $||\Delta||_\infty < 1$.



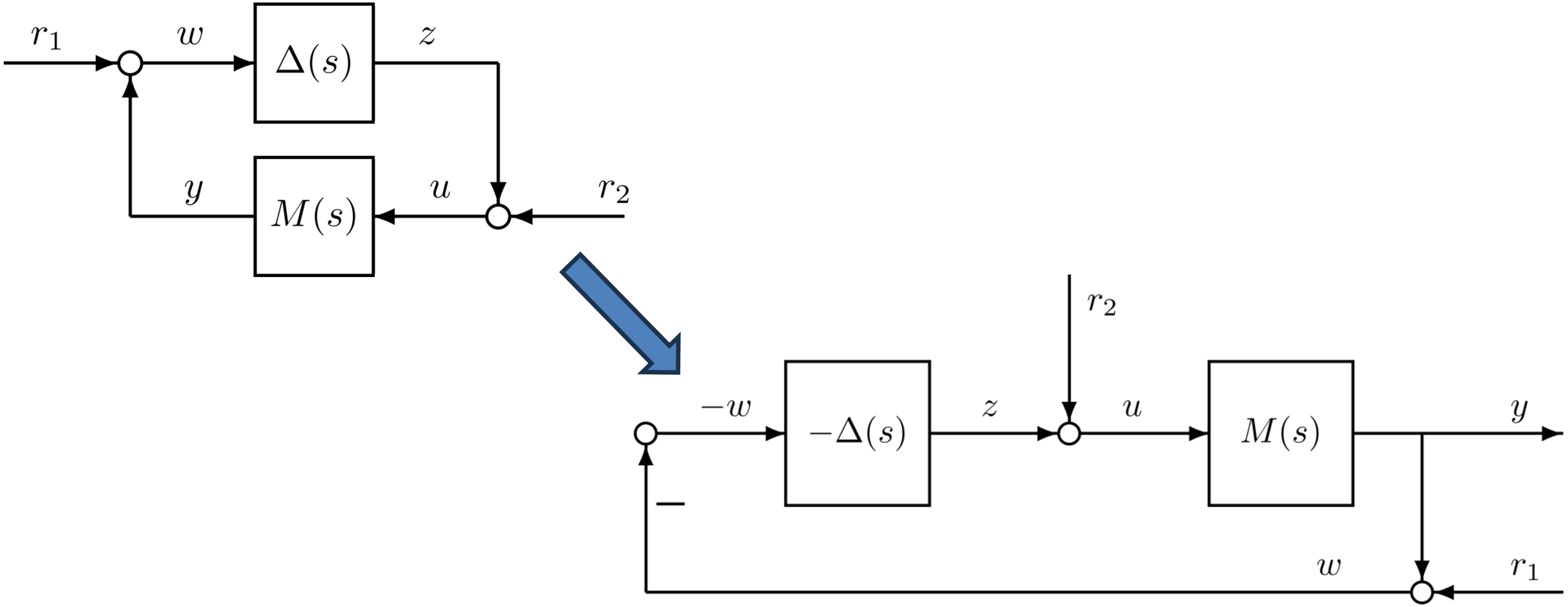
SISO Small Gain Condition

Consider the positive feedback system where $M(s)$ is stable.

A) If $||M||_\infty \leq 1$ then the feedback system is stable for all $\Delta(s)$ that are stable and norm-bounded $||\Delta||_\infty < 1$.

Proof Sketch:

1. Express as a negative feedback with loop $L(s) = -M(s)\Delta(s)$.



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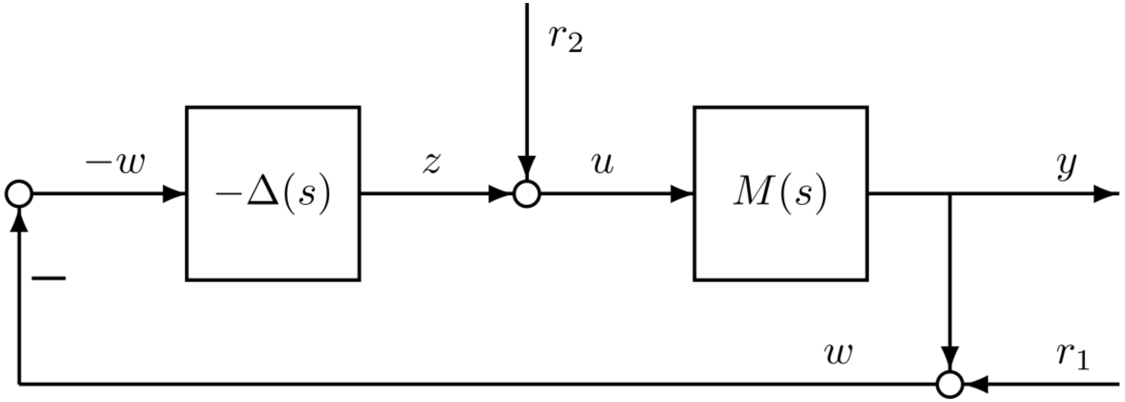
Proof Sketch:

1. Express as a negative feedback with loop $L(s) = -M(s)\Delta(s)$.

2. Both $M(s)$ and $\Delta(s)$ are stable so $L(s)$ is also stable.



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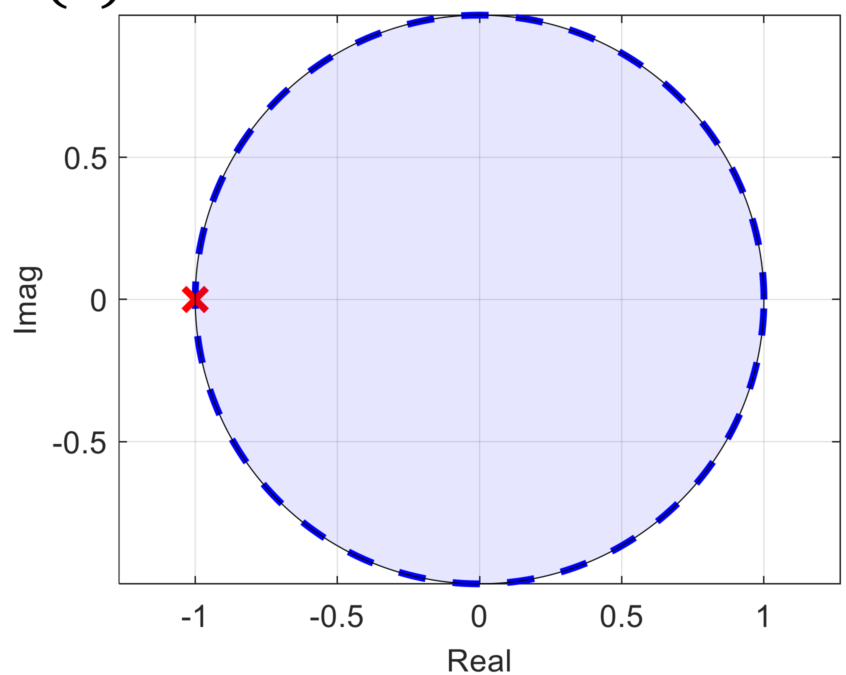
Proof Sketch:

1. Express as a negative feedback with loop $L(s) = -M(s)\Delta(s)$.
2. Both $M(s)$ and $\Delta(s)$ are stable so $L(s)$ is also stable.
3. The loop gain satisfies $|L(j\omega)| = |M(j\omega)| \cdot |\Delta(j\omega)| < 1$. The Nyquist plot lies strictly within the unit circle and hence it does not encircle $s = -1$.



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3. The loop $L(s)$ does not encircle $s = -1$.
4. The closed-loop is stable by the Nyquist theorem.

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Note: The conditions $||\mathrm{M}||_\infty \leq 1$ and $||\Delta||_\infty < 1$ ensure the loop gain is < 1 in step 3. Hence this is a "small gain" condition.

SISO Small Gain Condition

Consider the positive feedback system where $M(s)$ is stable.

B) If $||\mathrm{M}||_\infty > 1$ then there is a stable $\Delta(s)$ with $||\Delta||_\infty < 1$ such that the feedback system is unstable.

Proof Sketch:

1. $||\mathrm{M}||_\infty > 1 \Rightarrow$ There is $0 < \omega_0 < \infty$ such that $|M(j\omega_0)| > 1$.

SISO Small Gain Condition

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Proof Sketch:

1. $||\mathrm{M}||_\infty > 1 \Rightarrow$ There is $0 < \omega_0 < \infty$ such that $|M(j\omega_0)| > 1$.
2. Define the complex number $\delta_0 := \frac{1}{M(j\omega_0)}$. Note that $|\delta_0| < 1$.

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2. Define the complex number $\delta_0 := \frac{1}{M(j\omega_0)}$. Note that $|\delta_0| < 1$.
3. Construct a stable LTI system $\Delta(s)$ such that $||\Delta||_\infty \leq |\delta_0| < 1$ and $\Delta(j\omega_0) = \delta_0$.

- Can be done with a first-order system for $\Delta(s)$. [See Lemma S2 in "An Intro to Disk Margins", Seiler, Packard, Gahinet.]
- Can also be done with a gain+delay: $\Delta(s) = c_0\, e^{-s\tau_0}$ where $c_0 = |\delta_0|$ and $\tau_0 = -\frac{\angle\delta_0}{\omega_0}$. We have $\tau_0 > 0$ by choosing $\angle\delta_0 \in [-2\pi, 0)$.

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4. The loop (viewed in negative feedback) is $L(s) = -M(s)\Delta(s)$. This satisfies $L(j\omega_0) = -M(j\omega_0)\Delta(j\omega_0) = -1$. Recall the closed-loop is unstable when the loop intersects the critical -1 point.

SISO Small Gain Condition

Consider the positive feedback system where $M(s)$ is stable.

B) If $||\mathrm{M}||_\infty > 1$ then there is a stable $\Delta(s)$ with $||\Delta||_\infty < 1$ such that the feedback system is unstable.

The proof constructs a specific $\Delta(s)$ that causes instability. This uncertainty can be used in high-fidelity, nonlinear simulations for further investigation.

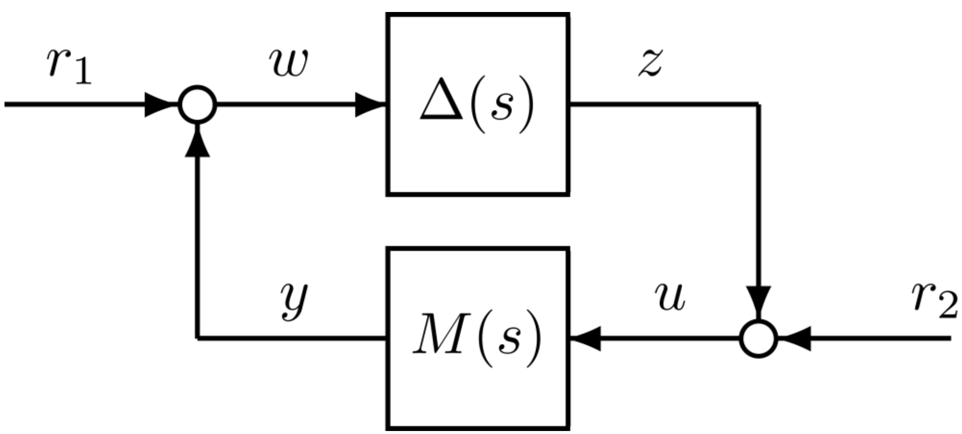
SISO Small Gain Condition

Theorem: Consider the positive feedback system below where $M(s)$ is stable. Let $m > 0$ be a given constant.

A) If $||\mathrm{M}||_\infty \leq \frac{1}{m}$ then the feedback system is stable for all $\Delta(s)$ that are stable and is norm-bounded $||\Delta||_\infty < m$.

B) If $||\mathrm{M}||_\infty > \frac{1}{m}$ then there is a stable $\Delta(s)$ with $||\Delta||_\infty < m$ such that the feedback system is unstable.

Thus, $m_{max} := \frac{1}{||M||_\infty}$ is the stability margin. The feedback is stable for all stable $\Delta(s)$ with $||\Delta||_\infty < m_{max}$.

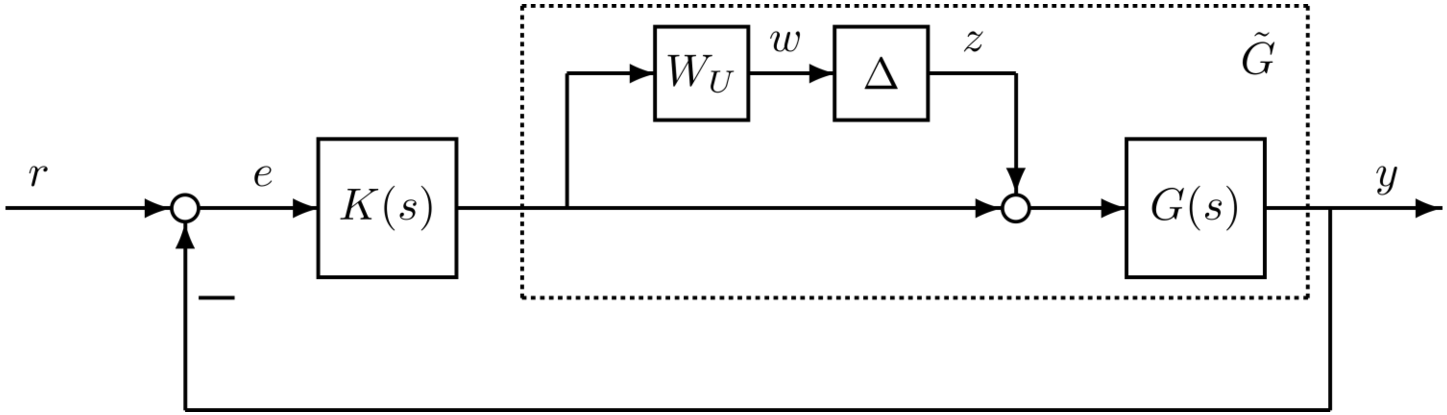


SISO Robust Stability Revisited

Def: A controller K achieves robust stability if it stabilizes every plant in $\tilde{G} \in \mathcal{M}$ where:

$$\mathcal{M} := \left\{ \tilde{G} = G(1 + \Delta W_U) \ : \ \|\Delta\|_\infty < 1 \right\}$$

This is a set of "multiplicative" uncertainty.

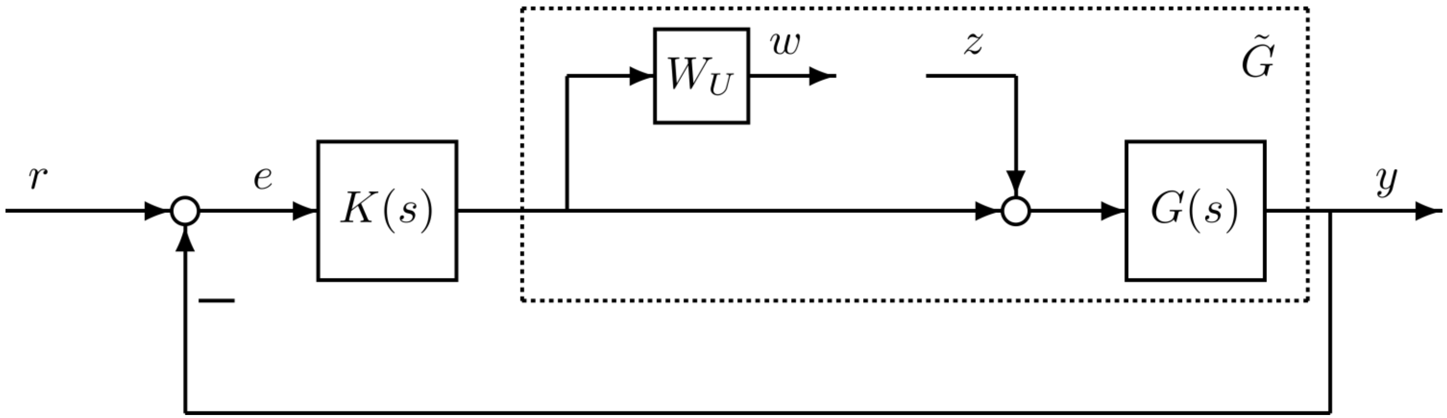


SISO Robust Stability Revisited

Theorem: A controller K achieves robust stability for the uncertainty set $\mathcal{M}$ if and only if

- K stabilizes G, i.e. achieves nominal stability, and
- $\|W_U T\|_\infty \leq 1$ where $T(s) = \frac{G(s)K(s)}{1+G(s)K(s)}$ is the nominal complementary sensitivity.

Proof: The transfer function from *z* to *w* is $M(s) := -W_U(s)T(s)$. By the assumptions, $M(s)$ is stable and $\|M\|_\infty \leq 1$. Apply the small gain theorem.



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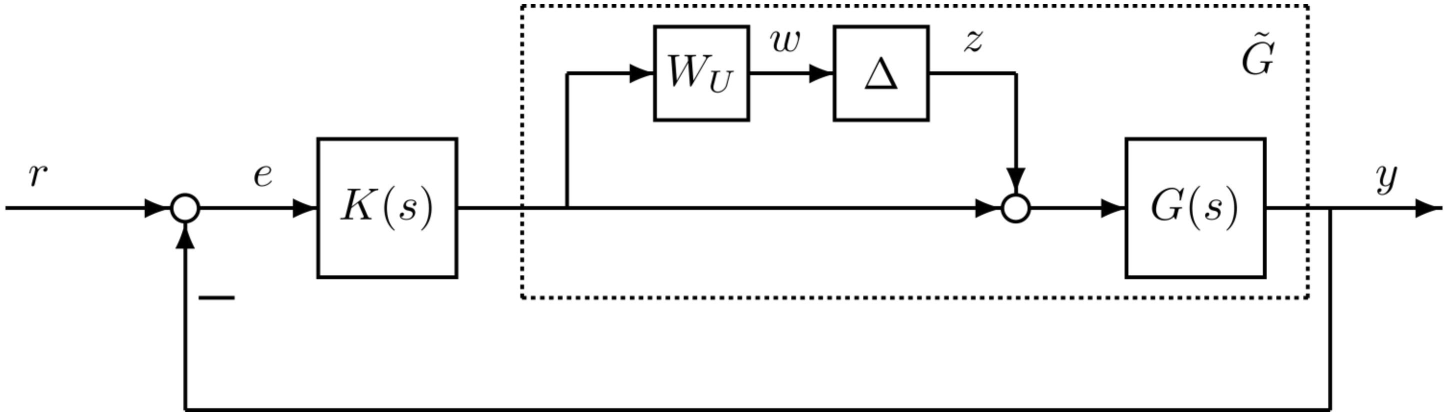
Thus, $m_{max} := \frac{1}{||W_U T||_\infty}$ is the stability margin. The feedback is stable for all stable $\Delta(s)$ with $||\Delta||_\infty < m_{max}$. The closed-loop is robustly stable with respect to $\mathcal{M}$ if and only if $m_{max} \geq 1$.



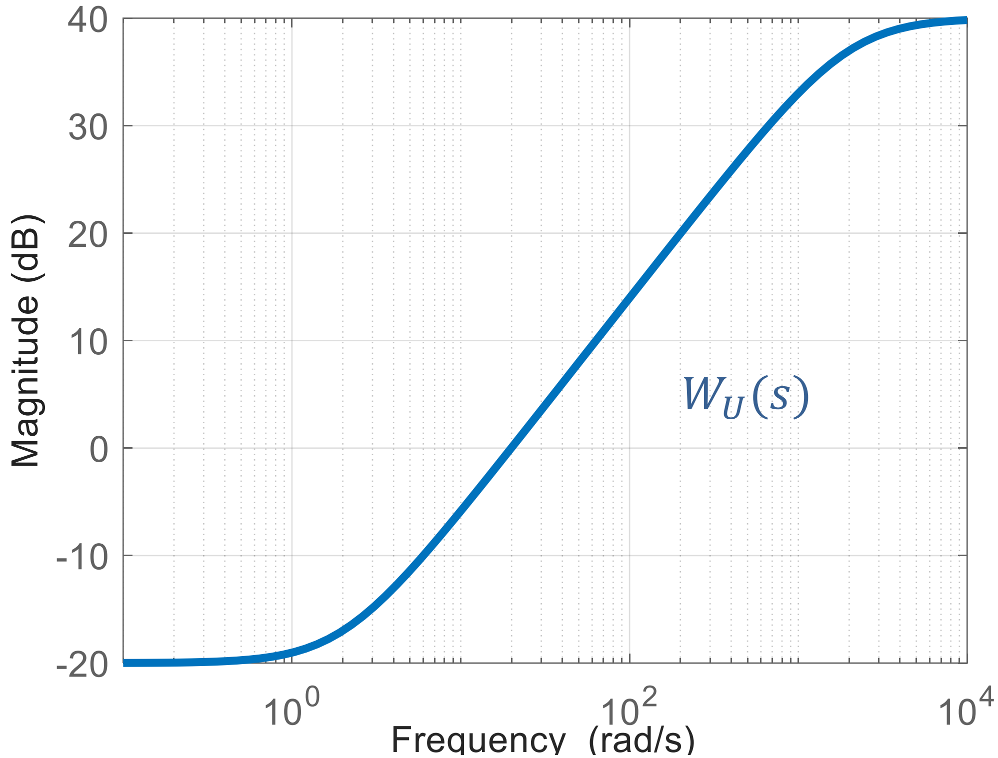
Example

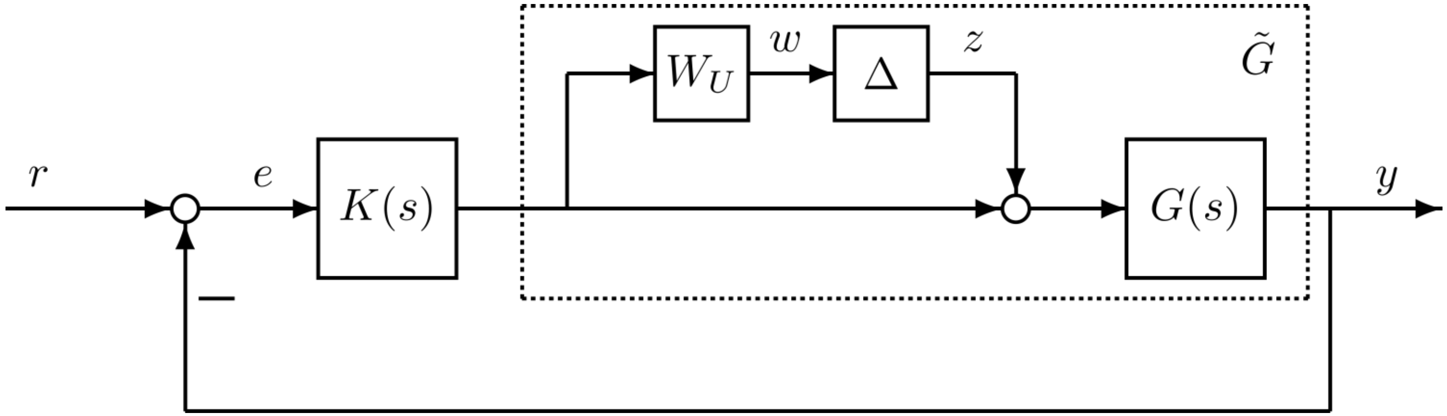
Plant: $G(s) = \frac{5}{s+2}$

Uncertainty Weight:

$$W_U(s) = \frac{s+2}{0.01s+20}$$

⇒10% uncertainty at low frequencies and rising above 2 rad/sec.





Example

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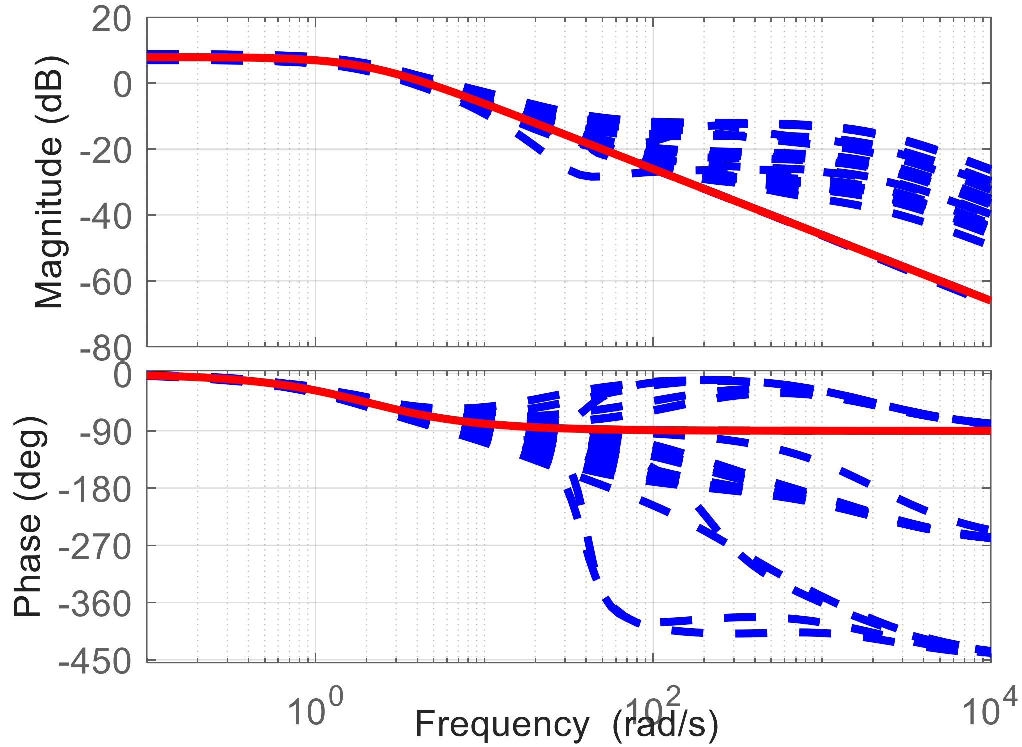
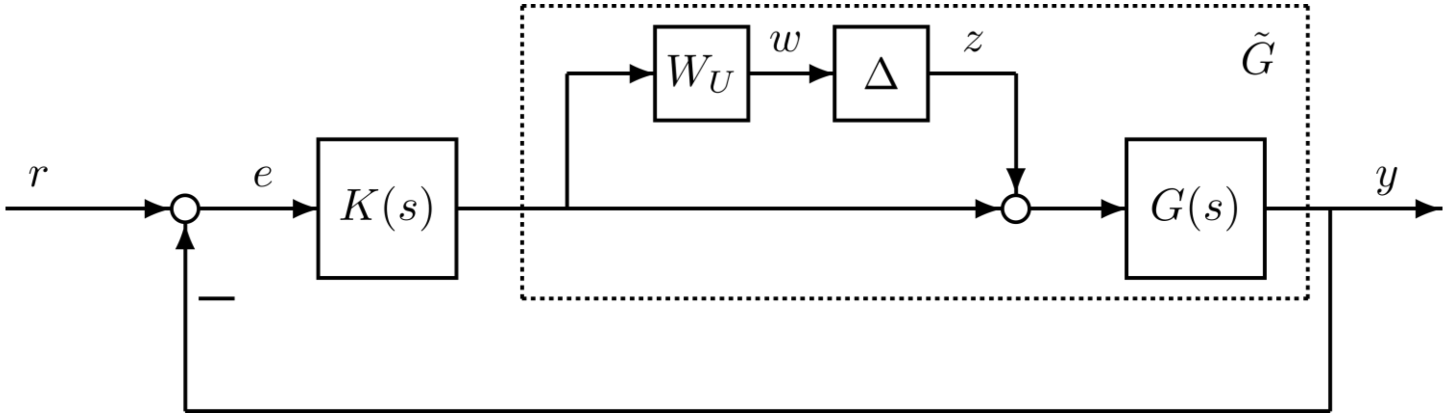
Bode plot of nominal plant $G(s)$ and samples of uncertainty.





Example

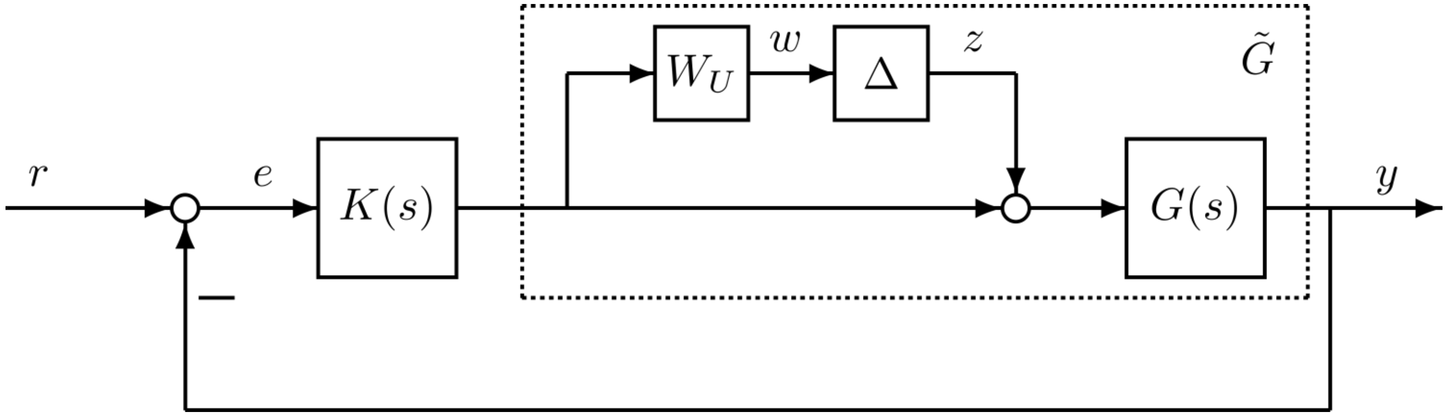
Plant: $G(s) = \frac{5}{s+2}$

Uncertainty Weight: $W_U(s) = \frac{s+2}{0.01s+20}$

PI Controller: $K_1(s) = \frac{4s+7.59}{3.32s}$

Loop bandwidth is 6 rad/s.

Nominal closed-loop is stable with phase margin = 91°.



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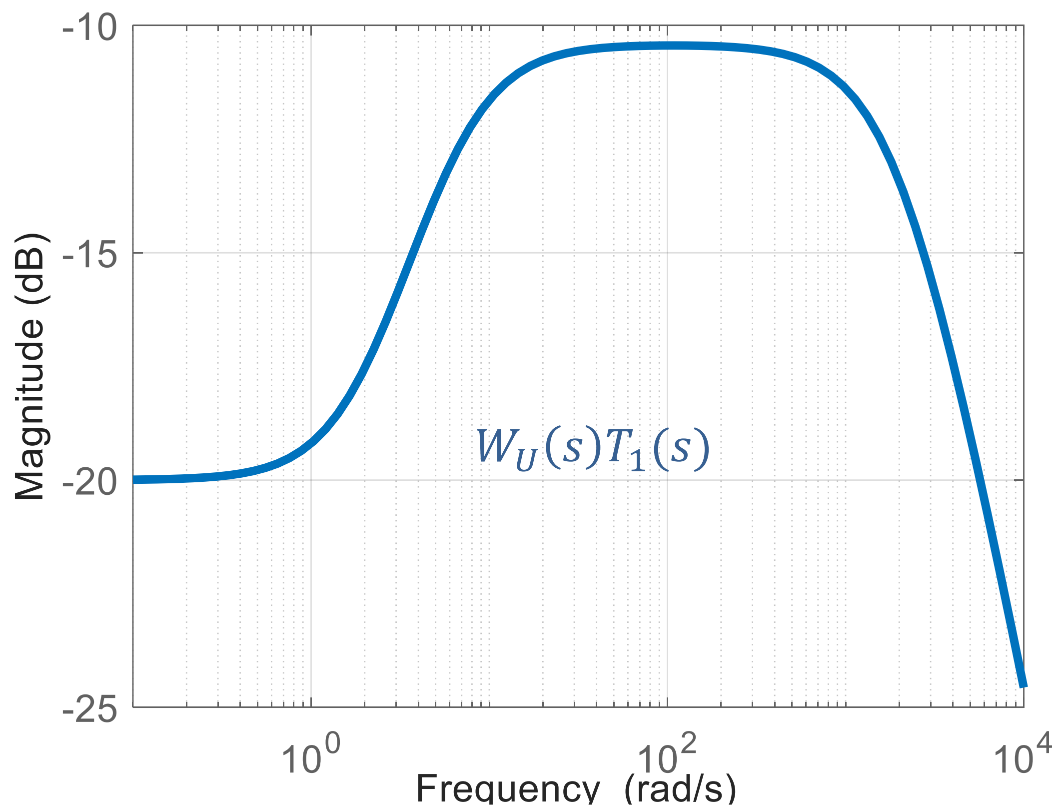
Nominal closed-loop is stable and $\|W_U T_1\|_\infty = 0.3 < 1.$ The closed-loop is robustly stable with margin $m_{max} = \frac{1}{0.3} = 3.3.$



Example

Plant: $G(s) = \frac{5}{s+2}$

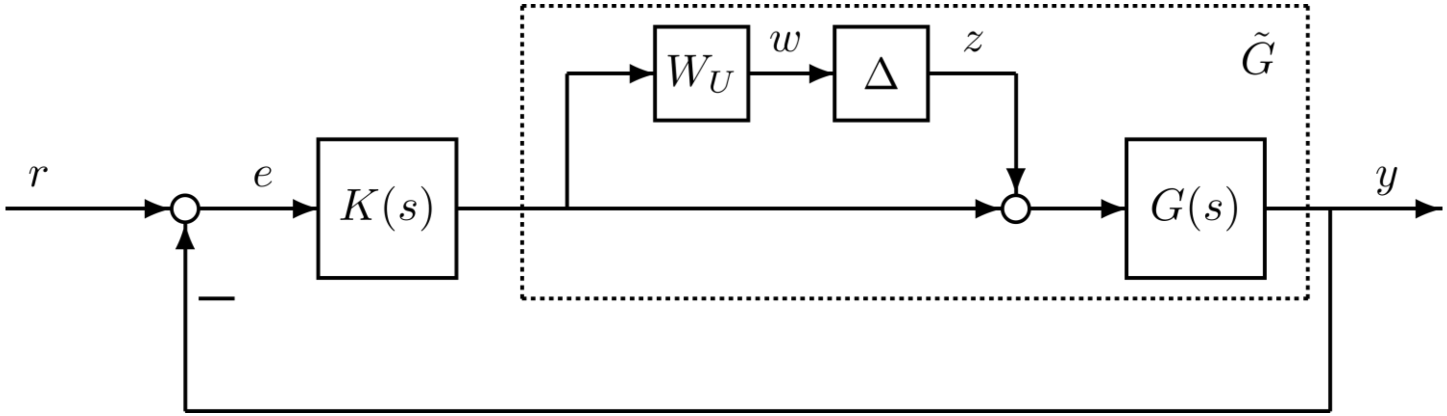
Uncertainty Weight: $W_U(s) = \frac{s+2}{0.01s+20}$

PI Controller: $K_2(s) = \frac{19.02s+180.4}{3.32s}$

Loop bandwidth is 30 rad/s.

Nominal closed-loop is stable with phase margin = 76°.

The loop bandwidth is at a higher frequency where the model uncertainty is larger.



Example

Plant: $G(s) = \frac{5}{s+2}$

Uncertainty Weight: $W_U(s) = \frac{s+2}{0.01s+20}$

PI Controller: $K_2(s) = \frac{19.02s+180.4}{3.32s}$

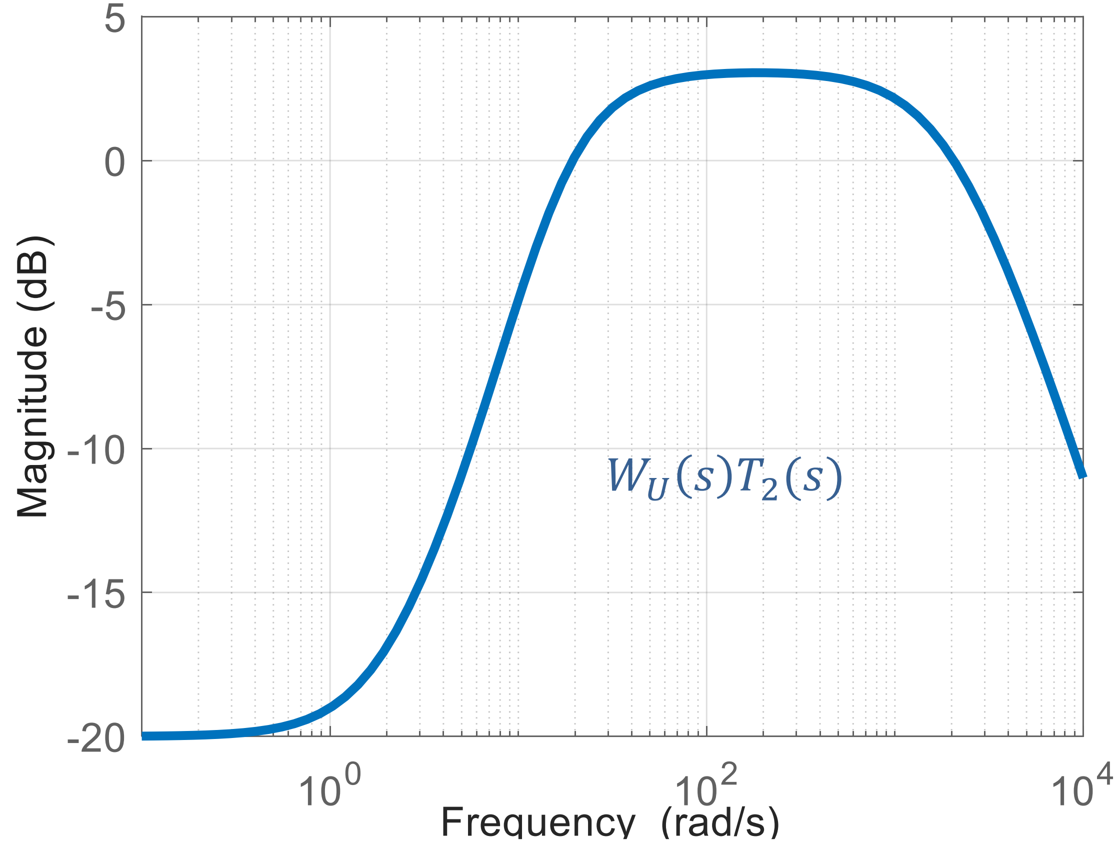
Nominal closed-loop

Is stable and

$\|W_U T_2\|_\infty = 1.4 > 1.$

The closed-loop

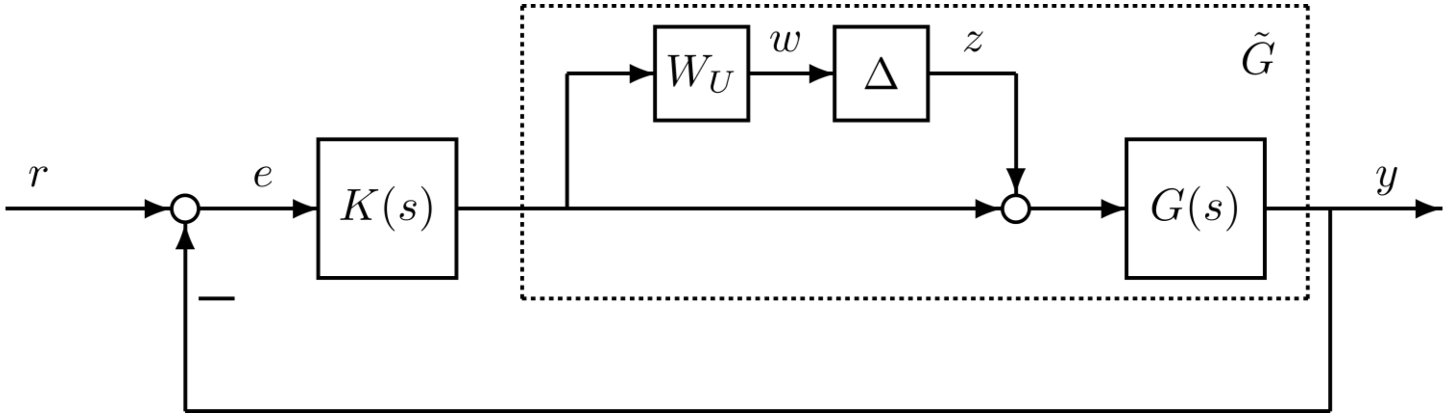
is <u>not</u> robustly stable.



Example

We have $|W_U(j\omega_0)T_2(j\omega_0)| = 1.42$ at ω_0=199 rad/s.

Set $\delta_0 := \frac{1}{-W_U(j\omega_0)T_2(j\omega_0)}$ and construct a first-order uncertainty $\Delta(s) = \frac{-0.704\, s\, +\, 0.202}{s+0.288}$.



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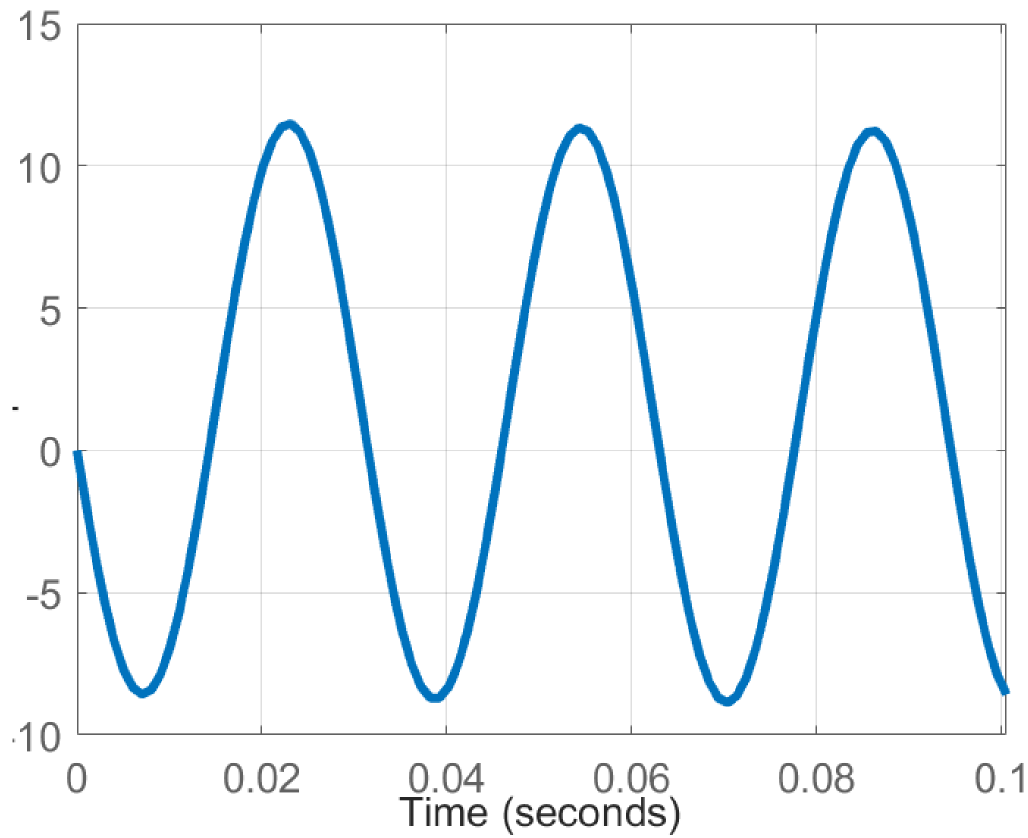
Set $\delta_0 := \frac{1}{-W_U(j\omega_0)T_2(j\omega_0)}$ and construct a first-order uncertainty $\Delta(s) = \frac{-0.704\, s + 0.202}{s+0.288}$.

- $||\Delta||_\infty = \frac{1}{1.42}$=$0.70 < 1$
- Closed-loop has poles on the imaginary axis at $\pm j\omega_0$.
- Step response oscillates at the frequency ω_0.



Symmetric Disk Margins Revisited

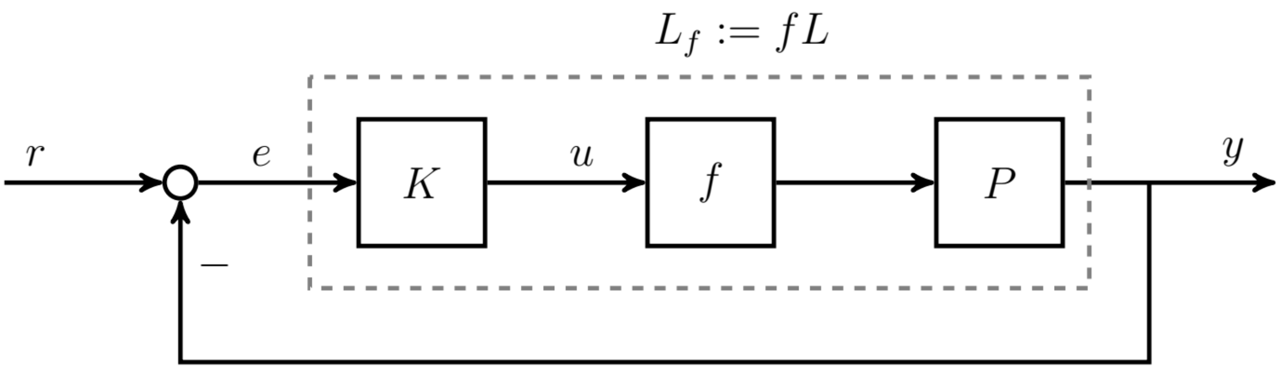
- **Definition:** A symmetric disk of uncertainty is:

$$D(\alpha) = \left\{ \frac{1+\frac{\delta}{2}}{1-\frac{\delta}{2}} : \delta \in \mathbb{C} \ \& \ |\delta| < \alpha \right\}$$

- **Definition:** The symmetric disk margin α_{max} is the largest value of α such that closed-loop with *is* stable for all complex perturbations $f \in D(\alpha)$.
- **Result:** Assume the feedback loop is nominally stable with $f=1$. The symmetric disk margin is:

$$\alpha_{max} = \frac{1}{\|S-0.5\|_\infty} = \frac{1}{0.5\|S-T\|_\infty}$$

This result is also a special case of the small gain theorem.

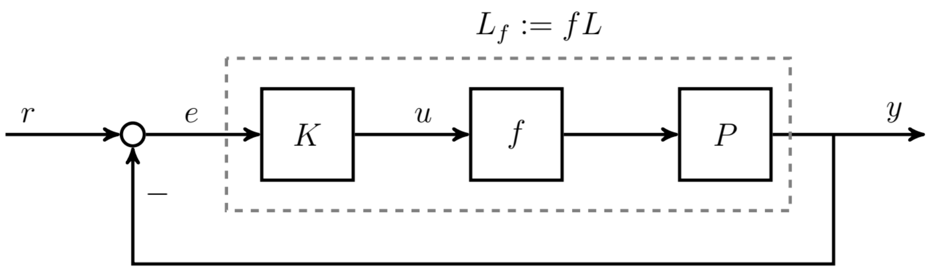


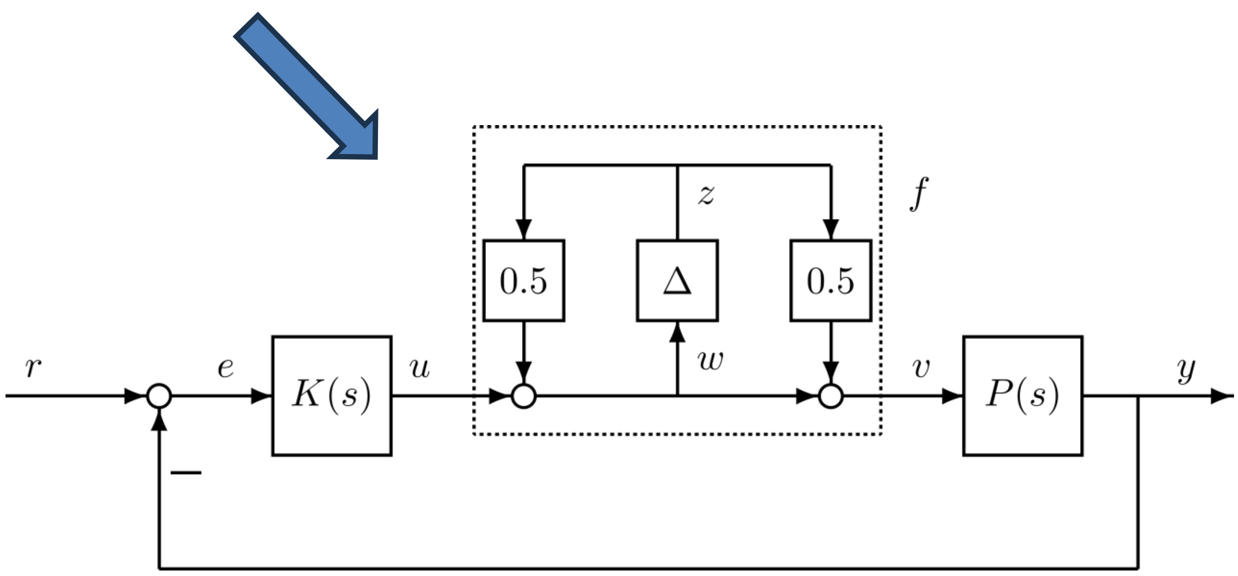
Symmetric Disk Margins Revisited

- **Definition:** A symmetric disk of uncertainty is:

$$D(\alpha) = \left\{ \frac{1+\frac{\delta}{2}}{1-\frac{\delta}{2}} \ : \ \delta \in \mathbb{C} \ \& \ |\delta| < \alpha \right\}$$

1. Replace f by $\frac{1+0.5\delta}{1-0.5\delta}$ in the block diagram.





Symmetric Disk Margins Revisited

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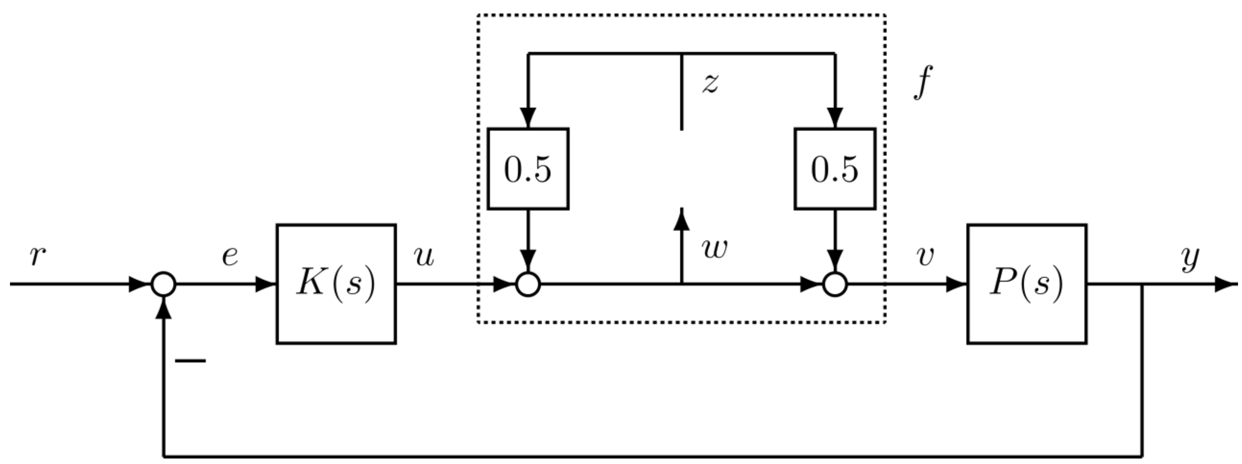
1. Replace f by $\frac{1+0.5\delta}{1-0.5\delta}$ in the block diagram.
2. The transfer function from *z* to *w* is $M(s) = \frac{1}{2}(S - T)$



Symmetric Disk Margins Revisited

- **Definition:** A symmetric disk of uncertainty is:

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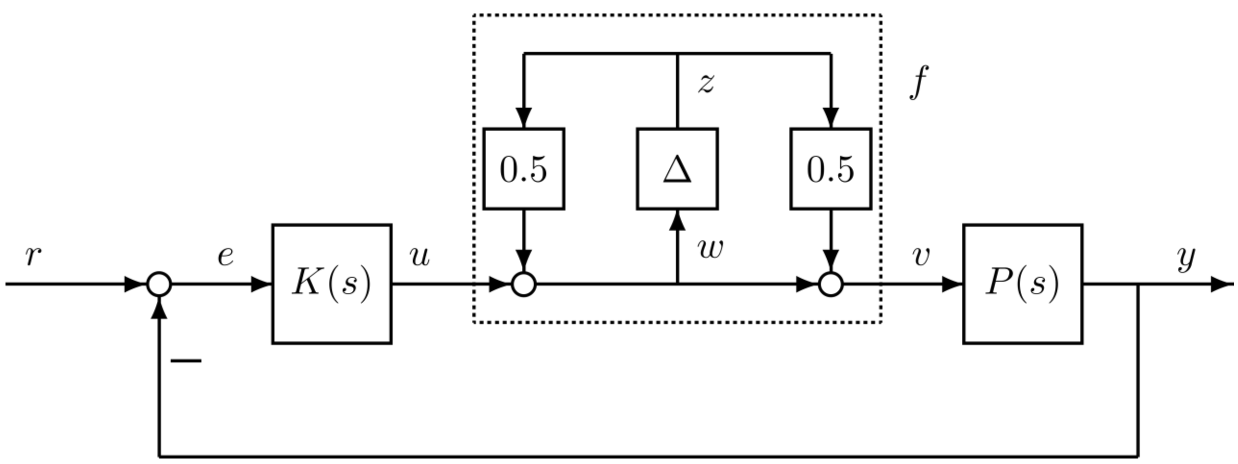
1. Replace f by $\frac{1+0.5\delta}{1-0.5\delta}$ in the block diagram.
2. The transfer function from *z* to *w* is $M(s) = \frac{1}{2}(S - T)$
3. By the assumptions, $M(s)$ is stable. By the small gain theorem the stability margin is

$$\alpha_{max} = \frac{1}{\|M\|_\infty} = \frac{1}{0.5\|S-T\|_\infty}$$



Problem

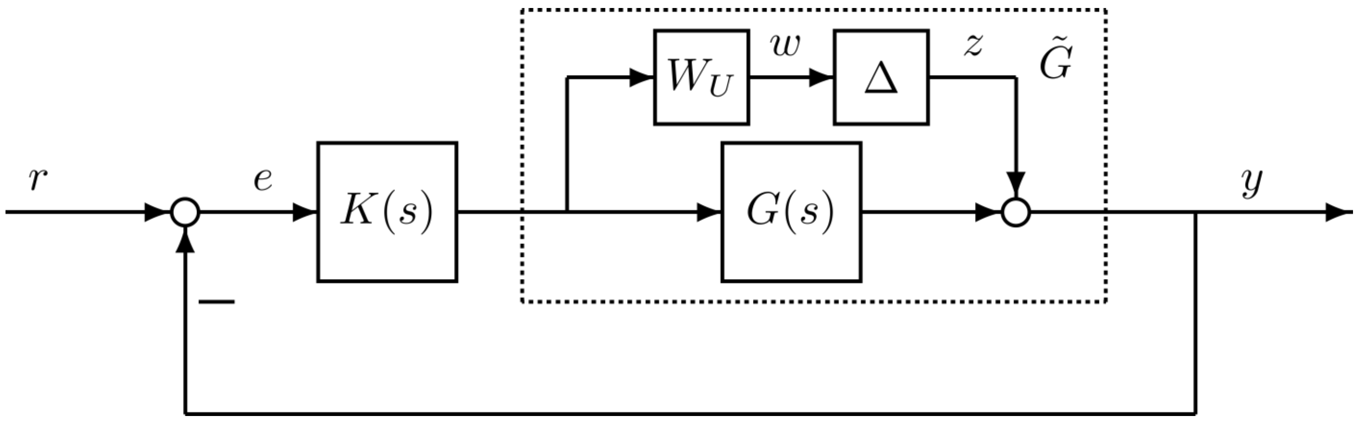
Def: A controller K achieves robust stability if it stabilizes every plant in $\tilde{G} \in \mathcal{M}$ where:

$$\mathcal{M} := \left\{ \tilde{G} = G + \Delta W_U \ : \ \|\Delta\|_\infty < 1 \right\}$$

This is a set of "additive" uncertainty.

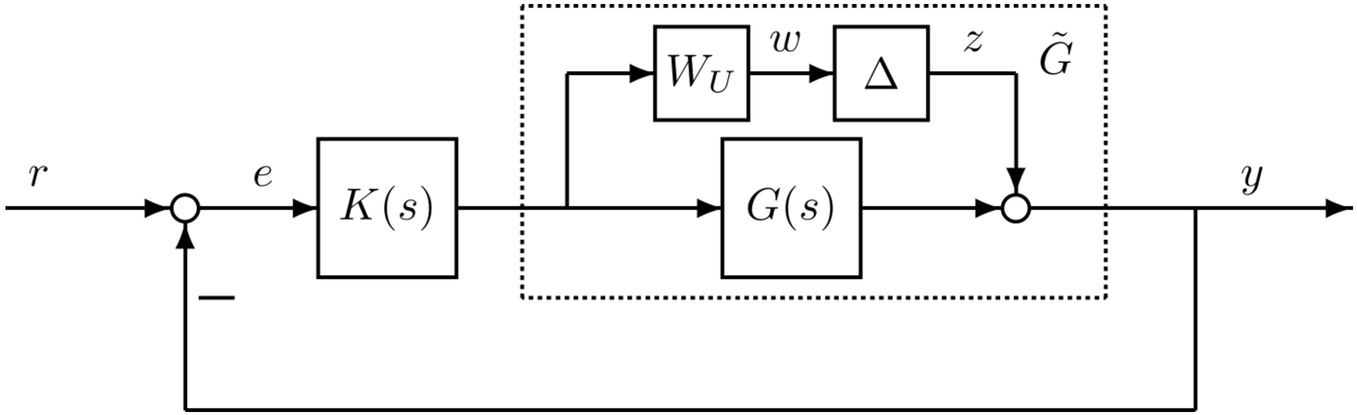
Question: What is a necessary and sufficient robust stability condition for this class of uncertainty?

Hint: Express this as a positive feedback connection of $\Delta(s)$ and some $M(s)$. Then apply the small gain theorem.



Solution

Question: What is a necessary and sufficient robust stability condition for this class of uncertainty?



Solution-Extra Space

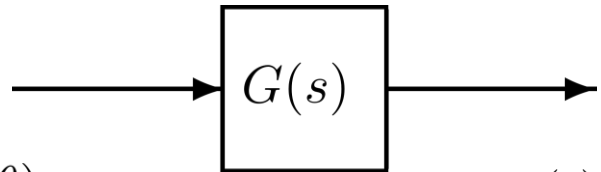
MIMO Frequency Response

Notes: “MIMO Frequency Response”

Revisit Lecture 1 of the second NASA workshop for details.

SISO Steady-State Sinusoidal Response

Assume *G* is a stable, SISO, LTI system.

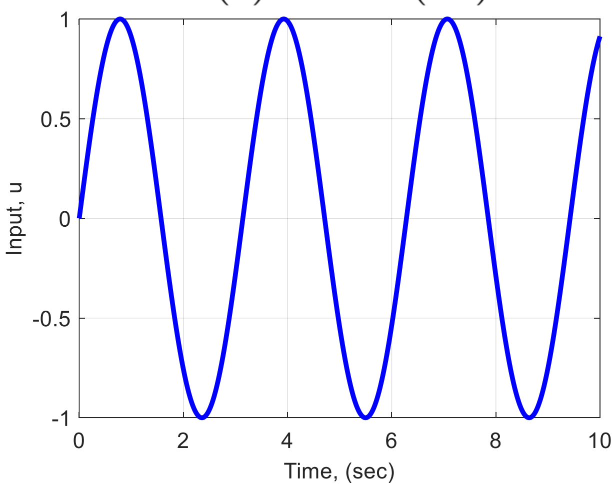


$$u(t) = \sin(\omega t) \qquad y(t) \to |G(j\omega)| \sin\left(\omega t + \angle G(j\omega)\right)$$

$$u(t) = A\sin(\omega t + \theta) \qquad y(t) \to A|G(j\omega)| \sin\left(\omega t + \theta + \angle G(j\omega)\right)$$

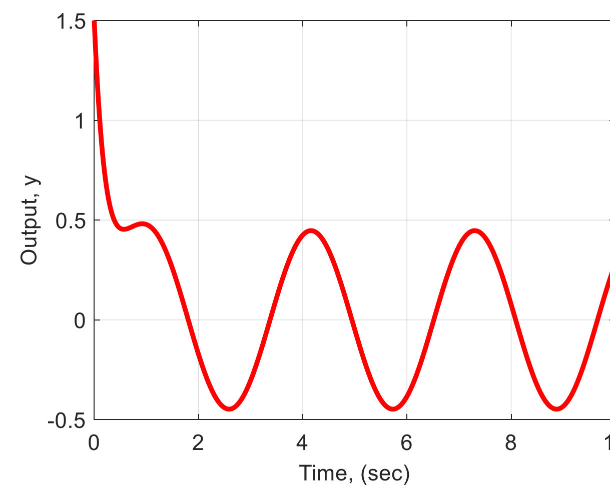
$$u(t) = Ae^{j(\omega t + \theta)} \qquad y(t) \to G(j\omega) \cdot Ae^{j(\omega t + \theta)}$$

$$u(t) = \sin(2t)$$



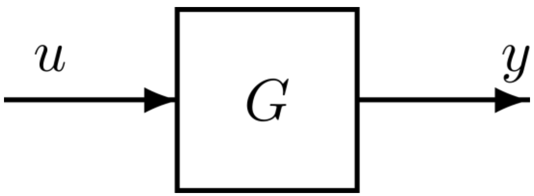
$$G(s) = \frac{2}{s+4}$$

$$y(0) = 1.5$$



MIMO Steady-State Sinusoidal Response

Assume *G* is a stable, MIMO, LTI system.



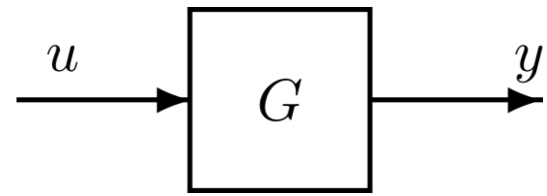
Consider the complex input:

$$u(t) = \bar{u} e^{j\omega t} \quad \text{where} \quad \bar{u} = \begin{bmatrix} u_1 e^{j\theta_1} \\ \vdots \\ u_m e^{j\theta_m} \end{bmatrix}$$

The complex-valued vector $\bar{u}$ has a direction in terms of amplitude and phase.

MIMO Steady-State Sinusoidal Response

Assume *G* is a stable, MIMO, LTI system.



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The steady-state output is:

$$y(t) \to \bar{y}e^{j\omega t} \quad \text{where} \quad \bar{y} = G(j\omega)\bar{u} = \begin{bmatrix} y_1 e^{j\phi_1} \\ \vdots \\ y_l e^{j\phi_l} \end{bmatrix}$$

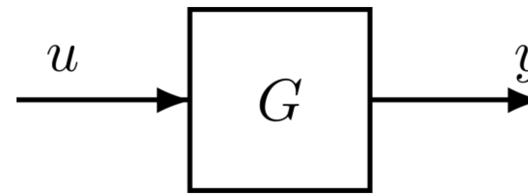
Reason: The general state-space solution is:

$$y(t) = Ce^{At}x_0 + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t)$$

Substitute for *u(t)*, perform the integration, and note that initial transient decays to zero because *G* is stable.

MIMO Steady-State Sinusoidal Response

Assume *G* is a stable, MIMO, LTI system.



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The transfer function matrix relates a sinusoidal input signal vector to a sinusoidal output signal vector.

The "Gain" of a MIMO System

- For SISO systems, $|G(j\omega)|$ is the amplification at frequency ω. **The "gain" of a SISO system varies with $\boldsymbol{\omega}$.**

- For MIMO systems, the I/O signals are related by the complex matrix $G(j\omega)$ but this mixes magnitude/phase and input directions.

- What is an appropriate measure for the gain of a MIMO system?

Singular Value Decomposition

Any matrix $M \in C^{n \times m}$ can be decomposed as $M = U\Sigma V^*$ where:

- $U \in C^{n \times n}$ is unitary, i.e. $U^*U = U\,U^* = I$
- $V \in C^{m \times m}$ is unitary, i.e. $V^*V = V\,V^* = I$
- Σ has entries σ_i only on the diagonal with $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_k \geq 0$ (where *k* is the smaller of *m* and *n*).

Comments:

- $\sigma_i(M) = \sqrt{\lambda_i(M^*M)} > 0$ are called singular values.
- Columns of U denote output directions.
- Columns of V denote input directions.
- *U* and *V* are "rotations" because they do not change the Euclidean norm of vectors, i.e. $||Ux||_2 = ||x||_2$ for any vector $x \in C^n$.

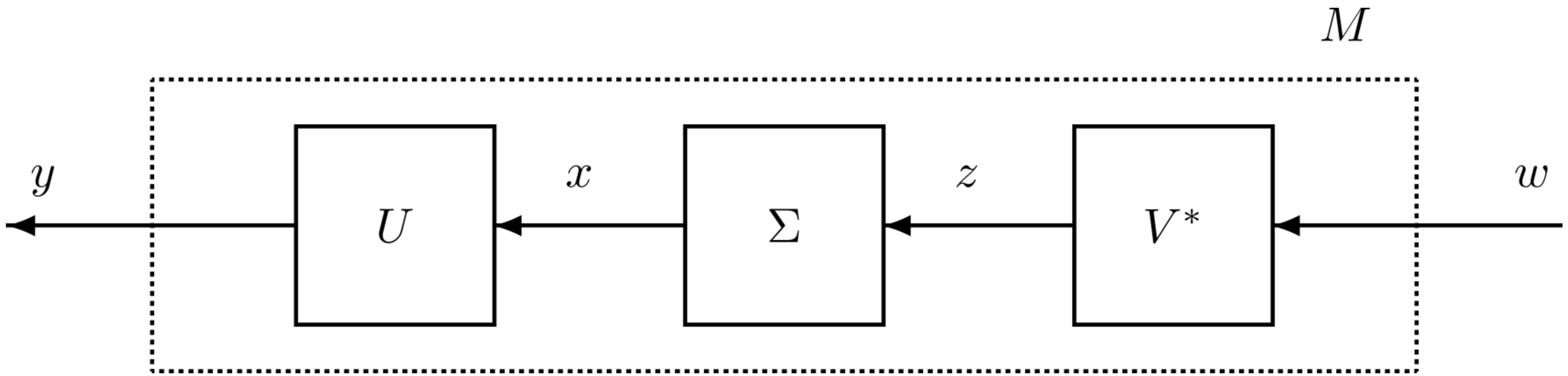
Singular Value Decomposition

Singular value denotes gain for that input/output direction

$$M = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} \begin{bmatrix} v_1 & v_2 \end{bmatrix}^* \longrightarrow \begin{array}{l} M v_1 = \sigma_1 u_1 \\ M v_2 = \sigma_2 u_2 \end{array}$$

Matrix multiplication *y=M x* can be interpreted as a rotation, scaling, and another rotation.

See `SVDEx.m` for a 2-by-2 example.



The "Gain" of a MIMO System Revisited

- For SISO systems, $|G(j\omega)|$ is the amplification at frequency ω. The "gain" of a SISO system varies with ω.

- **The "gain" of a MIMO system varies with input direction and with $\boldsymbol{\omega}$:**
  - Let $G(j\omega) = U\Sigma V^*$ be the SVD of the response at ω.
  - If the system is stable then the input $u(t) = Re(v_i e^{j\omega t})$ generates the output $y(t) \to \sigma_i \, Re(u_i e^{j\omega t})$.
  - The maximum gain at ω is given by $\sigma_1 = \bar{\sigma}(G(j\omega))$.

MIMO Small Gain Condition

Notes: Section 3 of “Introduction to MIMO Robustness”

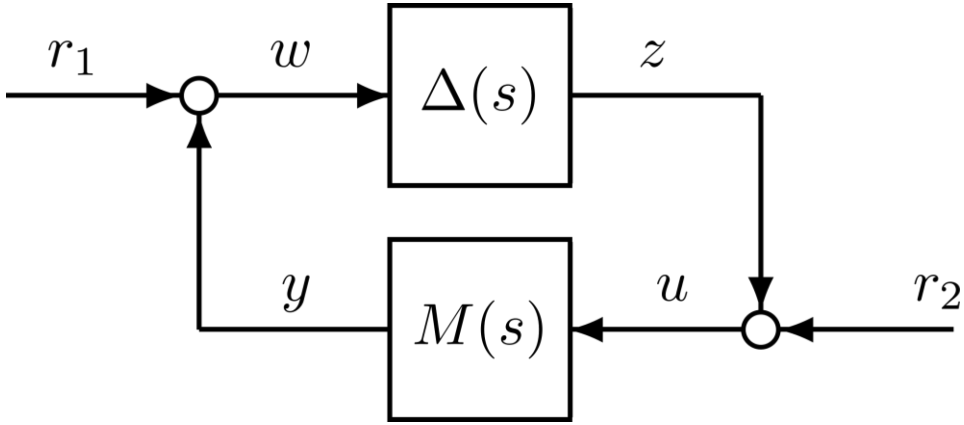
MIMO Small Gain Condition

Consider the *positive* feedback system below where:

- $M(s)$ is a known stable, MIMO, LTI system.
- $\Delta(s)$ is an unknown MIMO, LTI system, i.e. it is uncertain. Also assume that $\Delta(s)$ is stable and is norm-bounded:

$$||\Delta||_\infty := \max_{\omega \in \mathbb{R} \cup \{+\infty\}} \bar{\sigma}\big(\Delta(j\omega)\big) < 1$$

We will state a simple condition to prove stability of the feedback system.



MIMO Small Gain Condition

Theorem: Consider the positive feedback system below where $M(s)$ is stable.

A) If $||\mathrm{M}||_\infty \leq 1$ then the feedback system is stable for all $\Delta(s)$ that are stable and is norm-bounded $||\Delta||_\infty < 1$.

B) If $||\mathrm{M}||_\infty > 1$ then there is a stable $\Delta(s)$ with $||\Delta||_\infty < 1$ such that the feedback system is unstable.

This can be proved using a MIMO version of the Nyquist theorem.

The notes give an alternative linear algebra/state-space proof when Δ is a complex matrix. The result extends to the case where Δ is LTI by constructing systems that "interpolate" a complex matrix as in the SISO case.

MIMO Small Gain Condition

B) If $||\mathrm{M}||_\infty > 1$ then there is a stable $\Delta(s)$ with $||\Delta||_\infty < 1$ such that the feedback system is unstable.

Proof: We briefly describe the construction of the destabilizing complex matrix Δ.

1. $||\mathrm{M}||_\infty > 1 \Rightarrow$ There is $0 < \omega_0 < \infty$ such that $\bar{\sigma}[M(j\omega_0)] > 1$.

2 By the SVD of $M(j\omega_0)$ there are complex vectors *w, z* such that:

$$\bar{\sigma} \cdot w = M(j\omega_0) z \text{ and } \|w\|_2 = \|z\|_2 = 1.$$

3. Select $\Delta_0 := \frac{1}{\bar{\sigma}} z w^*$ and note that:

$$M(j\omega_0)\Delta_0 w = w \text{ and } \bar{\sigma}(\Delta_0) < 1$$

4. Thus $I - M(j\omega_0)$ is singular and you can use this to show the feedback loop has a pole on the imaginary axis at $s = j\omega_0$.

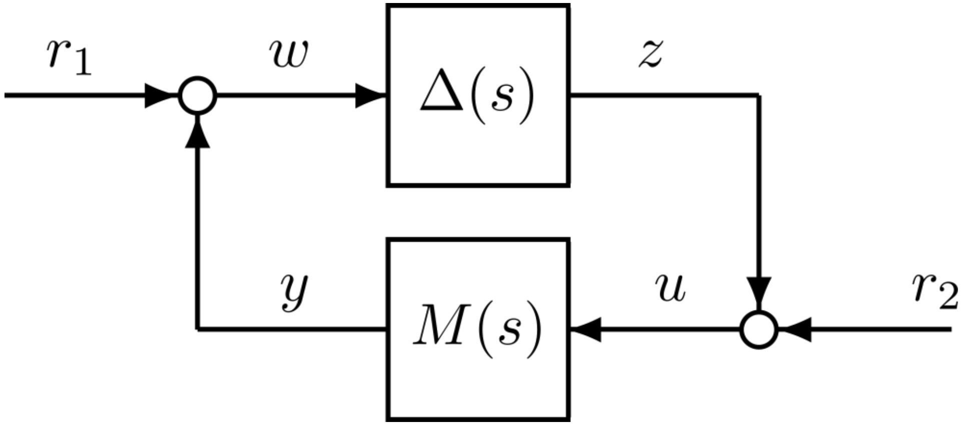
MIMO Small Gain Condition

Theorem: Consider the positive feedback system below where $M(s)$ is stable. Let $m > 0$ be a given constant.

A) If $||\mathrm{M}||_{\infty} \leq \frac{1}{m}$ then the feedback system is stable for all $\Delta(s)$ that are stable and is norm-bounded $||\Delta||_{\infty} < m$.

B) If $||\mathrm{M}||_{\infty} > \frac{1}{m}$ then there is a stable $\Delta(s)$ with $||\Delta||_{\infty} < m$ such that the feedback system is unstable.

Thus, $m_{max} := \frac{1}{||M||_{\infty}}$ is the stability margin. The feedback is stable for all stable $\Delta(s)$ with $||\Delta||_{\infty} < m_{max}$.



Robust Stability With Unstructured Uncertainty

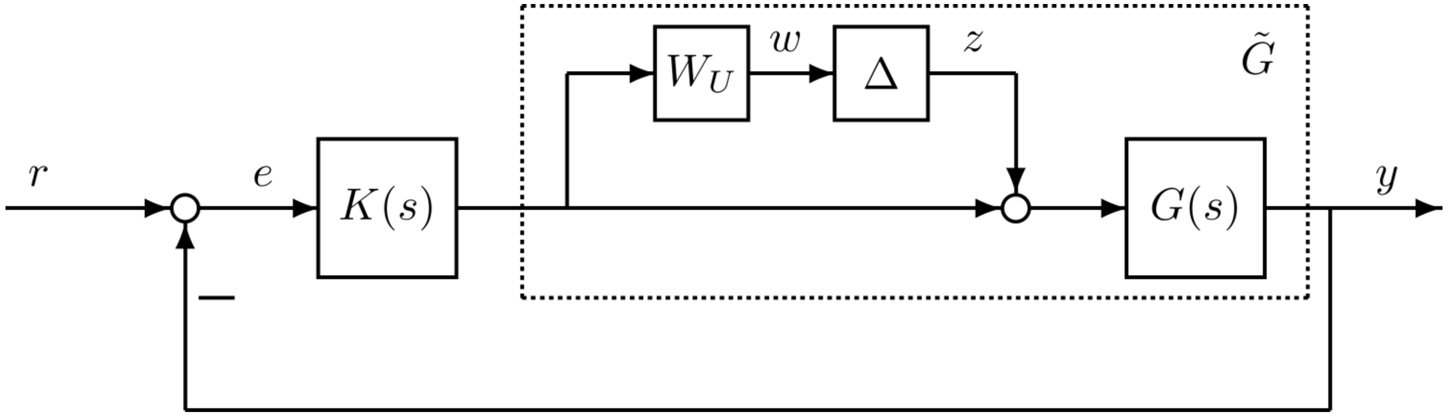
Notes: Section 4 of “Introduction to MIMO Robustness”

MIMO Robust Stability

Def: A controller *K* achieves robust stability if it stabilizes every plant in $\tilde{G} \in \mathcal{M}$ where:

$$\mathcal{M} := \left\{ \tilde{G} = G(I + \Delta W_U) \; : \; \|\Delta\|_\infty < 1 \right\}$$

This is a set of "multiplicative" uncertainty.

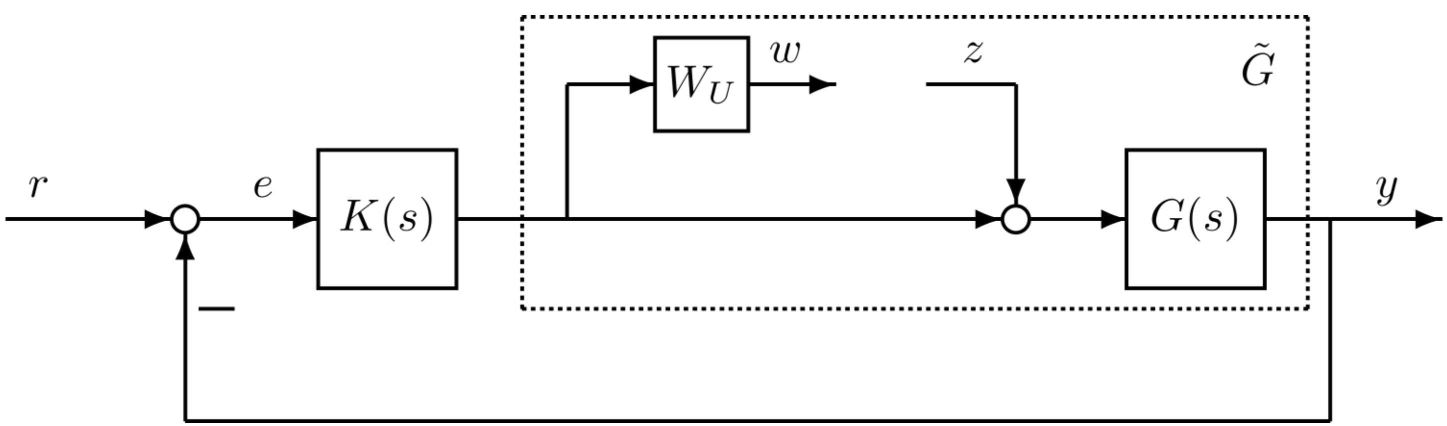


MIMO Robust Stability

Theorem: A controller K achieves robust stability for the uncertainty set $\mathcal{M}$ if and only if

- K stabilizes G, i.e. achieves nominal stability, and
- $\|W_U T_I\|_\infty \le 1$ where $T_I(s) = K(s)G(s)[I + K(s)G(s)]^{-1}$ is the nominal input complementary sensitivity.

Proof: The transfer function from *z* to *w* is $M(s) := -W_U(s)T_I(s)$. By the assumptions, $M(s)$ is stable and $\|M\|_\infty \le 1$. Apply the small gain theorem.



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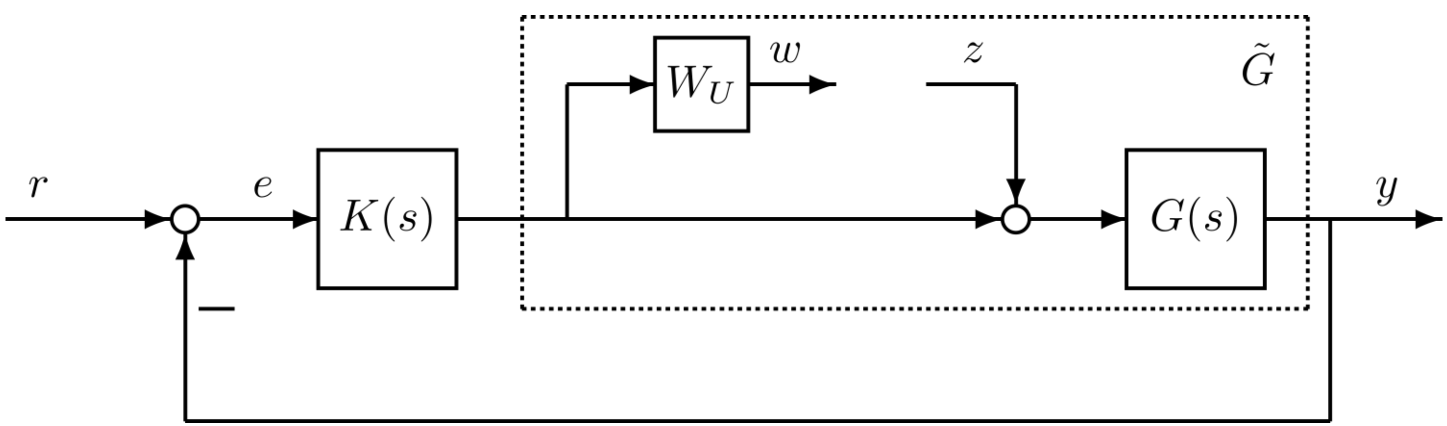
Thus, $m_{max} := \frac{1}{||W_U T_I||_\infty}$ is the stability margin. The feedback is stable for all stable $\Delta(s)$ with $||\Delta||_\infty < m_{max}$. The closed-loop is robustly stable with respect to $\mathcal{M}$ if and only if $m_{max} \geq 1$.



Unstructured Vs. Structured Uncertainty

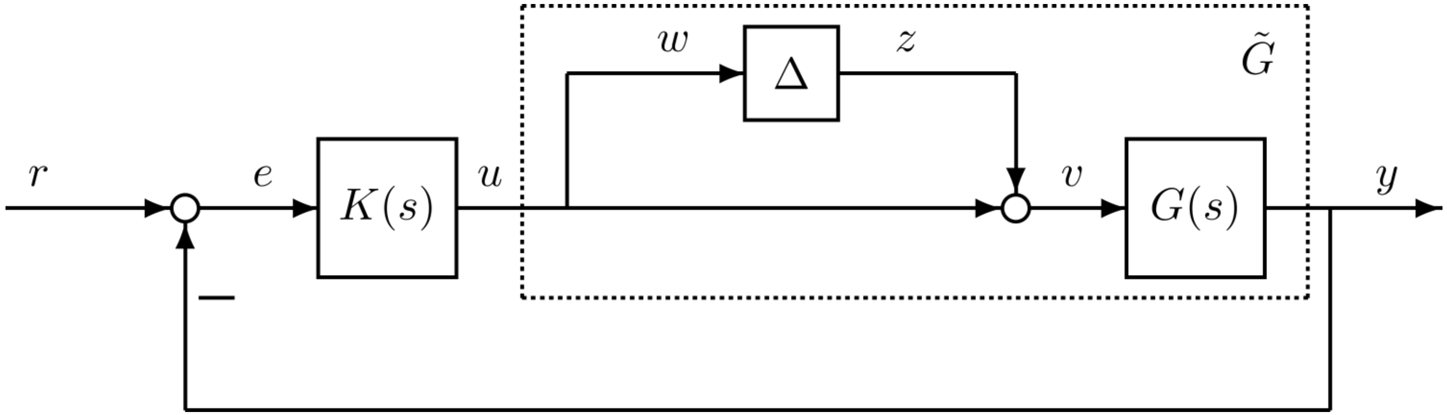
Consider the special case with $W_U(s) = I$.

Unstructured: $\Delta(s)$ is "full", i.e. all entries can be non-zero. This can model cross-coupling in the uncertainty.

Example: In the 2-by-2 case,

$$v = (I + \Delta)u \quad \Rightarrow \quad \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 + \Delta_{11} & \Delta_{12} \\ \Delta_{21} & 1 + \Delta_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Control command u_1 can affect plant input v_2 and vice versa.



Unstructured Vs. Structured Uncertainty

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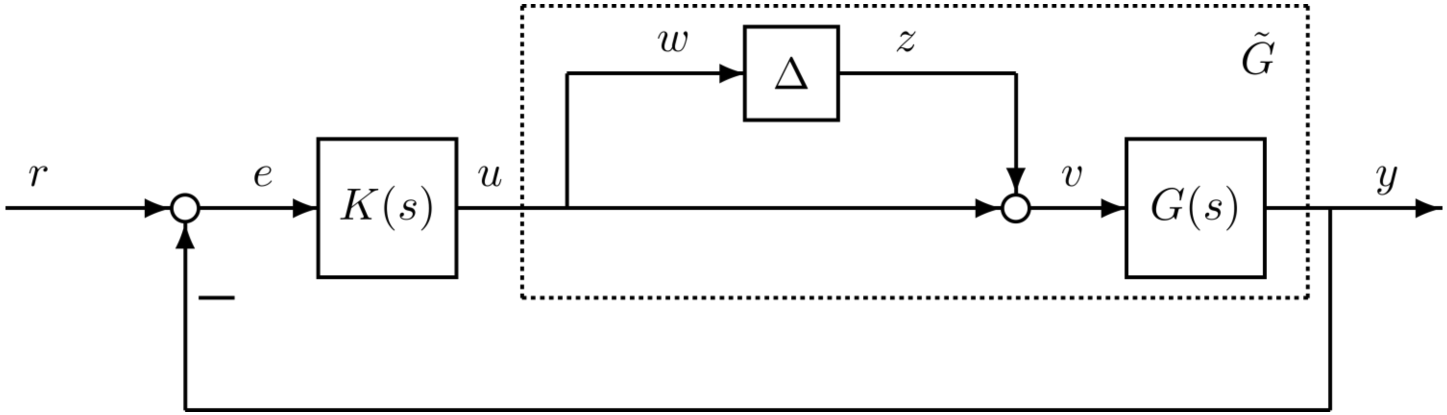
Unstructured: $\Delta(s)$ is "full", i.e. all entries can be non-zero. This can model cross-coupling in the uncertainty.

Multi-Loop: $\Delta(s)$ has structure, e.g. it is diagonal.

Example: In the 2-by-2 case, if Δ is diagonal

$$v = (I + \Delta)u \;\Rightarrow\; \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 + \Delta_{11} & 0 \\ 0 & 1 + \Delta_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

This corresponds to uncertainty in each channel (multi-loop).



Spinning Satellite Example

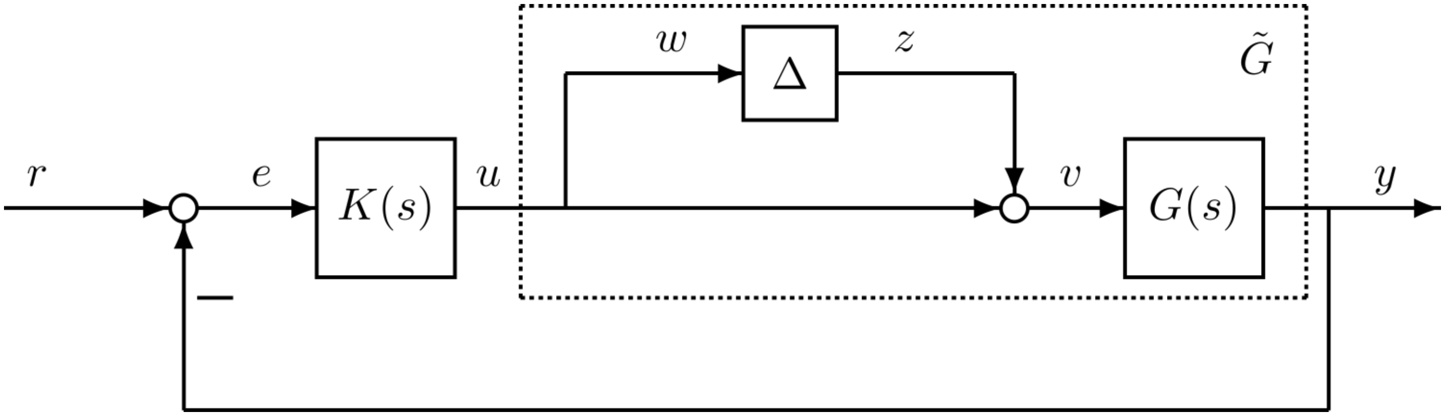
Plant and Controller:

$$G(s) := \frac{1}{s^2 + a^2} \begin{bmatrix} s - a^2 & a(s+1) \\ -a(s+1) & s - a^2 \end{bmatrix} \text{ and } K(s) := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

with *a = 10.* The dynamics represent a simplified model for a spinning satellite.

References:

- Doyle, Robustness of Multiloop Linear Feedback Systems, '78 CDC.
- Additional details in Section 3.7 of Skogestad and Postlethwaite or Section 9.6 of Zhou, Doyle, Glover.



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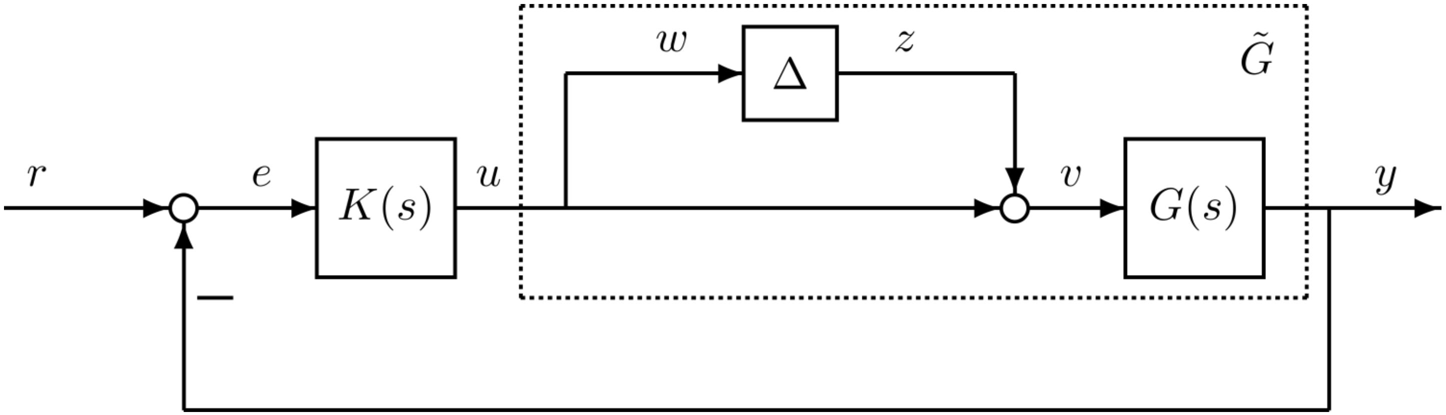
with *a = 10.* The dynamics represent a simplified model for a spinning satellite.

We showed that this system is sensitive to multi-loop uncertainty. It must also be sensitive to unstructured uncertainty.



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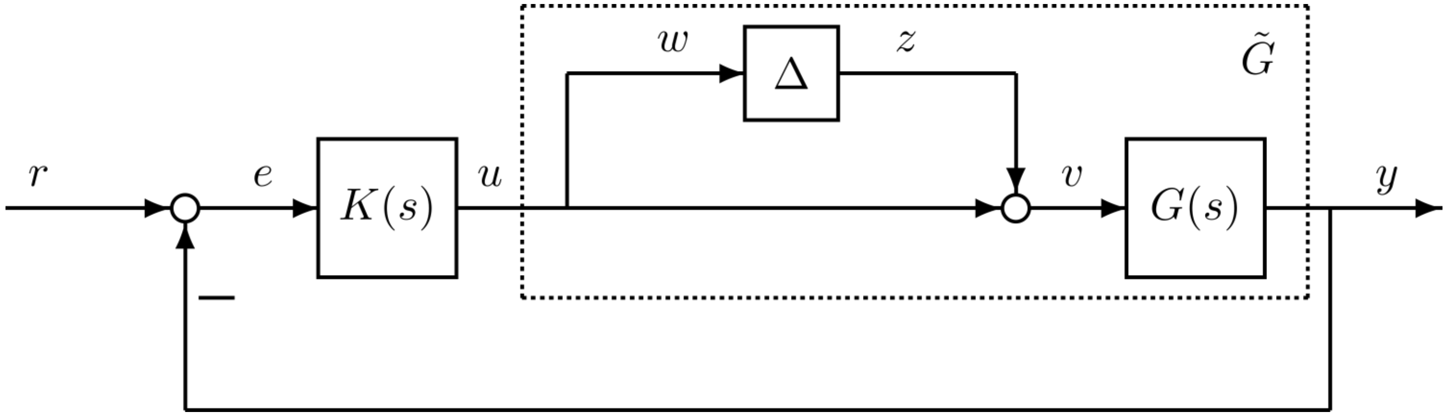
with *a = 10.* The dynamics represent a simplified model for a spinning satellite.

$$||T_I||_\infty = 10.05 \Rightarrow m_{max} = \frac{1}{||T_I||_\infty} = 0.0995$$

The feedback loop is stable if and only if $||\Delta||_\infty < 0.0995$.

The system is sensitive to unstructured uncertainty as expected.



Conclusions

The small gain theorem is stated for a generic feedback loop involving a known part $M(s)$ and an uncertainty $\Delta(s)$.

- This provides a necessary and sufficient condition to prove robust stability in terms of the gain (H_∞ norm) of $M(s)$.
- The proof constructs a destabilizing uncertainty $\Delta(s)$ that can be studied further in nonlinear simulations.

The small gain theorem can be adapted, with notational changes, to MIMO systems. It treats the uncertainty as "unstructured".

We will derive conditions for "structured" uncertainty later.