GRAVITATIONAL MULTIBODY DYNAMICS: A QUICK-START GUIDE

Course plan:

1) **Choosing the right model**
2) **Going with the flow**
3) **Periodic motion**
4) **Stability and orbit manifolds**
5) **From CR3BP to ephemeris**

Davide Guzzetti

May-June 2026, NESC Academy Series

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ABSOLUTE RESTRICTED THREE BODY PROBLEM

We consider three bodies:

- m_1, m_2: primary and secondary bodies (massive) move under mutual gravitation
- $m_3 \rightarrow 0$: third body (infinitesimal mass, doesn't influence m_1, m_2)

$$\ddot{\mathbf{r}} = Gm_1 \frac{\mathbf{r}_1 - \mathbf{r}}{\|\mathbf{r}_1 - \mathbf{r}\|^3} + Gm_2 \frac{\mathbf{r}_2 - \mathbf{r}}{\|\mathbf{r}_2 - \mathbf{r}\|^3}$$

Lecture 2:

GOING WITH THE FLOW

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THE CIRCULAR RESTRICTED THREE BODY PROBLEM

n=1 implicit

$$\frac{d^2x}{dt^2} - 2\frac{dy}{dt} = x - \frac{(1-\mu)(x+\mu)}{r_1^3} - \frac{\mu(x-1+\mu)}{r_2^3}$$

$$\frac{d^2y}{dt^2} + 2\frac{dx}{dt} = y - \frac{(1-\mu)y}{r_1^3} - \frac{\mu y}{r_2^3}$$

$$\frac{d^2z}{dt^2} = -\frac{(1-\mu)z}{r_1^3} - \frac{\mu z}{r_2^3}$$

HOW TO SOLVE THE CR3BP INITIAL VALUE PROBLEM?

- What does it mean for system dynamics be chaotic?
- Permissible regions of motion
- Equilibrium solutions
- Maps

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WHAT DOES IT MEAN FOR SYSTEM DYNAMICS TO BE CHAOTIC?

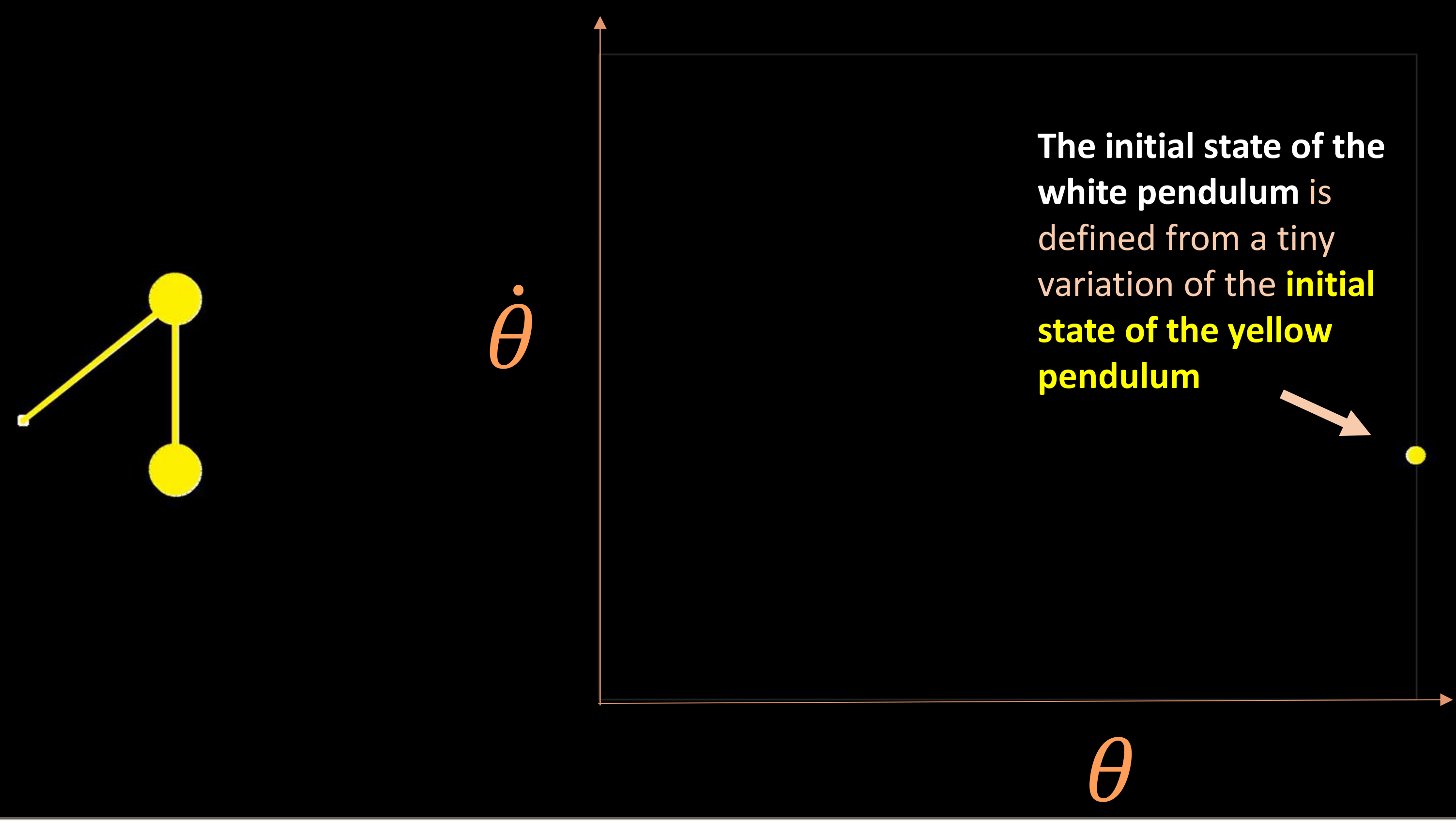
For a map $f: X \rightarrow X$ on a metric space X, the system is often called **chaotic in the sense of Devaney** if:

- f is topologically transitive → *there is mixing, given enough time*
- periodic points are dense in X → *every open region of the state space contains at least one periodic orbit*
- f **has sensitive dependence on initial conditions**. → *can be derived from conditions above (Banks at al.)*

Robert L. Devaney, *An Introduction to Chaotic Dynamical Systems*, 2nd ed., Addison-Wesley, 1989.

WHAT DOES IT MEAN FOR SYSTEM DYNAMICS TO BE CHAOTIC?

The trajectory of the white pendulum is not a tiny variation of the **trajectory of the yellow pendulum**



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DERIVATION OF THE JACOBI CONSTANT FROM THE EQUATIONS OF MOTION

$$\dot{x}\cdot\left(\ddot{x} - 2\dot{y} = \frac{\partial U^*}{\partial x}\right)$$

$$+\ \dot{y}\left(\ddot{y} + 2\dot{x} = \frac{\partial U^*}{\partial y}\right)$$

$$+\ \dot{z}\left(\ddot{z} = \frac{\partial U^*}{\partial z}\right) =$$

$$\frac{1}{2}\frac{d}{dt}\underbrace{\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)}_{v^2} = \frac{d}{dt}U^*(x,y,z)$$

$$U^* = \frac{(1-\mu)}{d} + \frac{\mu}{r} + \frac{1}{2}(x^2+y^2)$$

this quantity is conserved!

$$\Rightarrow \frac{d}{dt}\left(2U^* - v^2\right) = 0$$

ZERO VELOCITY CURVES

$$\text{JC} = 2U^* - v^2$$

REARRANGE

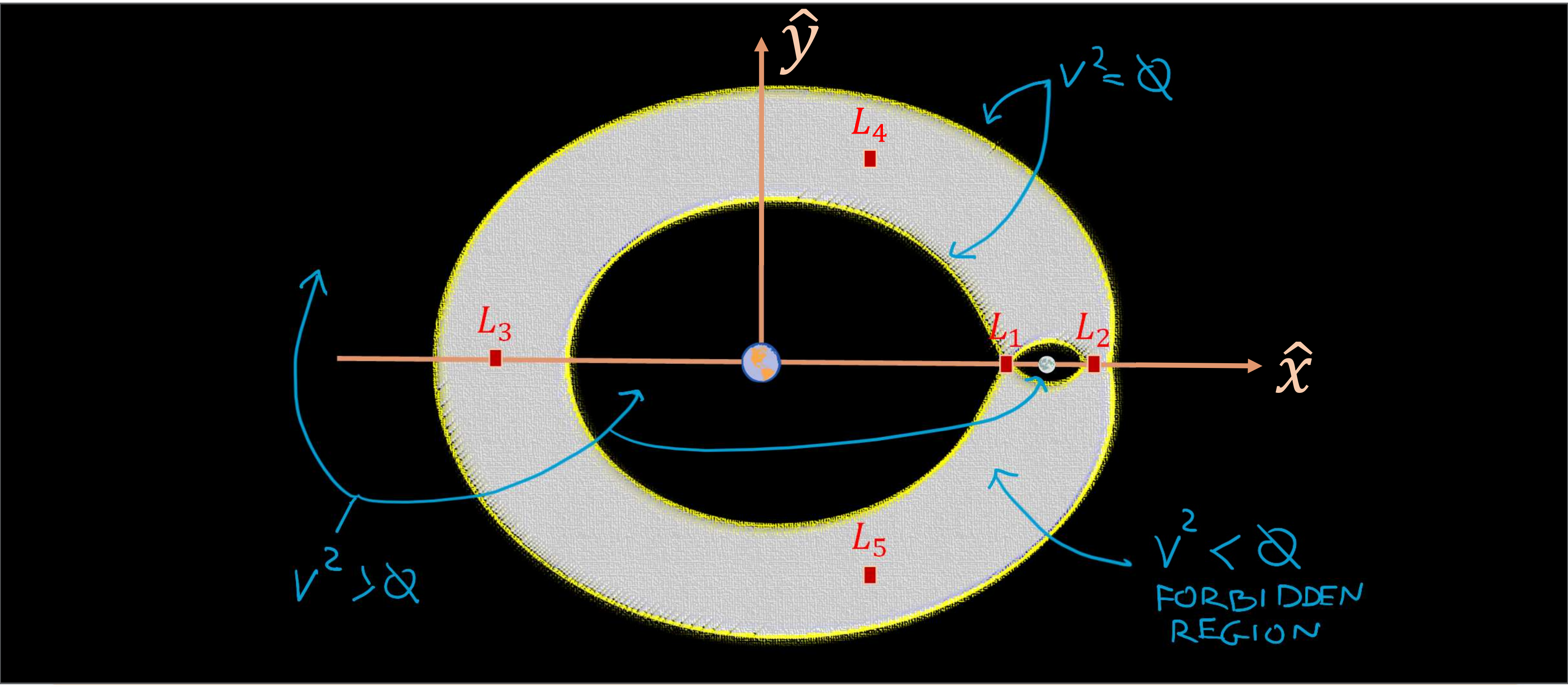
$$V^2 = 2U^* - JC = 0$$

ASSUME PLANAR DYNAMICS $U^* = U^*(x, y, z = 0)$

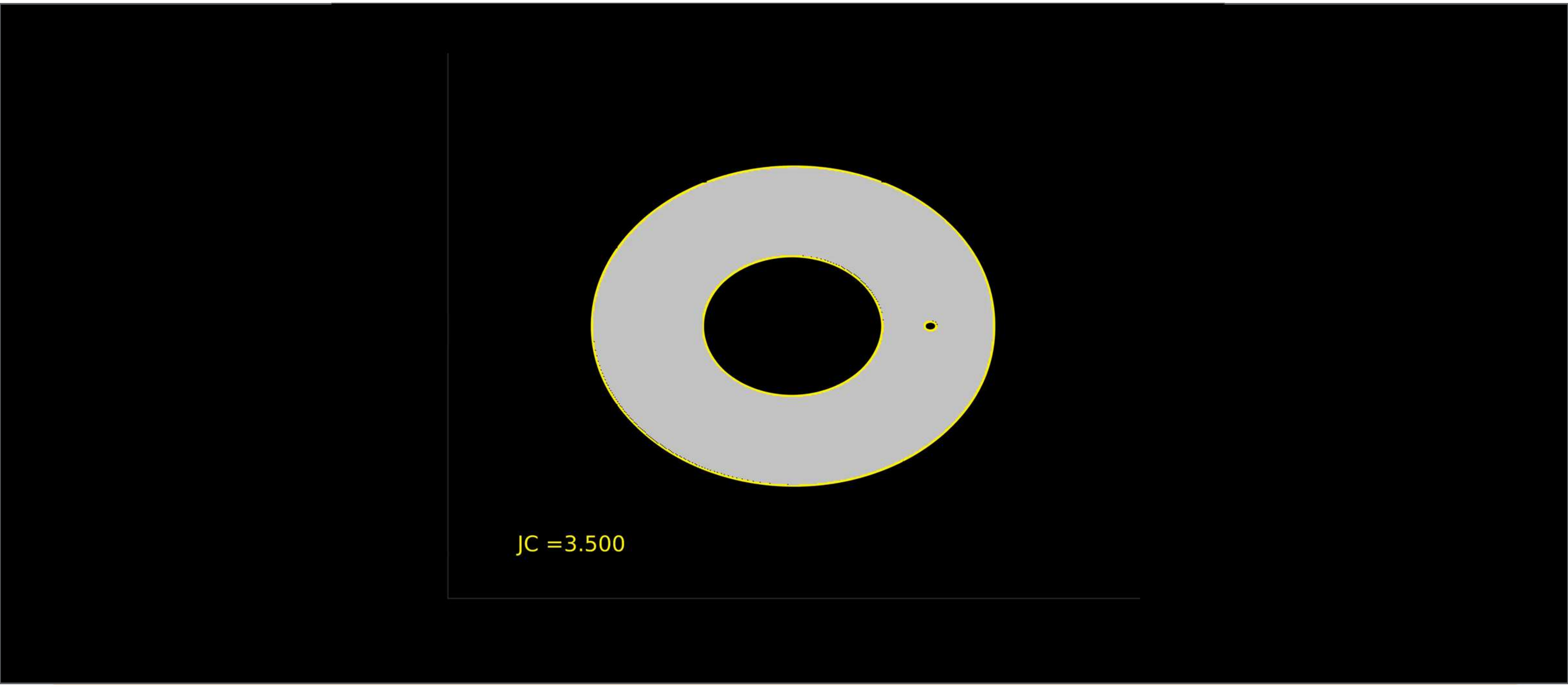
$$2U^*(x, y) - C = 0$$

EQN OF A CURVE WHERE $V^2 = 0$

ZERO VELOCITY CURVES AS QUALITATIVE SOLUTION



ZVC VARIATION WITH CHANGING JACOBI CONSTANT



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EQUILIBRIUM SOLUTIONS

INCLUDE FICTITIOUS FORCES

$$\sum \vec{F} = \vec{0}$$

rotating Frame

RHS

$$\frac{d\vec{x}}{dt} = \vec{f}(\vec{x}, t) = \vec{0}$$

acceleration = 0

velocity = 0

$$\frac{dU^*}{d\vec{x}}(\vec{x}, \cancel{t}) = \vec{0}$$

FCN OF POSITION ONLY

(not velocity, not time)

Collinear Points

$$\frac{dU^*}{dx}(x) = 0 \rightarrow -\frac{(1-\mu)}{(x+\mu)^2} + \frac{\mu}{(x-1+\mu)^2} + x = 0$$

L1: $x_{L_1} = 1-\mu-\gamma_1 \;,\; \gamma_1 > 0$

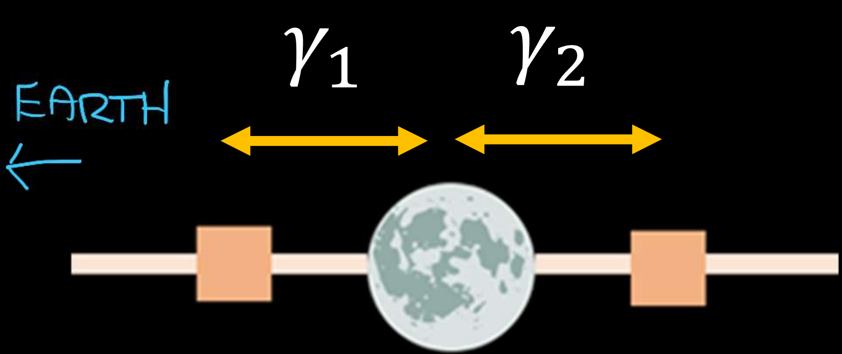
$$\gamma_1^5 - (3-\mu)\gamma_1^4 + (3-2\mu)\gamma_1^3 - \mu\gamma_1^2 + 2\mu\gamma_1 - \mu = 0$$

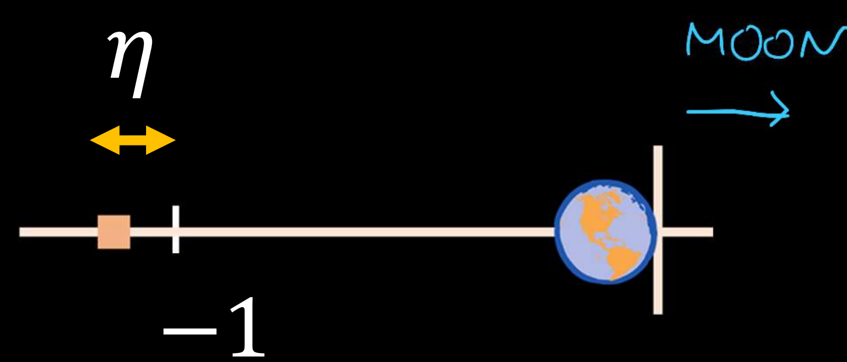
L2: $x_{L_2} = 1-\mu+\gamma_2 \;,\; \gamma_2 > 0$

$$\gamma_2^5 + (3-\mu)\gamma_2^4 + (3-2\mu)\gamma_2^3 - \mu\gamma_2^2 - 2\mu\gamma_2 - \mu = 0$$

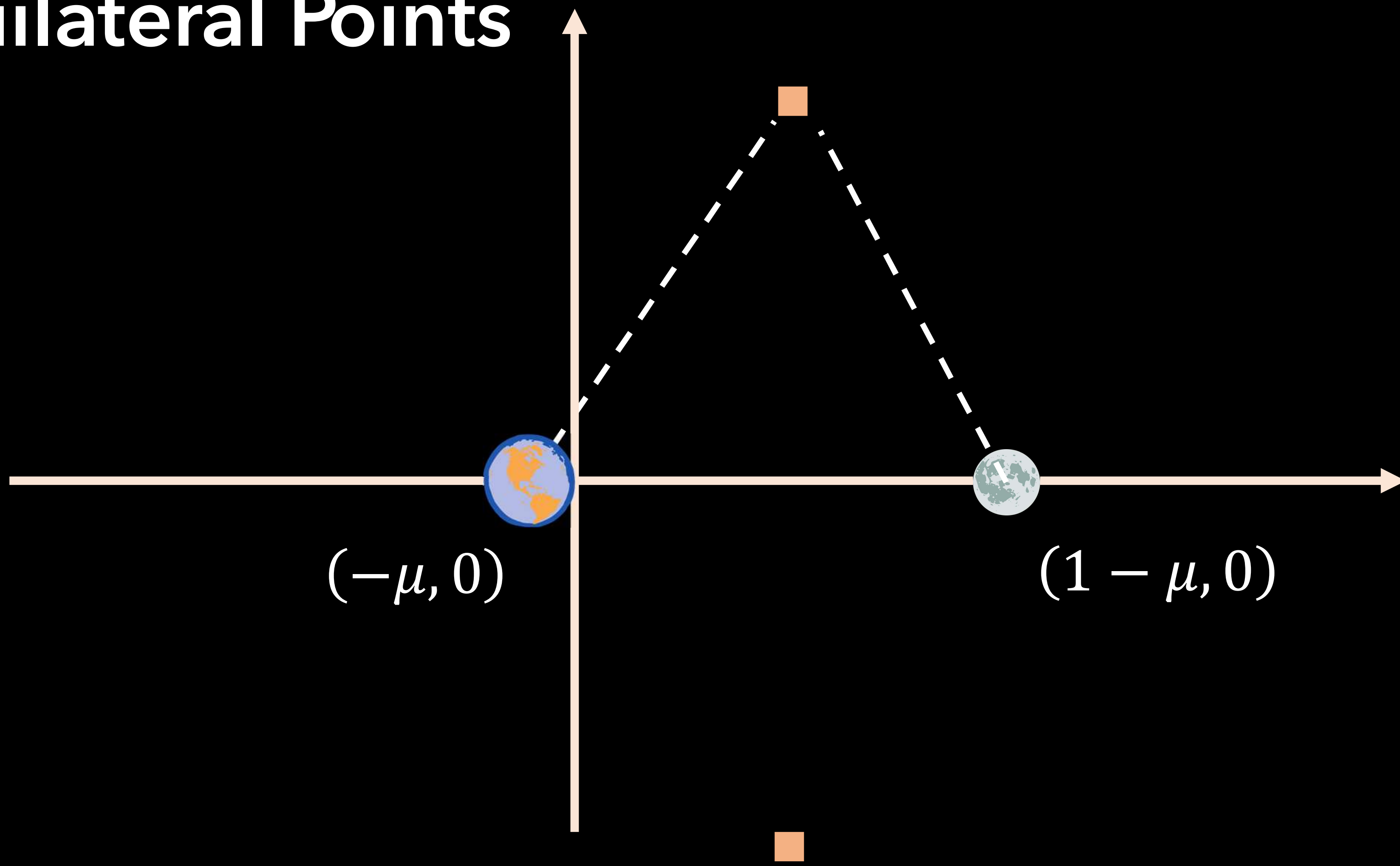
L3: $x_{L_3} = -1-\eta \;,\; \eta > 0$

$$\eta^5 + (7+\mu)\eta^4 + (19+6\mu)\eta^3 + (24+13\mu)\eta^2 + 2(6+7\mu)\eta + 7\mu = 0$$





Equilateral Points



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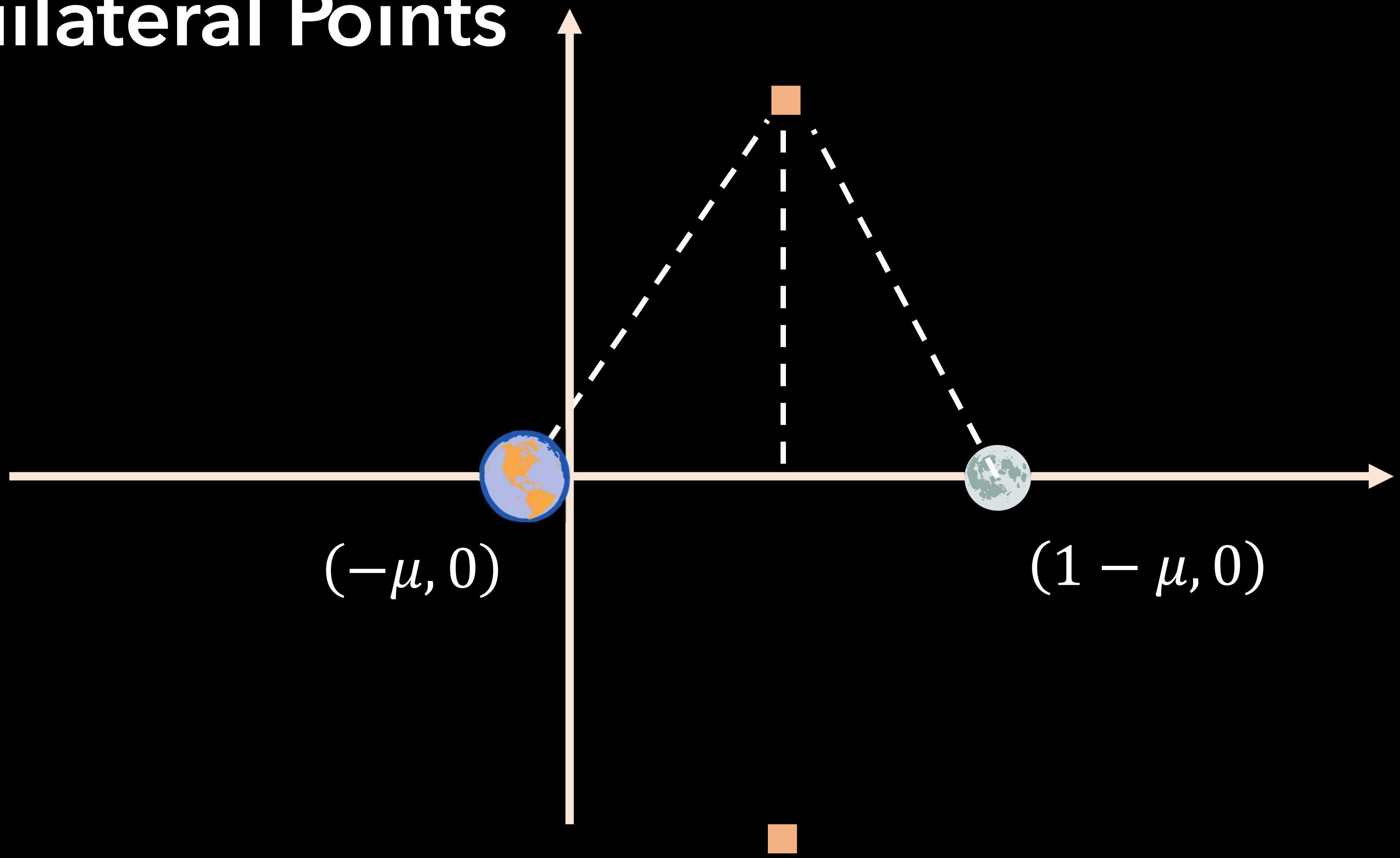
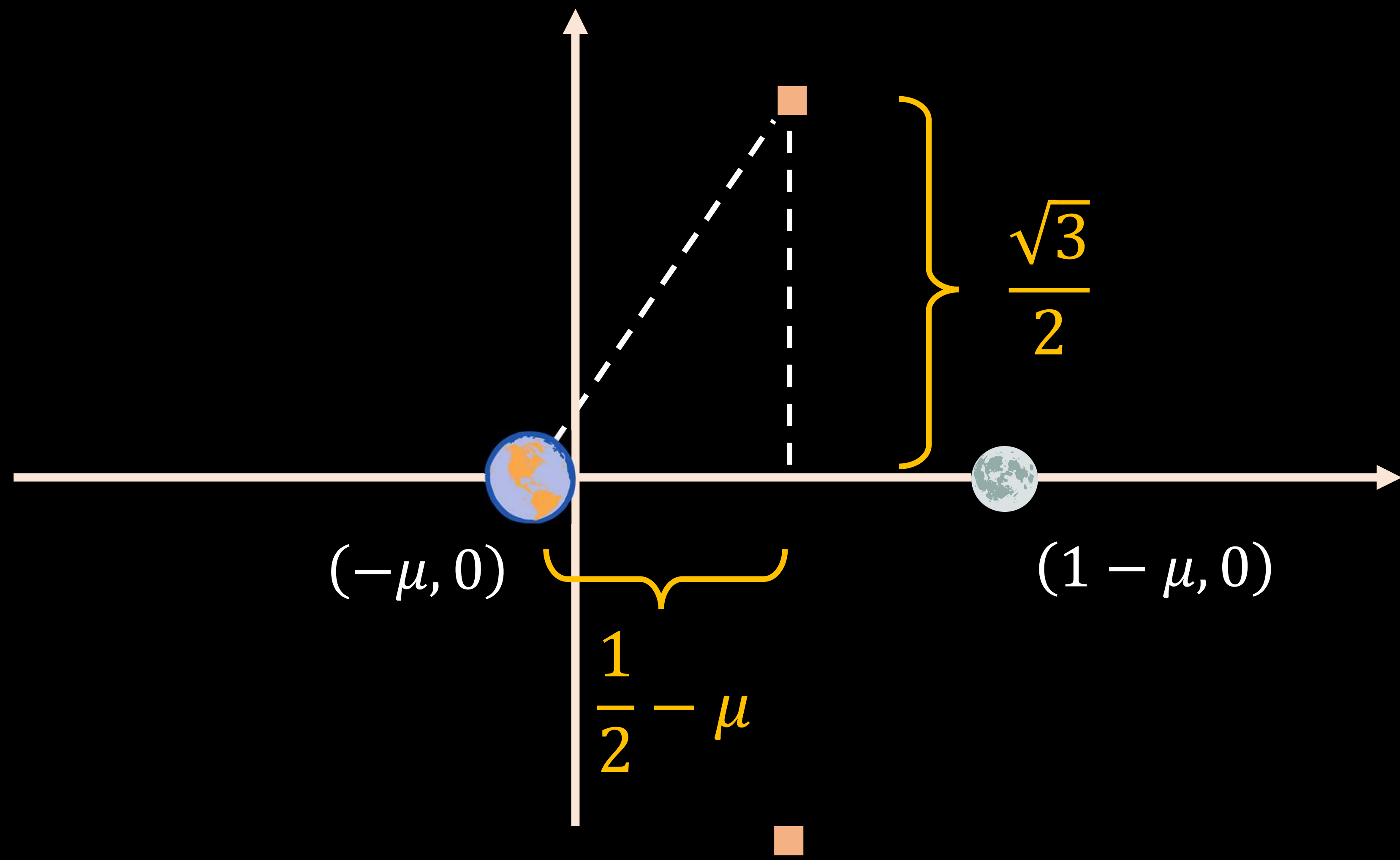
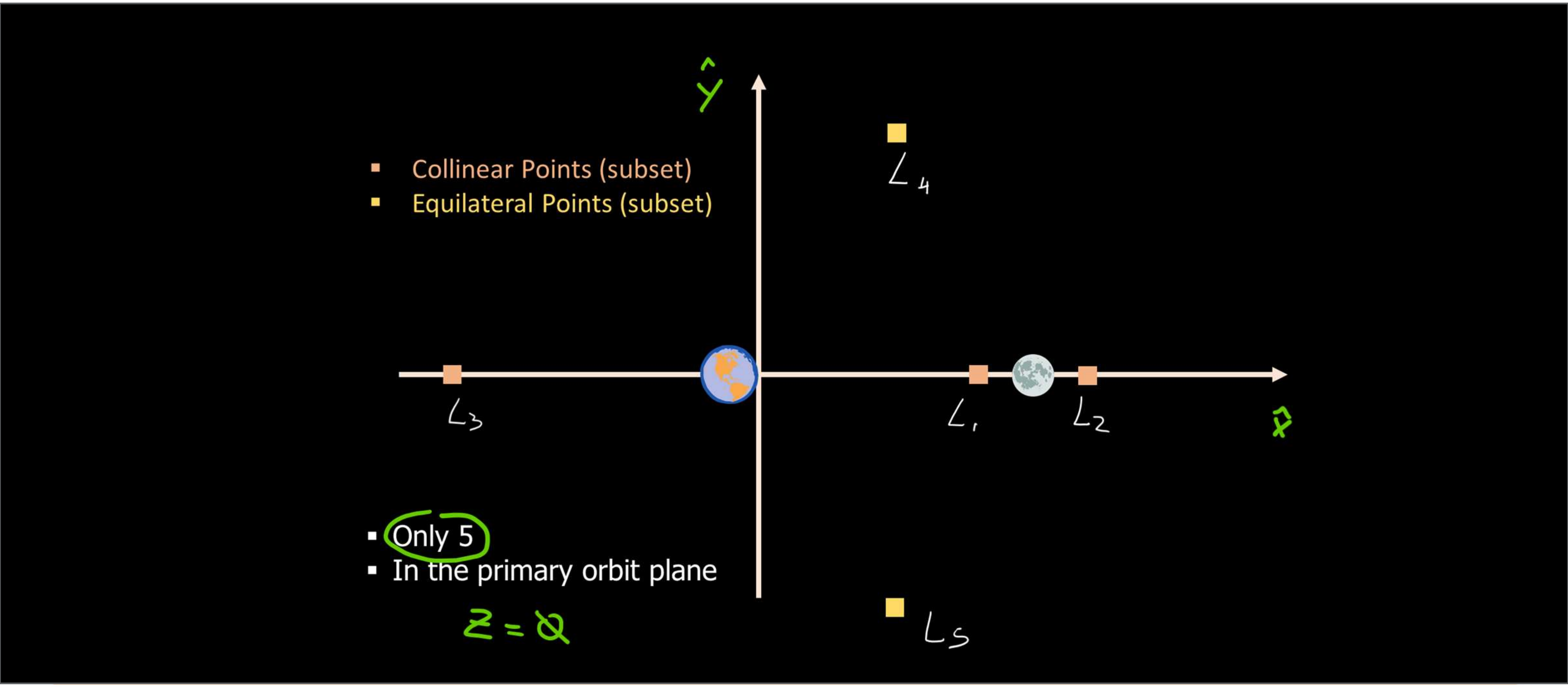




$\frac{\sqrt{3}}{2}$
$(-\mu, 0)$
$(1-\mu, 0)$
$\frac{1}{2} - \mu$

THE FIVE EQUILIBRIUM POINTS



LOCATION OF EQUILIBRIUM POINTS IN NOTABLE CELESTIAL SYSTEMS

Equilibrium point	Approximate distance from Earth	
Earth-Moon L_1	326,400 km (58,000 km from Moon)	*10x GEO*
Earth-Moon L_2	448,900 km (64,500 km from Moon)	
Earth-Moon L_3	384,400 km	
Earth-Moon L_4, L_5	384,400 km	
Sun-Earth L_1	1,491,500 km	*4x Earth-Moon*
Sun-Earth L_2	1,501,500 km	*4x Earth-Moon*
Sun-Earth L_3	2 AU	
Sun-Earth L_4, L_5	1 AU	

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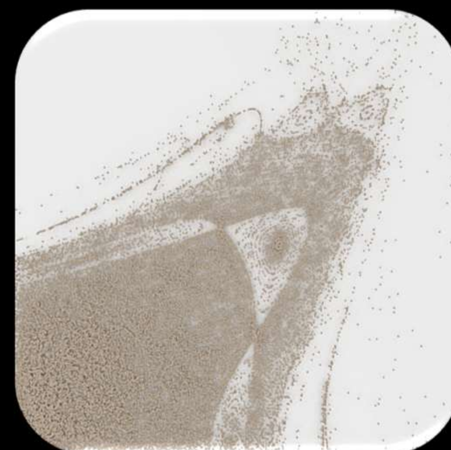
Describing the dynamical flow in two steps

π

Formal definition

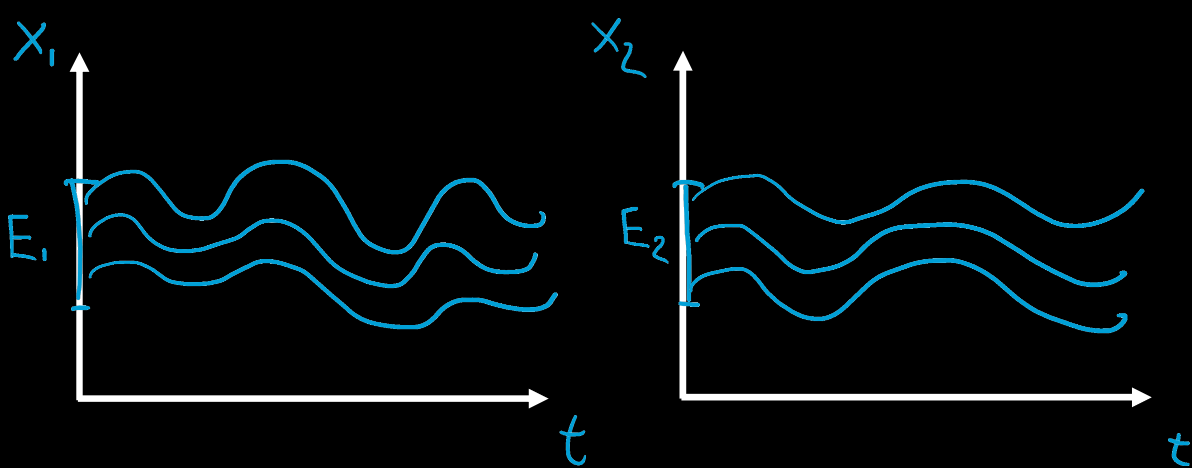


Visual trace

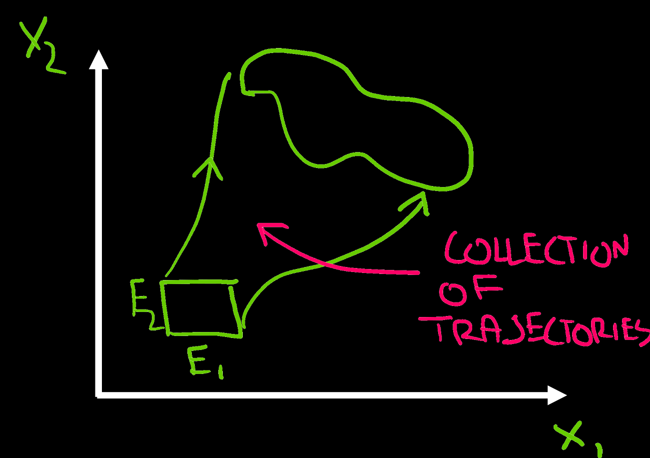
Dynamical flow

IVP $\begin{cases} \dot{x}_1 = f_1(x_1, x_2) \\ \dot{x}_2 = f_2(x_1, x_2) \end{cases}$ with $x_1(t=t_0) \in E_1$, $x_2(t=t_0) \in E_2$

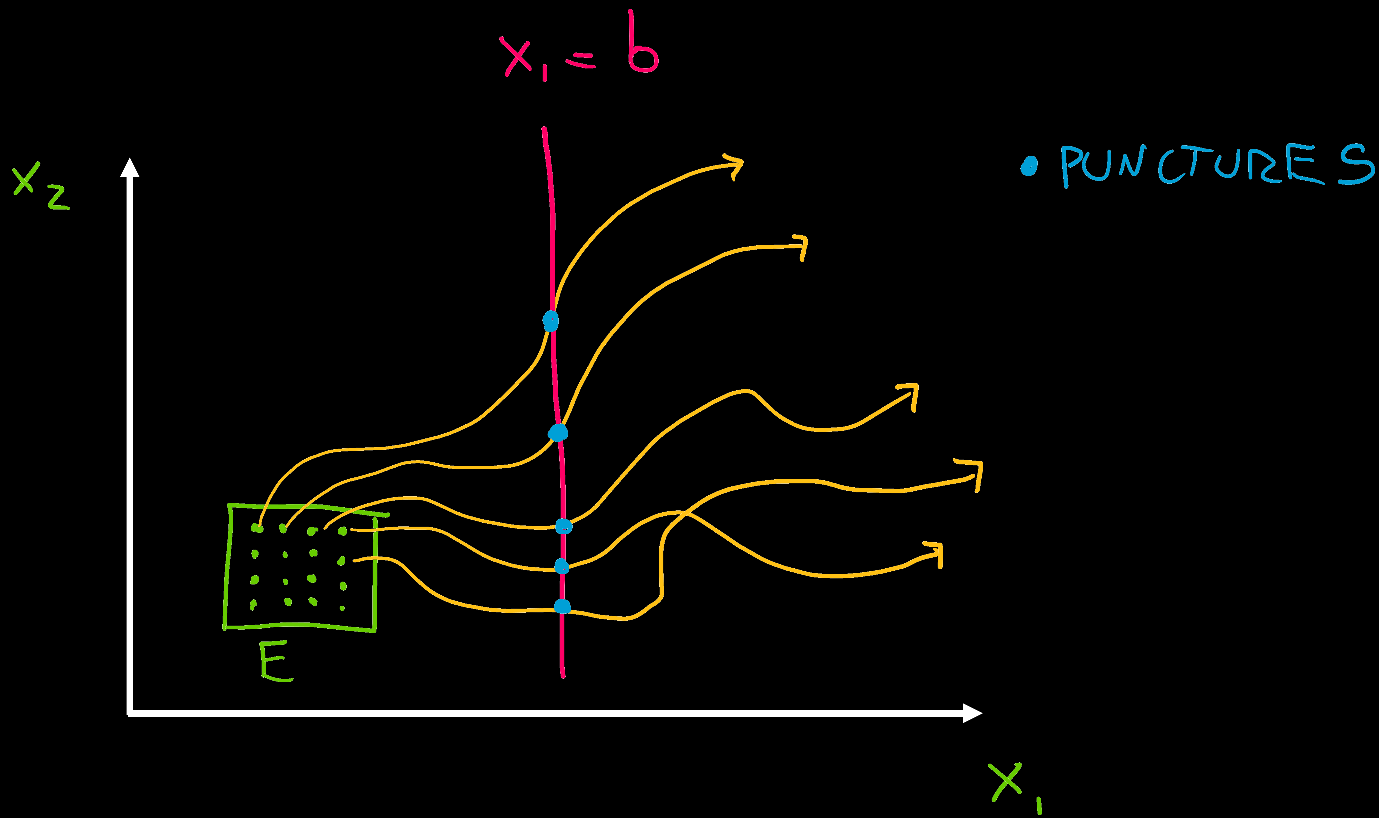
SOLUTION AS TIME EVOLUTION OF SYSTEM STATES



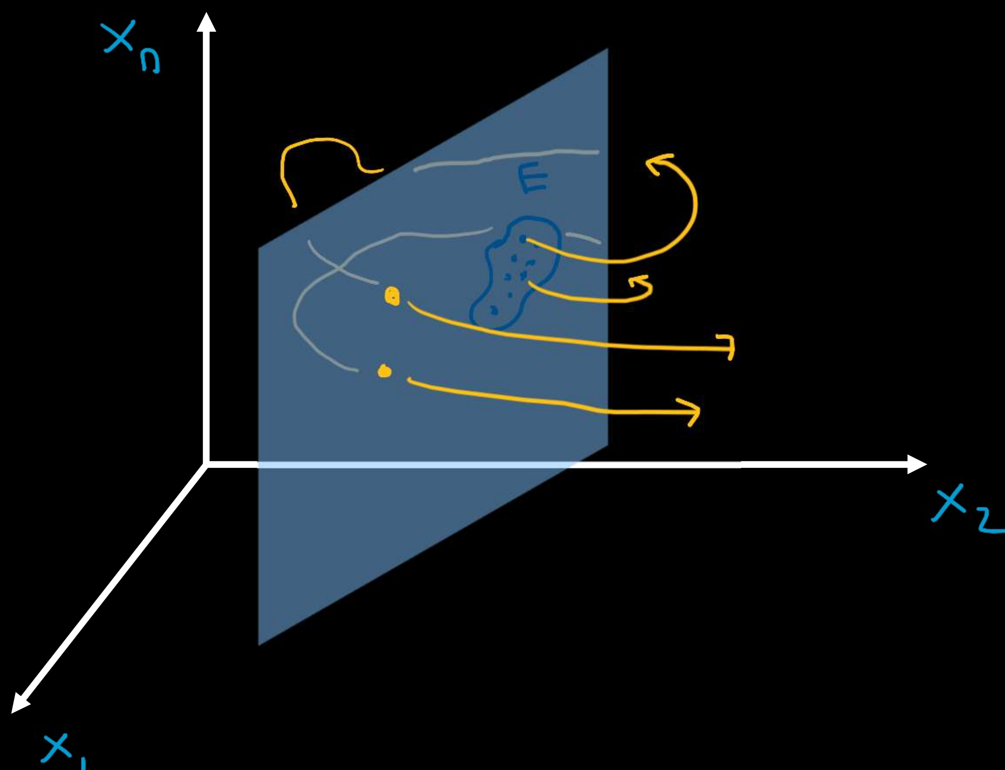
SOLUTION AS EVOLUTION OF $E_1 \cup E_2$



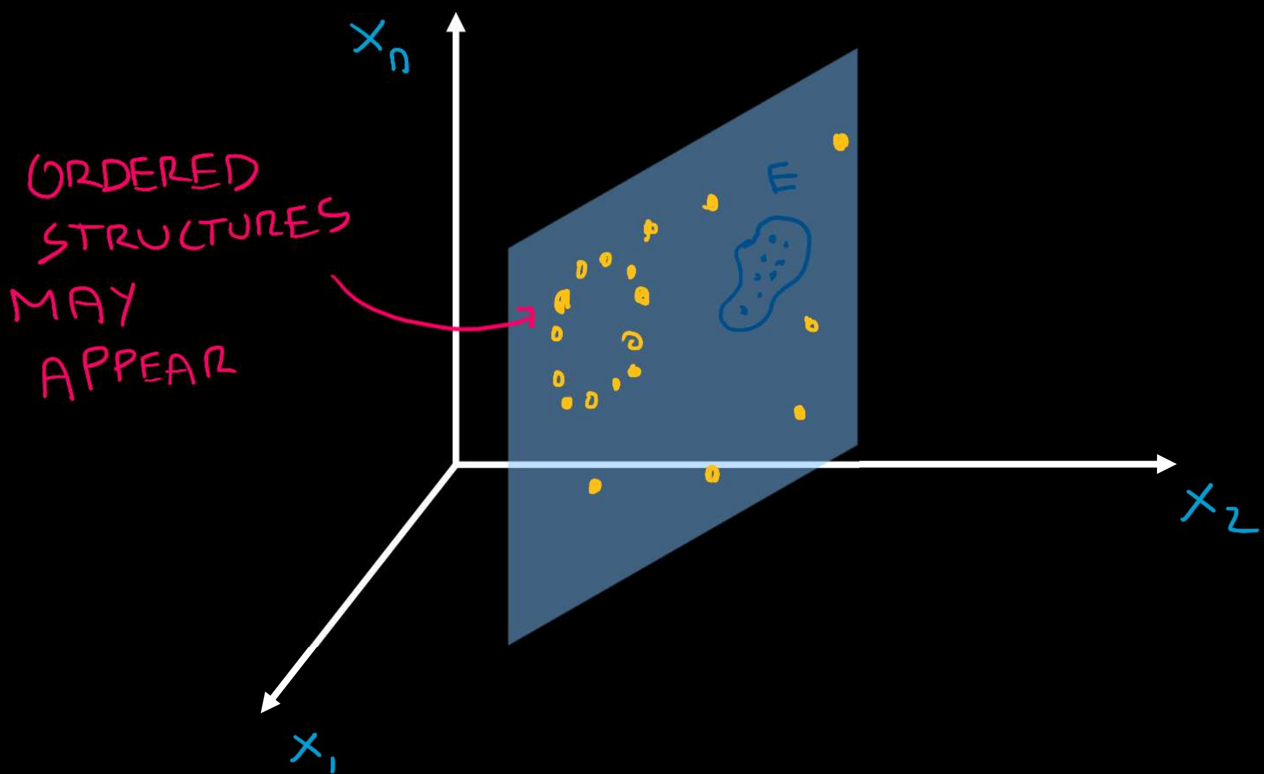
Section of the flow



In n-dimensions

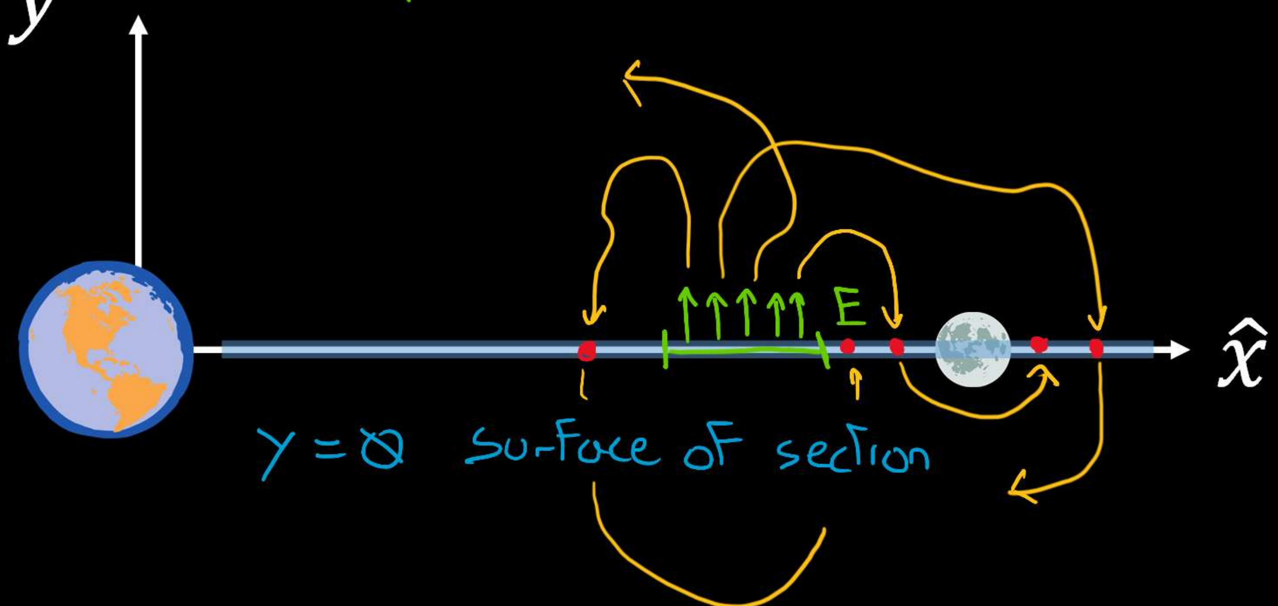


Poincaré map



Example in CR3BP

$$E : \left\{ x, y, \dot{x}, \dot{y} \;\middle|\; x_L \leq x \leq x_U,\ y = 0,\ \dot{x} = 0,\ JC(x, y, \dot{x}, \dot{y}) = \tilde{C} \right\}$$



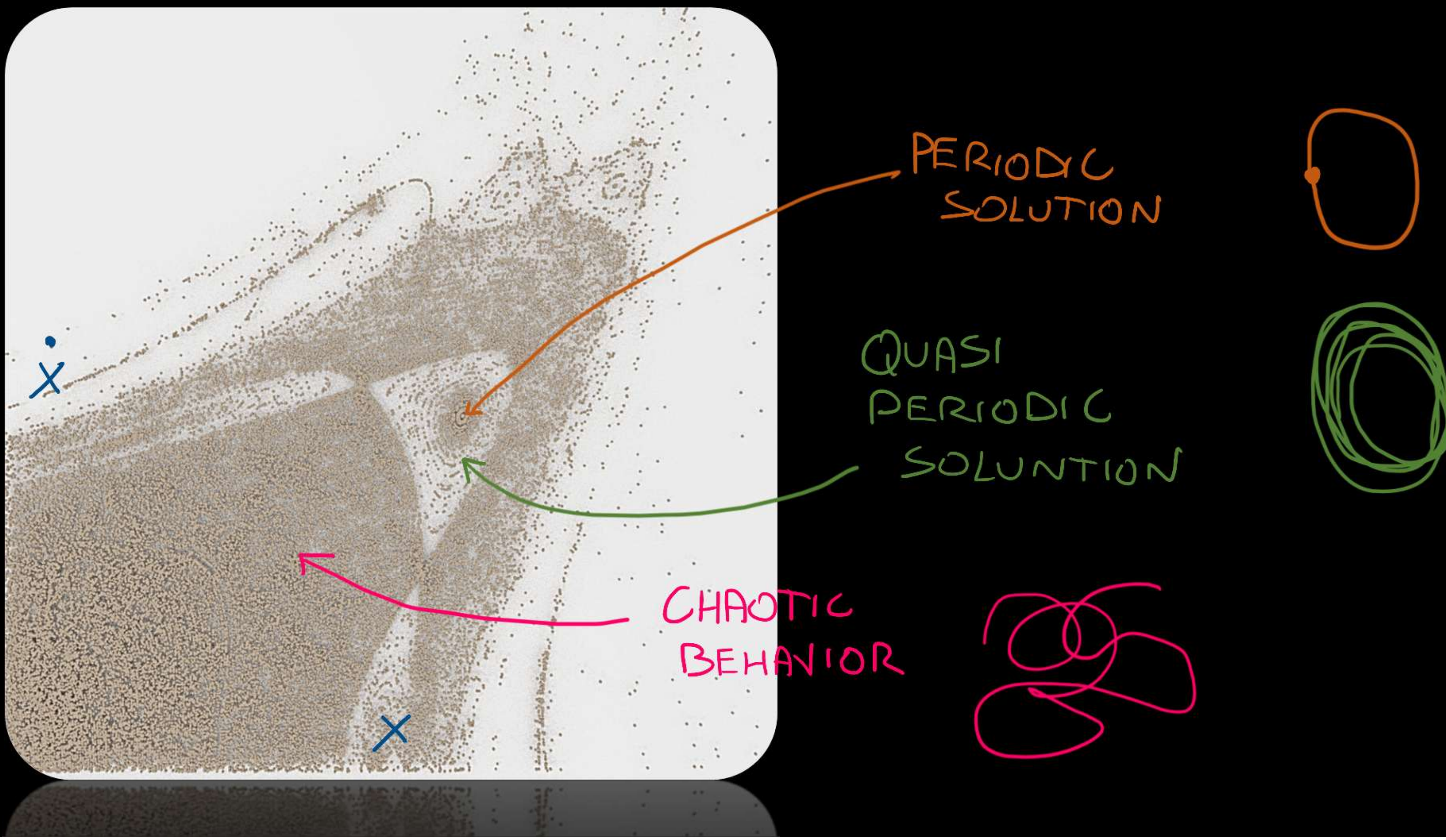
1) DEFINE A SURFACE OF SECTION

2) DEFINE A SET OF ICs E

3) PROPAGATE ICs ∈ E

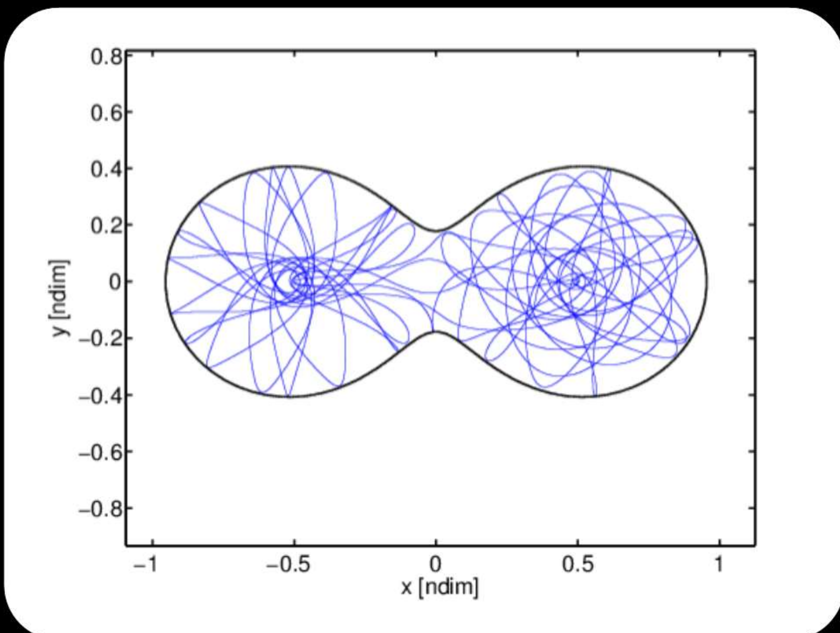
4) RECORD PUNCTURES

Example in CR3BP



Example in CR3BP

CHAOTIC



PERIODIC (IN RED)
QUASI-PERIODIC (IN BLUE)