GRAVITATIONAL MULTIBODY DYNAMICS: A QUICK-START GUIDE

Course plan:

1) **Choosing the right model**
2) **Going with the flow**
3) **Periodic motion**
4) **Stability and orbit manifolds**
5) **From CR3BP to ephemeris**

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May-June 2026, NESC Academy Series

AUBURN

Lecture 1:

CHOOSING THE RIGHT MODEL

Gravitational Multibody Dynamics: a Quick-start Guide

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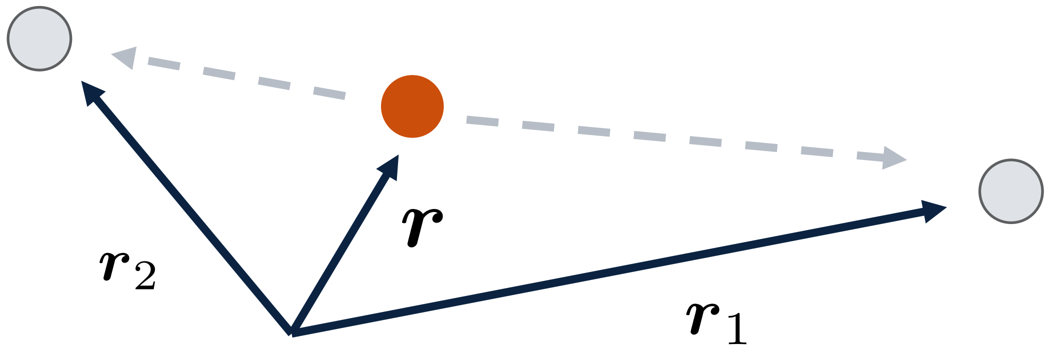
WHAT MULTIBODY GRAVITATIONAL DYNAMICS ARE

Multibody gravitational dynamics studies how two or more bodies evolve under their mutual gravitational attraction

→ **In mission design, this often reduces to studying the motion of a massless particle under the gravitational influence of other celestial bodies.**



ABSOLUTE RESTRICTED THREE BODY PROBLEM

We consider three bodies:

- m_1, m_2: primary and secondary bodies (massive) move under mutual gravitation
- $m_3 \rightarrow 0$: third body (infinitesimal mass, doesn't influence m_1, m_2)

$$\ddot{\mathbf{r}} = G m_1 \frac{\mathbf{r}_1 - \mathbf{r}}{\|\mathbf{r}_1 - \mathbf{r}\|^3} + G m_2 \frac{\mathbf{r}_2 - \mathbf{r}}{\|\mathbf{r}_2 - \mathbf{r}\|^3}$$

DOMINANT FORCES ON THE SPACECRAFT: WHICH FORCES MATTER MOST?

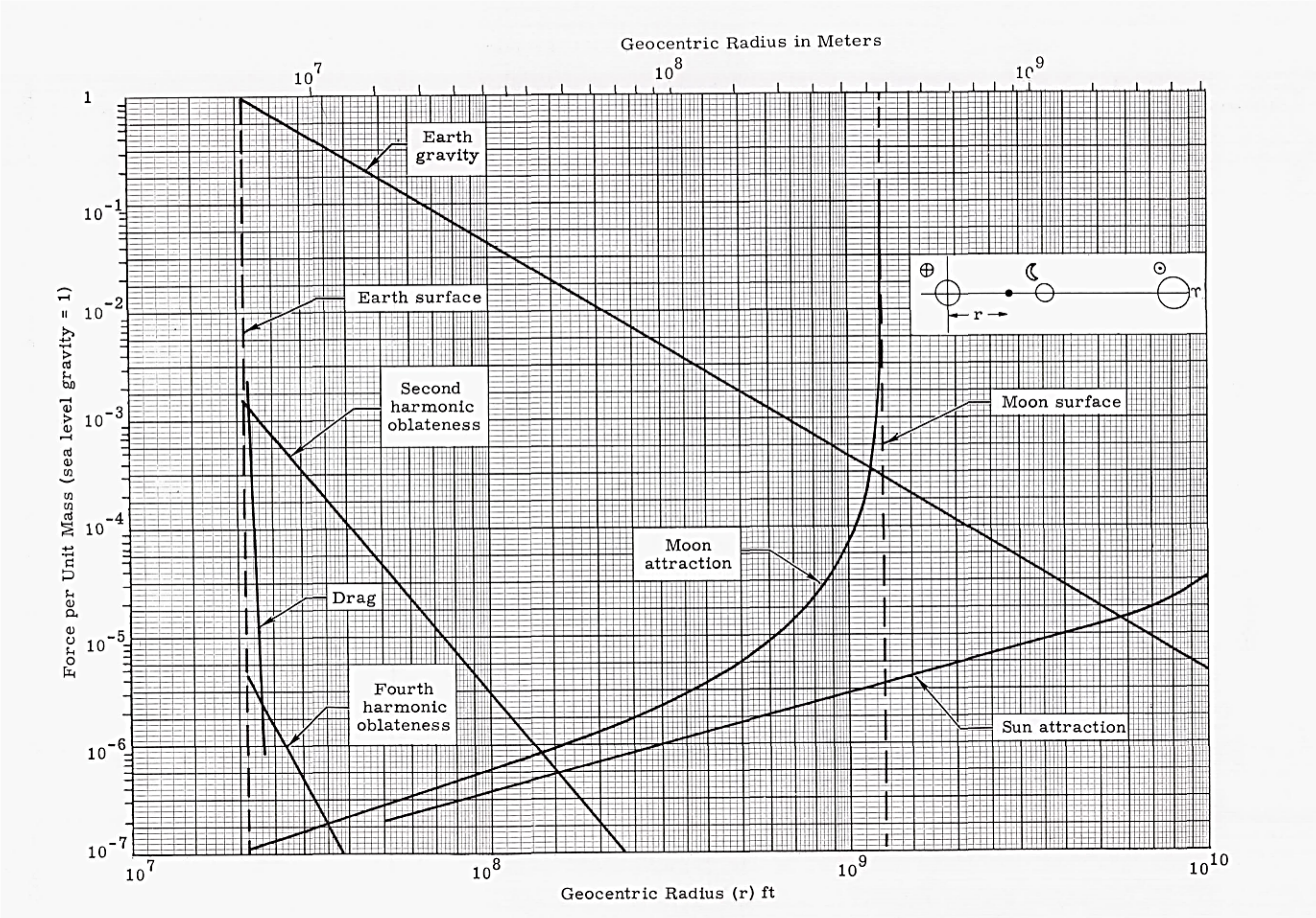


Fig. 1. Comparison of Perturbation Magnitudes (for equinoctial lunar conjunction)

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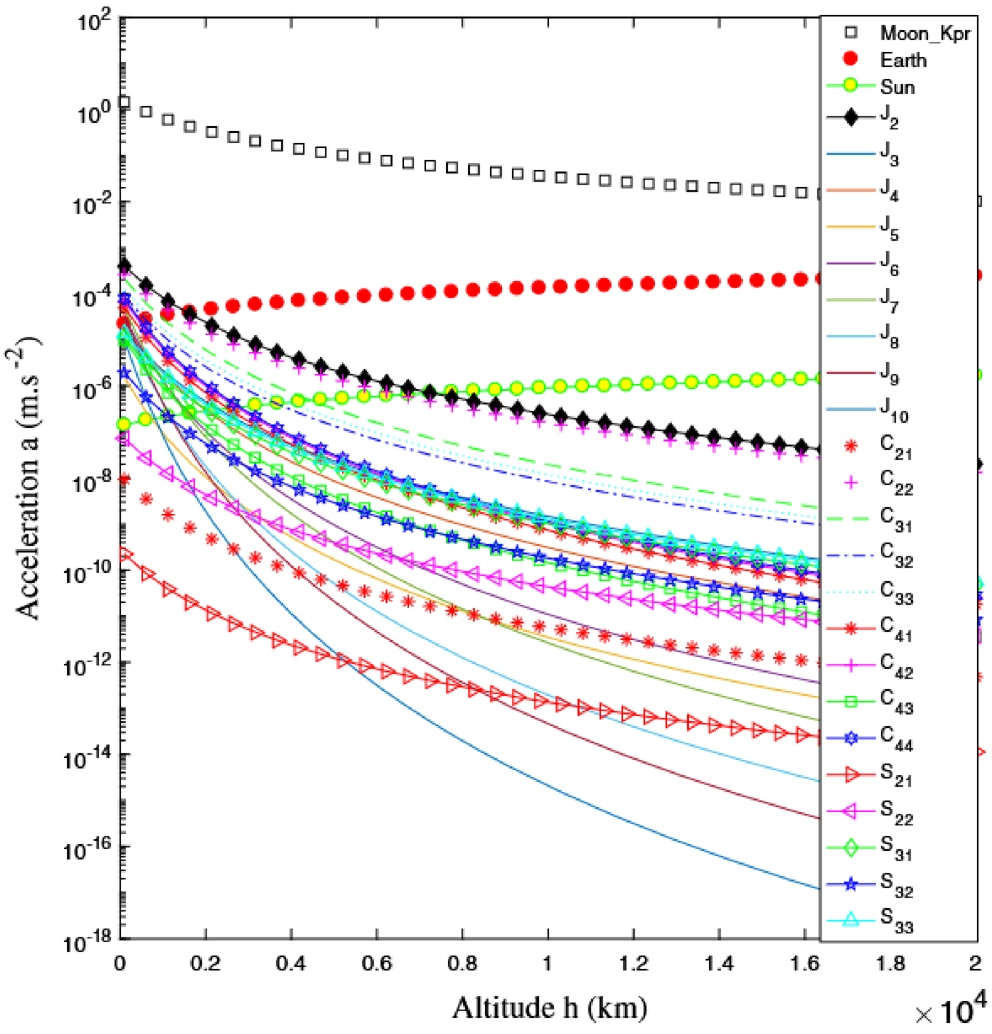

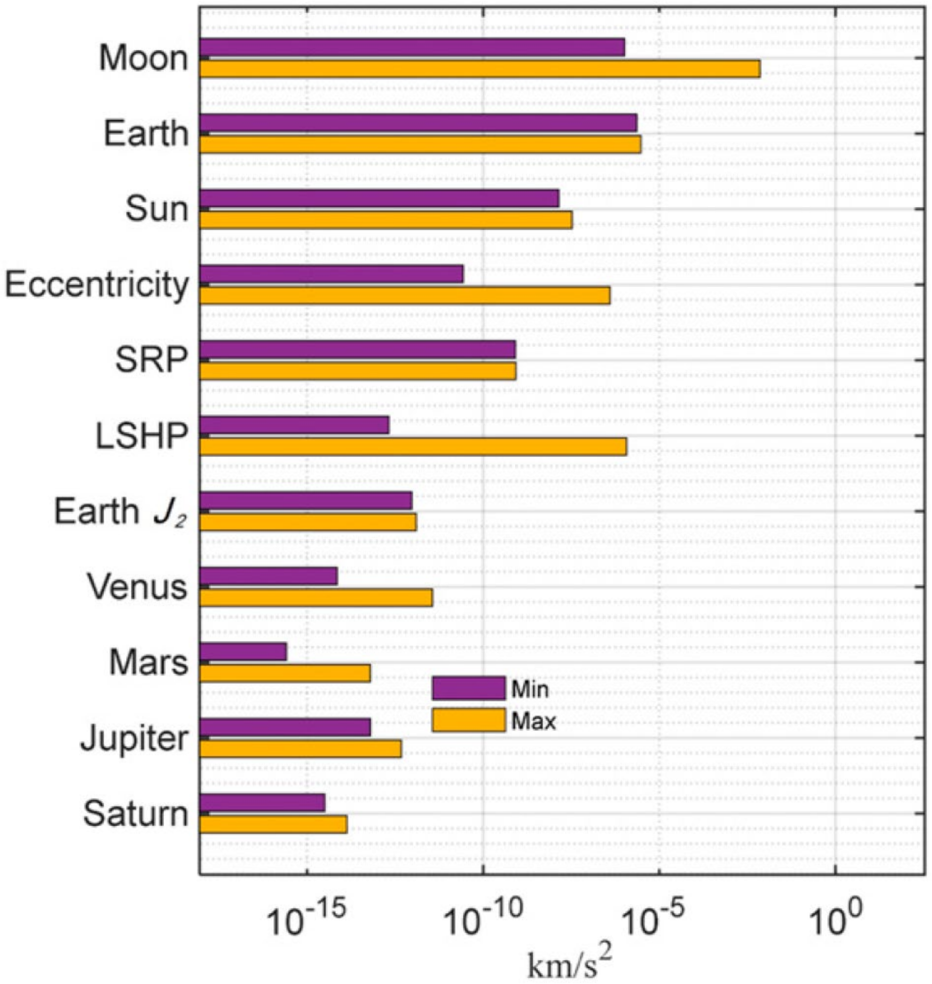

Left:
Tao Nie and Pini Gurfil
Lunar frozen orbits revisited
Figure 1.

Right:
Chongrui Du and Olga L. Starinova
Orbital perturbation analysis and generation of nominal near rectilinear halo orbits using low-thrust propulsion
Figure 24

ABSOLUTE RESTRICTED THREE BODY PROBLEM

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Relative formulation doesn't help much here…

STEP 1: NORMALIZED MOTION OF PRIMARIES IN ELLIPTICAL ORBIT

m_1, m_2 orbit their common center of mass in ellipses with eccentricity *e* and semi-major axis *a*

Normalize distances by semilatus rectum:

→ Distance (LU) = Distance (NDU) * p (LU)

Normalize time by characteristic time *T*:

$$T = \sqrt{\frac{p^3}{G(m_1 + m_2)}}$$

→ Time (TU) = Time (NDU) * *T*

→ This removes [ ] from the equation. Equivalent: choose *T* such that *G* (NDU) = [ ]

Normalize mass by

[ ]

Define mass parameter

[ ]

STEP 2: CHANGE TO ROTATING-PULSATING FRAME

Define a rotating and scaling frame:

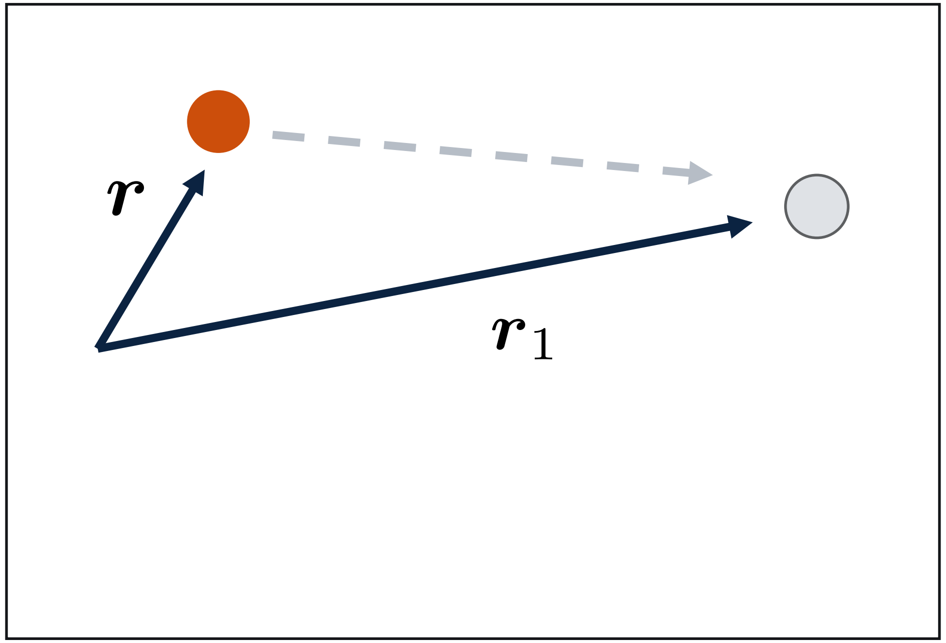
- Rotates at angular position *f(t)*
- Origin is at the center of mass of the primaries
- *x-axis* always points from m_1 to m_2, *z-axis* orthogonal to orbit plane
- Scales by $r_{12}(f)$ to maintain constant positions for the primaries in the rotating frame (fixed on the x-axis)

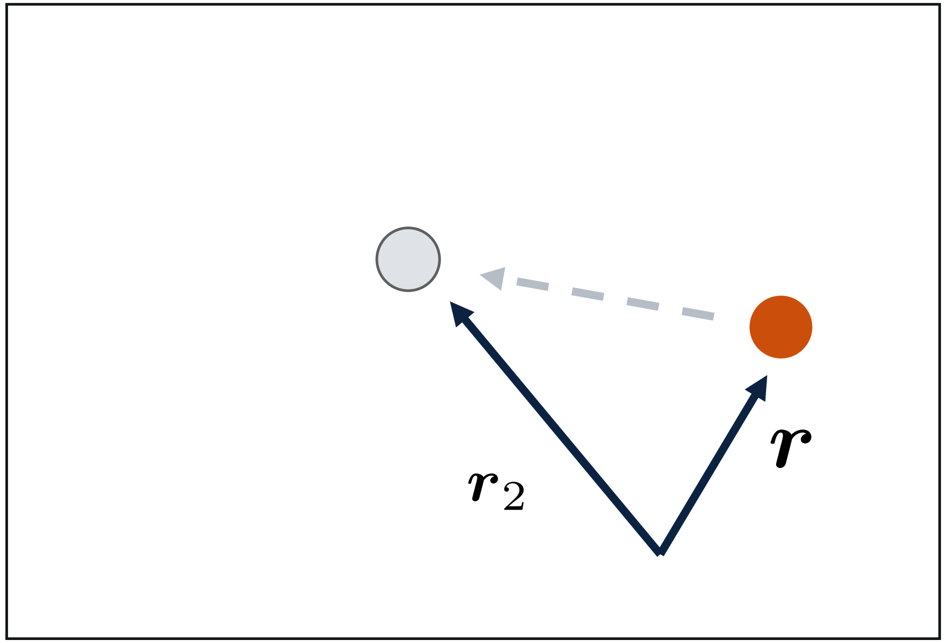
Draw frame

Let $\mathbf{r}=\mathbf{r_R}r_{12}(f)$, where $\mathbf{r_R}=(x,y,z)$ is position in rotating frame

The normalized instantaneous separation (distance) between the primaries is:

$$r_{12}(f) = \frac{1}{1 + e\cos(f)}$$





$$\frac{d^2(r_{12}\mathbf{r}_{\mathbf{R}})}{dt^2} = \frac{1}{r_{12}^2}\left((1-\mu)\frac{\mathbf{r}_{1R}-\mathbf{r}_{\mathbf{R}}}{|\mathbf{r}_{1R}-\mathbf{r}_{\mathbf{R}}|^3} + \mu\frac{\mathbf{r}_{2R}-\mathbf{r}_{\mathbf{R}}}{|\mathbf{r}_{2R}-\mathbf{r}_{\mathbf{R}}|^3}\right)$$

STEP 2: CHANGE TO ROTATING-PULSATING FRAME

Pulsation → apply chain rule

$$\frac{d\mathbf{r}}{dt} = \dot{r}_{12}\,\mathbf{r}_R + r_{12}\,\frac{d\mathbf{r}_R}{dt}$$

$$\frac{d^2\mathbf{r}}{dt^2} = \ddot{r}_{12}\,\mathbf{r}_R + 2\dot{r}_{12}\,\frac{d\mathbf{r}_R}{dt} + r_{12}\,\frac{d^2\mathbf{r}_R}{dt^2}$$

STEP 2: CHANGE TO ROTATING-PULSATING FRAME

Rotation → apply transport theorem (or BKE)

$$\left(\frac{d\mathbf{r_R}}{dt}\right)_{\text{inertial}} = \left(\frac{d\mathbf{r_R}}{dt}\right)_{\text{rot}} + \boldsymbol{\omega} \times \mathbf{r_R}$$

$$\left(\frac{d^2\mathbf{r_R}}{dt^2}\right)_{\text{inertial}} = \left(\frac{d^2\mathbf{r_R}}{dt^2}\right)_{\text{rot}} + 2\boldsymbol{\omega} \times \left(\frac{d\mathbf{r_R}}{dt}\right)_{\text{rot}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r_R}) + \dot{\boldsymbol{\omega}} \times \mathbf{r_R}$$

$$\frac{d^2(r_{12}\mathbf{r_R})}{dt^2} = \frac{1}{r_{12}^2}\left((1-\mu)\frac{\mathbf{r}_{1R}-\mathbf{r_R}}{|\mathbf{r}_{1R}-\mathbf{r_R}|^3} + \mu\frac{\mathbf{r}_{2R}-\mathbf{r_R}}{|\mathbf{r}_{2R}-\mathbf{r_R}|^3}\right)$$

$$\boxed{\ddot{\mathbf{r}} = \ddot{r}_{12}\,\mathbf{r}_R + 2\dot{r}_{12}\,\dot{\mathbf{r}}_R + r_{12}\,\ddot{\mathbf{r}}_R + 2r_{12}\,\boldsymbol{\omega}\times\dot{\mathbf{r}}_R + \dot{r}_{12}\,\boldsymbol{\omega}\times\mathbf{r}_R + r_{12}\,\dot{\boldsymbol{\omega}}\times\mathbf{r}_R + r_{12}\,\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}_R)}$$

$$\ddot{\mathbf{r}} = \ddot{r}_{12}\,\mathbf{r}_R + 2\dot{r}_{12}\,\dot{\mathbf{r}}_R + r_{12}\,\ddot{\mathbf{r}}_R + 2r_{12}\,\dot{f}\,\hat{\mathbf{z}}\times\dot{\mathbf{r}}_R + \dot{r}_{12}\,\dot{f}\,\hat{\mathbf{z}}\times\mathbf{r}_R + r_{12}\,\ddot{f}\,\hat{\mathbf{z}}\times\mathbf{r}_R + r_{12}\,\dot{f}\,\hat{\mathbf{z}}\times\left(\dot{f}\,\hat{\mathbf{z}}\times\mathbf{r}_R\right)$$

Simplify this expression by

1) Multiply both sides by r_{12}^3

2) Substitute 2BP relationships
(valid under current normalization)

$$r_{12}^2\dot{f} = 1 \qquad \dot{r}_{12} = -\frac{1}{2}r_{12}^3\ddot{f} \qquad \ddot{r}_{12} = \frac{1}{r_{12}^3} - \frac{1}{r_{12}^2}$$

STEP 3: WRITE EQUATIONS IN TERMS OF TRUE ANOMALY

$$\frac{d}{dt} = \frac{df}{dt} \cdot \frac{d}{df}, \qquad \frac{d^2}{dt^2} = \left(\frac{df}{dt}\right)^2 \frac{d^2}{df^2} + \frac{d^2 f}{dt^2} \cdot \frac{d}{df}$$

ER3BP IN NONUNIFORMLY ROTATING, ISOTOPICALLY PULSATING, BARYCENTRIC COORDINATE

$$\frac{d^2x}{df^2} - 2\frac{dy}{df} = \frac{1}{r_{12}(f)}\left[x - \frac{(1-\mu)(x+\mu)}{r_1^3} - \frac{\mu(x-1+\mu)}{r_2^3}\right]$$

$$\frac{d^2y}{df^2} + 2\frac{dx}{df} = \frac{1}{r_{12}(f)}\left[y - \frac{(1-\mu)y}{r_1^3} - \frac{\mu y}{r_2^3}\right]$$

$$\frac{d^2z}{df^2} = \frac{1}{r_{12}(f)}\left[-\frac{(1-\mu)z}{r_1^3} - \frac{\mu z}{r_2^3}\right]$$

Lagrangian Coherent Structures in the Elliptic Restricted Three-Body Problem by E. Gawlik provides an elegant and compact full derivation using complex coordinates, note normalization by semi-latus rectum ($p = a(1-e^2)$ for $e < 1$).

REDUCTION TO CR3BP

For e = *0*: $r_{12}(f) = 1$ and $df/dt = 1 \rightarrow df = dt$

Characteristic quantities also coincide (primary period → 2π, mean motion → 1)

$$\frac{d^2x}{dt^2} - 2\frac{dy}{dt} = x - \frac{(1-\mu)(x+\mu)}{r_1^3} - \frac{\mu(x-1+\mu)}{r_2^3}$$

$$\frac{d^2y}{dt^2} + 2\frac{dx}{dt} = y - \frac{(1-\mu)y}{r_1^3} - \frac{\mu y}{r_2^3}$$

$$\frac{d^2z}{dt^2} = -\frac{(1-\mu)z}{r_1^3} - \frac{\mu z}{r_2^3}$$

EFFECT OF NEGLECTING THE SECONDARY BODY

Open lecture001_livescript001

Given the same initial state vector, this script propagates one trajectory in the full CR3BP model and another with the **secondary body's gravity removed**

$$\frac{d^2x}{dt^2} - 2\frac{dy}{dt} = x - \frac{(1-\mu)(x+\mu)}{r_1^3} - \frac{\mu(x-1+\mu)}{r_2^3}$$

$$\frac{d^2y}{dt^2} + 2\frac{dx}{dt} = y - \frac{(1-\mu)y}{r_1^3} - \frac{\mu y}{r_2^3}$$

$$\frac{d^2z}{dt^2} = -\frac{(1-\mu)z}{r_1^3} - \frac{\mu z}{r_2^3}$$

- How do we measure the difference between two trajectories?
- How quickly does that difference grow over time?
- Where does neglecting the secondary body remain acceptable?
- How does the acceptable error depend on the mission objective?

FROM PERTURBATIONS TO DYNAMICAL STRUCTURES

Two-body viewpoint

- dominant central body
- conic trajectories
- orbital elements as the main language
- perturbations added afterward

Multibody viewpoint

- multiple gravitating bodies matter simultaneously
- selected perturbing forces become part of the dynamical model
- two-body as special case
- new solutions and new dynamical structures

Which forces matter most? Comparing magnitudes is a useful starting point, but it does not provide the full answer.