GRAVITATIONAL MULTIBODY DYNAMICS: A QUICK-START GUIDE

Course plan:

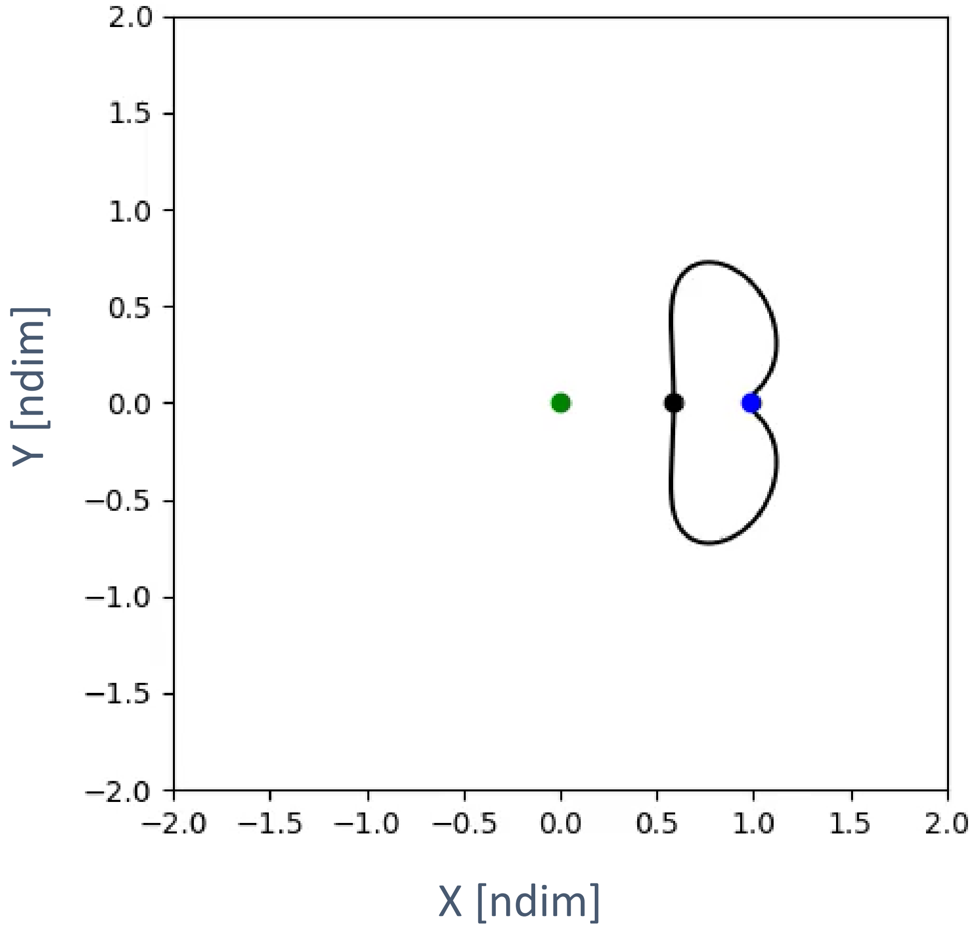
1) **Choosing the right model**
2) **Going with the flow**
3) **Periodic motion**
4) **Stability and orbit manifolds**
5) **From CR3BP to ephemeris**

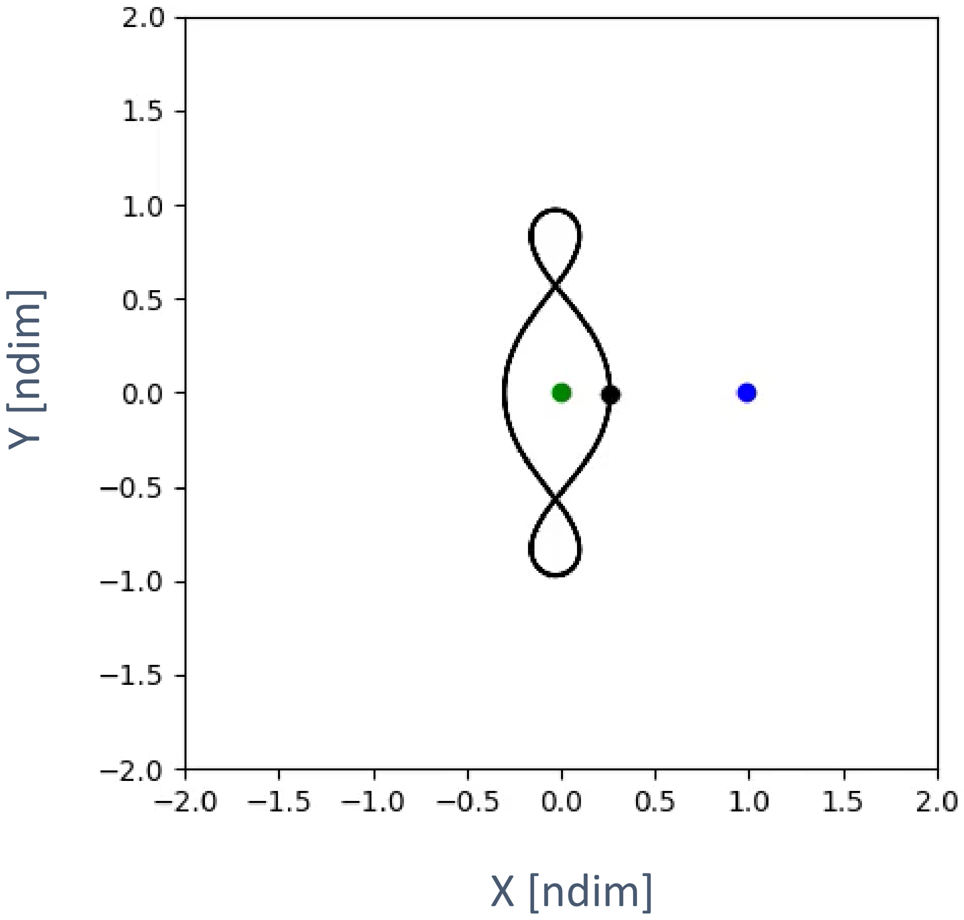
Davide Guzzetti

May-June 2026, NESC Academy Series

AUBURN

PERIODICITY IN DIFFERENT FRAMES





Lecture 3:

PERIODIC MOTION

Gravitational Multibody Dynamics: a Quick-start Guide

Davide Guzzetti

May-June 2026, NESC Academy Series

AUBURN

DEFINITION OF A PERIODIC ORBIT

Let the state evolve under an autonomous vector field

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \qquad \mathbf{x} \in \mathbb{R}^n$$

The flow map advances the initial condition to its state at time t

$$\varphi_t(\mathbf{x}_0) = \mathbf{x}(t; \mathbf{x}_0)$$

A trajectory

$$\Gamma = \{\mathbf{x}(t; \mathbf{x}_0) : 0 \leq t < T\}$$

is periodic if there exists a period $T > 0$ such that the time-T flow returns the state to itself

$$\varphi_T(\mathbf{x}_0) = \mathbf{x}_0$$

AUTONOMOUS SYSTEMS AND PHASE INVARIANCE

For an autonomous system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

if x(t) is periodic, then any phase shift
x(t + tau) is also periodic

$$\varphi_T(\mathbf{x}(t)) = \mathbf{x}(t) \quad \Longrightarrow$$

$$\varphi_T(\mathbf{x}(t+\tau)) = \mathbf{x}(t+\tau)$$

Every point on the orbit may serve as an initial condition → a periodic orbit corresponds to a continuum of fixed points of the time-T flow map.

$$\varphi_T(\mathbf{x}^*) = \mathbf{x}^*, \qquad \mathbf{x}^* \in \Gamma$$

ORIGIN AND TAXONOMY

Collinear point

L_1

Lyapunov
Halo
Axial
Vertical
…

L_2

L_3

Equilateral point

L_4

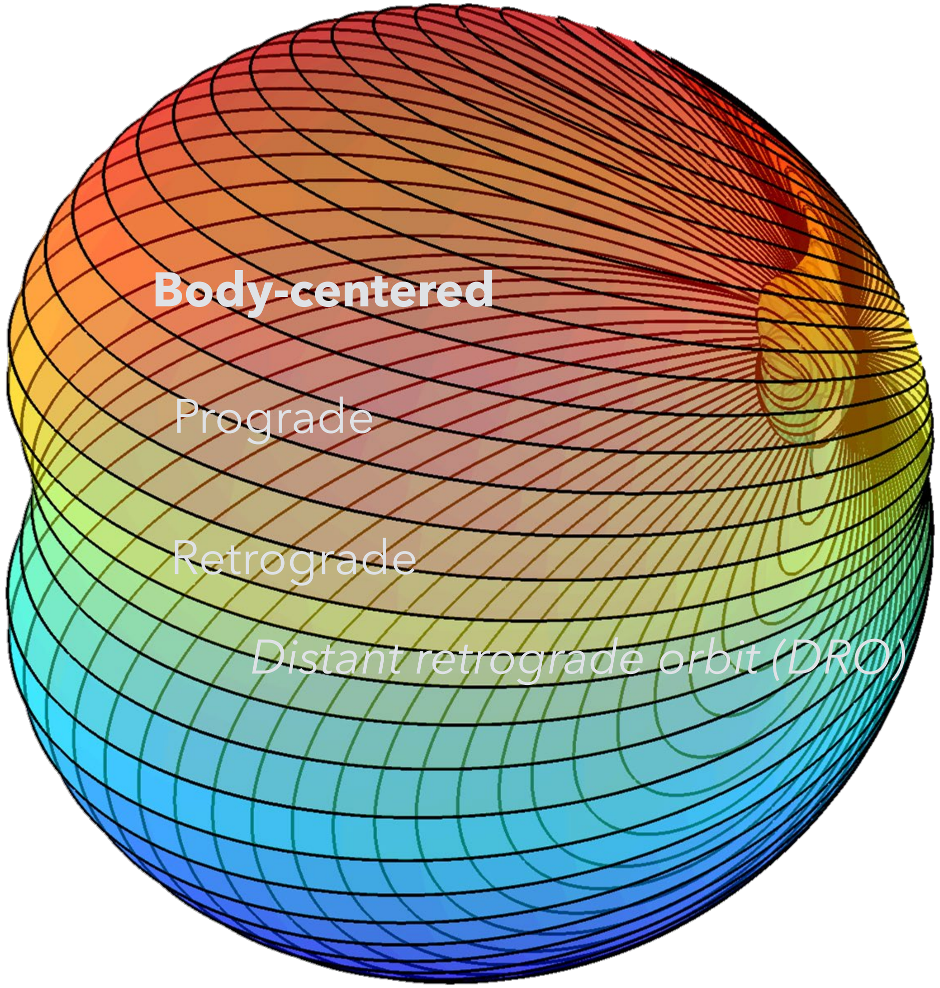
Short period
Long period
Vertical
…

L_5

Resonance

Interior

Exterior



Body-centered

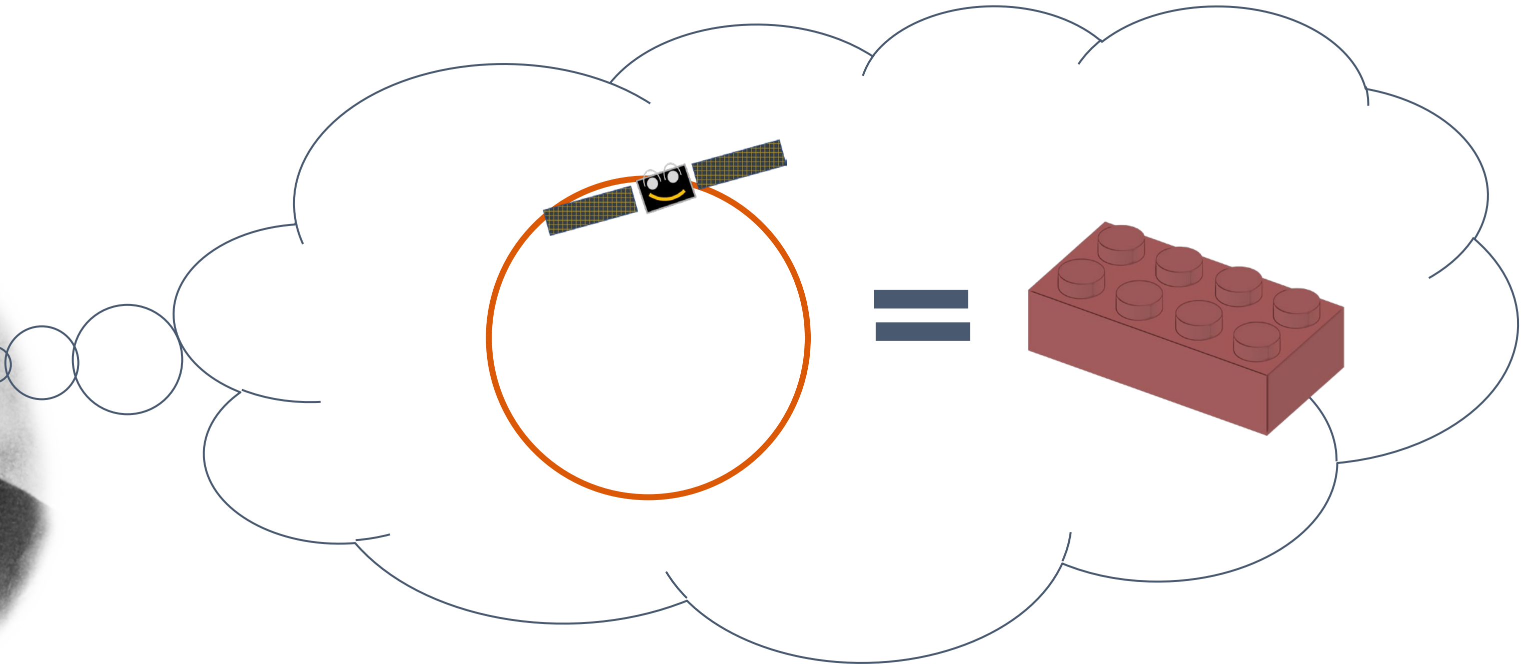
Prograde

Retrograde

Distant retrograde orbit (DRO)

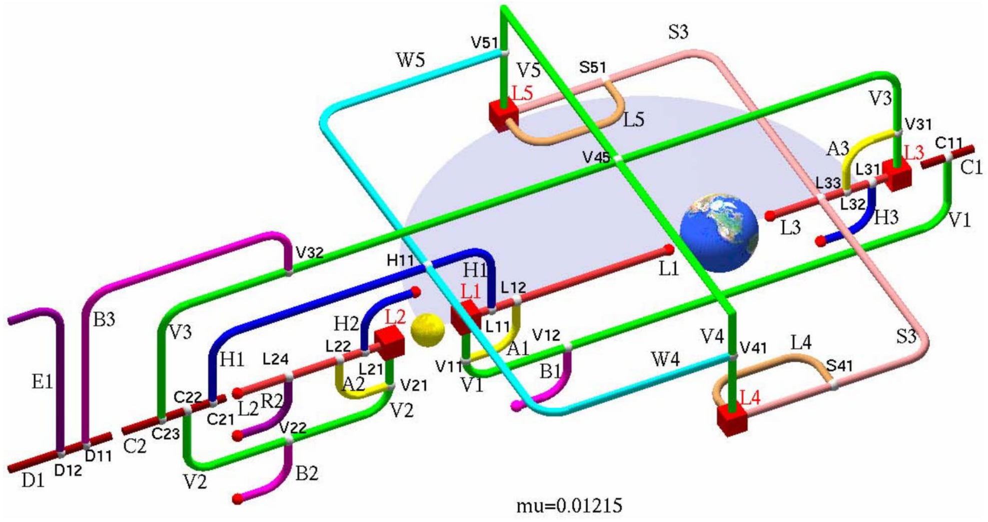
Guzzetti, D., Bosanac, N., Haapala, A., Howell, K., and Folta, D., "Rapid Trajectory Design in the Earth-Moon Ephemeris System via an Interactive Catalog of Periodic and Quasi-Periodic Orbits," Acta Astronautica, 126, Sep-Oct 2016, https://doi.org/10.1016/j.actaastro.2016.06.029

IMPORTANCE



Credit: Wikipedia.org

- Unit blocks of the underlying dynamical structure → for a nonlinear system, a general solution may be hard to describe globally
- Known to be infinitely bounded (assuming no perturbations)
- Aid mission design → organize nearby trajectories, generate invariant manifolds, define repeatable transfer opportunities, reveal local phase-space geometry, and anchor trajectory design
- Guide trajectory design through dynamical models of increasing fidelity



Doedel, Eusebius J., et al. "Elemental periodic orbits associated with the libration points in the circular restricted 3-body problem." International Journal of Bifurcation and Chaos 17.08 (2007): 2625-2677.

LOCAL LINEAR DYNAMICS

Suppose system has an eq. pt.

$$\mathbf{f}(\mathbf{x}_e) = \mathbf{0}$$

Define a perturbation away from eq. pt

$$\delta\mathbf{x} = \mathbf{x} - \mathbf{x}_e$$

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}_e + \delta\mathbf{x})$$

Expand

$$\mathbf{f}(\mathbf{x}_e + \delta\mathbf{x}) = \mathbf{f}(\mathbf{x}_e) + D\mathbf{f}(\mathbf{x}_e)\delta\mathbf{x} + \mathcal{O}(\|\delta\mathbf{x}\|^2)$$

The leading-order motion is governed by the Jacobian

$$\delta\dot{\mathbf{x}} = A\delta\mathbf{x} + \mathcal{O}(\|\delta\mathbf{x}\|^2), \qquad A = D\mathbf{f}(\mathbf{x}_e)$$

$$\delta\dot{\mathbf{x}} = A\delta\mathbf{x}$$

GENERAL LINEAR SOLUTION

The solution of the linearized system is generated by the matrix exponential

$$\delta\dot{\mathbf{x}} = A\delta\mathbf{x} \longrightarrow \delta\mathbf{x}(t) = e^{At}\delta\mathbf{x}_0$$

When A is diagonalizable, the solution decomposes into modal components

$$A\mathbf{v}_i = \lambda_i \mathbf{v}_i \longrightarrow \delta\mathbf{x}(t) = \sum_i c_i e^{\lambda_i t}\mathbf{v}_i$$

Pure imaginary eigenvalues produce oscillatory, periodic modes

$$\lambda_{1,2} = \pm i\omega \longrightarrow e^{\pm i\omega t} = \cos(\omega t) \pm i\sin(\omega t)$$

$$T_{\text{lin}} = \frac{2\pi}{\omega}$$

Real-valued motion produced by a complex conjugate eigenpair (two-dimensional oscillatory subspace)

$$\delta\mathbf{x}(t) = a\,\mathrm{Re}(\mathbf{v}e^{i\omega t}) + b\,\mathrm{Im}(\mathbf{v}e^{i\omega t})$$

Choose initial conditions along a pure-imaginary mode to obtain periodic linear motion

LINEAR CENTER MOTION

Near a collinear libration point in the CR3BP, the linearized dynamics contain one real hyperbolic pair, an imaginary in-pane center pair and, and an imaginary out-of-plane center pair

$$\lambda, \quad -\lambda, \quad +i\omega_p, \quad -i\omega_p, \quad +i\omega_v, \quad -i\omega_v$$

To generate bounded periodic motion near the libration point, suppress the hyperbolic modes and retain center motion

$$\delta\mathbf{x}(t) = c_u e^{\lambda t}\mathbf{v}_u + c_s e^{-\lambda t}\mathbf{v}_s + \delta\mathbf{x}_p(t) + \delta\mathbf{x}_v(t)$$

$$c_u = c_s = 0$$

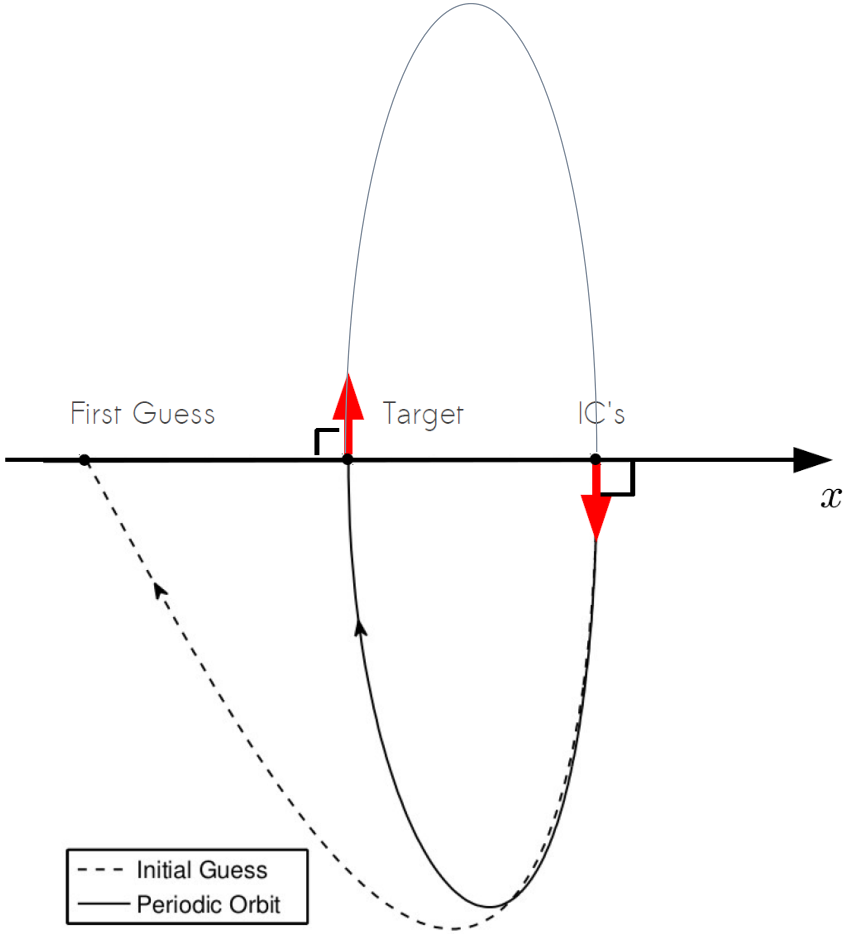
The real-valued oscillation is built from the real and imaginary parts of the complex eigenvector

$$\delta\mathbf{x}_p(t) = A_p\left[\mathrm{Re}(\mathbf{v}_p)\cos(\omega_p t) - \mathrm{Im}(\mathbf{v}_p)\sin(\omega_p t)\right]$$

At t=0 (*imposing a phase convention*)

$$\boxed{\mathbf{x}_0 = \mathbf{x}_e + A_p\,\mathrm{Re}(\mathbf{v}_p)}$$

LINEAR GUESS FAILS NONLINEARLY



$$\delta \mathbf{x}_{\text{lin}}(T_0) = \delta \mathbf{x}_{\text{lin}}(0)$$

linear

$$\mathbf{x}_0 = \mathbf{x}_e + A_p \,\text{Re}(\mathbf{v}_p)$$

$$\varphi_{T_0}(\mathbf{x}_0) \neq \mathbf{x}_0$$

nonlinear

PERIODICITY CONSTRAINT EQUATION

Define the periodicity constraint using the flow map

$$\mathbf{F}(\mathbf{x}_0, T) = \varphi_T(\mathbf{x}_0) - \mathbf{x}_0$$

A periodic orbit is a **root** of this constraint

$$\mathbf{x}_0 \text{ such that } \mathbf{F}(\mathbf{x}_0, T) = \mathbf{0}$$

The unknowns are the initial condition and the period

$$\mathbf{z} = \begin{bmatrix} \mathbf{x}_0 \\ T \end{bmatrix}$$

In an autonomous system, the problem is underdetermined because phase along the orbit is arbitrary, so we add a phase condition

$$g(\mathbf{x}_0) = 0$$

$$\mathbf{G}(\mathbf{x}_0, T) = \begin{bmatrix} \varphi_T(\mathbf{x}_0) - \mathbf{x}_0 \\ g(\mathbf{x}_0) \end{bmatrix} = \mathbf{0}$$

LINEARIZATION OF PERIODICITY CONSTRAINT

Correct the initial condition and period with a first-order update

$$\mathbf{x}_0 \to \mathbf{x}_0 + \delta\mathbf{x}_0, \qquad T \to T + \delta T$$

Expand about the current solution

$$\mathbf{F}(\mathbf{x}_0, T) = \boldsymbol{\varphi}_T(\mathbf{x}_0) - \mathbf{x}_0$$

$$\mathbf{F}(\mathbf{x}_0 + \delta\mathbf{x}_0, T + \delta T) \approx \mathbf{F}(\mathbf{x}_0, T) + \boxed{\frac{\partial \mathbf{F}}{\partial \mathbf{x}_0}}\delta\mathbf{x}_0 + \boxed{\frac{\partial \mathbf{F}}{\partial T}}\delta T$$

The derivative of the flow with respect to the initial state is the STM

$$\Phi(T, 0) = \frac{\partial \boldsymbol{\varphi}_T}{\partial \mathbf{x}_0} \longrightarrow \boxed{\frac{\partial \mathbf{F}}{\partial \mathbf{x}_0} = \Phi(T, 0) - I}$$

The derivative of the flow with respect to final time is the vector field evaluated at the final state

$$\boxed{\frac{\partial \mathbf{F}}{\partial T} = \frac{\partial}{\partial T}\boldsymbol{\varphi}_T(\mathbf{x}_0) = \mathbf{f}(\mathbf{x}(T))}$$

$$\longrightarrow \quad \delta\mathbf{F} = \left[\Phi(T, 0) - I\right]\delta\mathbf{x}_0 + \mathbf{f}(\mathbf{x}(T))\delta T$$

NEWTON CORRECTION FOR PERIODIC ORBITS

The Newton step drives the periodicity defect to zero

$$\mathbf{F} + [\Phi(T,0) - I]\,\delta\mathbf{x}_0 + \mathbf{f}(\mathbf{x}(T))\delta T = \mathbf{0}$$

In matrix form

$$\begin{bmatrix}\Phi(T,0) - I & \mathbf{f}(\mathbf{x}(T))\end{bmatrix}\begin{bmatrix}\delta\mathbf{x}_0 \\ \delta T\end{bmatrix} = -\mathbf{F}$$

Augmented with phase constraints

$$\begin{bmatrix}\Phi(T,0) - I & \mathbf{f}(\mathbf{x}(T)) \\ \nabla g(\mathbf{x}_0)^T & 0\end{bmatrix}\begin{bmatrix}\delta\mathbf{x}_0 \\ \delta T\end{bmatrix} = -\begin{bmatrix}\boldsymbol{\varphi}_T(\mathbf{x}_0) - \mathbf{x}_0 \\ g(\mathbf{x}_0)\end{bmatrix}$$

The corrected initial state and period are updated iteratively

$$\mathbf{x}_0^{(k+1)} = \mathbf{x}_0^{(k)} + \epsilon\,\delta\mathbf{x}_0$$

$$T^{(k+1)} = T^{(k)} + \epsilon\,\delta T$$

SYMMETRY-BASED HALF-PERIOD TARGETING

For planar orbits, symmetric about the x-axis, select an initial condition that lies on the symmetry plane

$$\mathbf{x}_0(\dot{y}_0) = \begin{bmatrix} x_0^{\text{fix}} & 0 & 0 & 0 & \dot{y}_0 & 0 \end{bmatrix}^T \qquad \mathbf{u} = \begin{bmatrix} \dot{y}_0 \\ t_f \end{bmatrix}$$

At half period, the trajectory crosses the symmetry plane again with the appropriate velocity symmetry

$$y(t_f) = 0, \qquad \dot{x}(t_f) = 0 \qquad t_f = \frac{T}{2}$$

Turn periodic-orbit correction into a smaller targeting problem

$$\mathbf{C}(\mathbf{u}) = \begin{bmatrix} y(t_f) \\ \dot{x}(t_f) \end{bmatrix} = \mathbf{0}$$

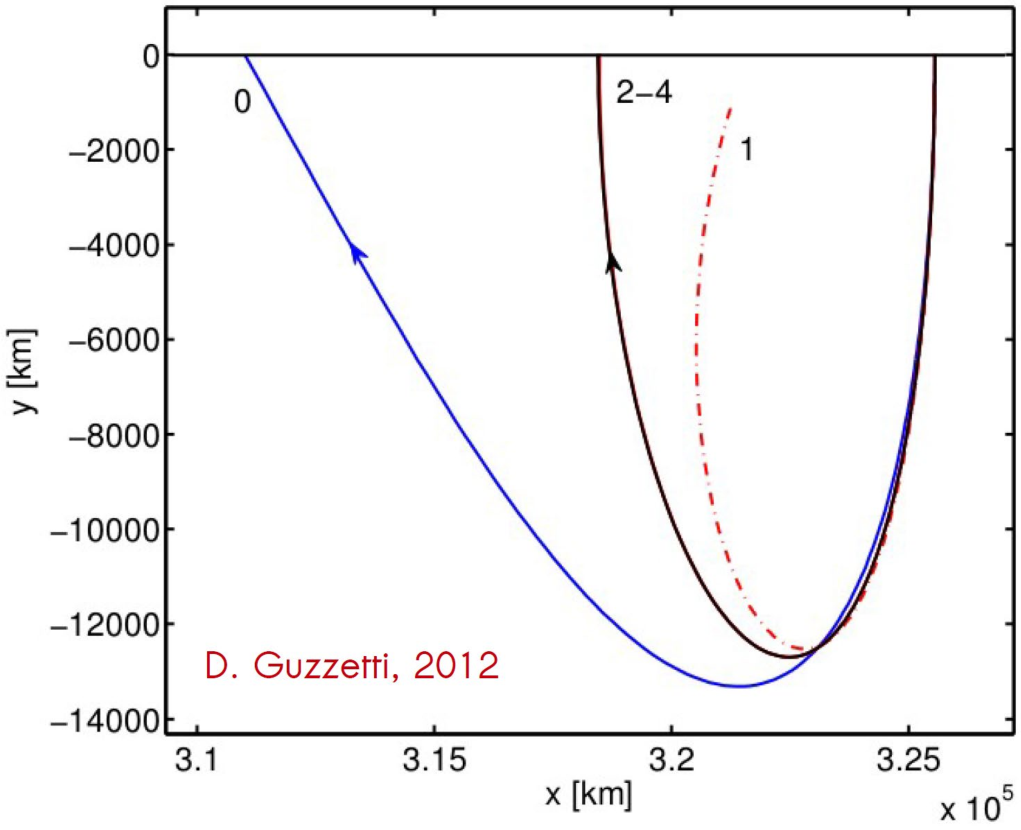
$$D\mathbf{C}\,\delta\mathbf{u} = -\mathbf{C}$$

$$\begin{bmatrix} \Phi_{2,5} & \dot{y}_f \\ \Phi_{4,5} & \ddot{x}_f \end{bmatrix} \begin{bmatrix} \delta\dot{y}_0 \\ \delta t_f \end{bmatrix} = -\begin{bmatrix} y_f \\ \dot{x}_f \end{bmatrix}$$

The corrected initial state and period are updated iteratively

$$\begin{bmatrix} \dot{y}_0^{(k+1)} \\ t_f^{(k+1)} \end{bmatrix} = \begin{bmatrix} \dot{y}_0^{(k)} \\ t_f^{(k)} \end{bmatrix} + \epsilon \begin{bmatrix} \delta\dot{y}_0 \\ \delta t_f \end{bmatrix}$$

SYMMETRY-BASED HALF-PERIOD TARGETING



RESOURCES

https://www.nasa.gov/stem-content/designing-the-moonshot-introduction-to-multi-body-dynamics/

Designing the Moonshot: Introduction to Multi-Body Dynamic

‹ Back to search page

AUDIENCE
Students

GRADE LEVELS
Higher Education

SUBJECT
Mathematics, Physical Science, Space Science, Forces and Motion, Physics, Missions to Planets and Moons

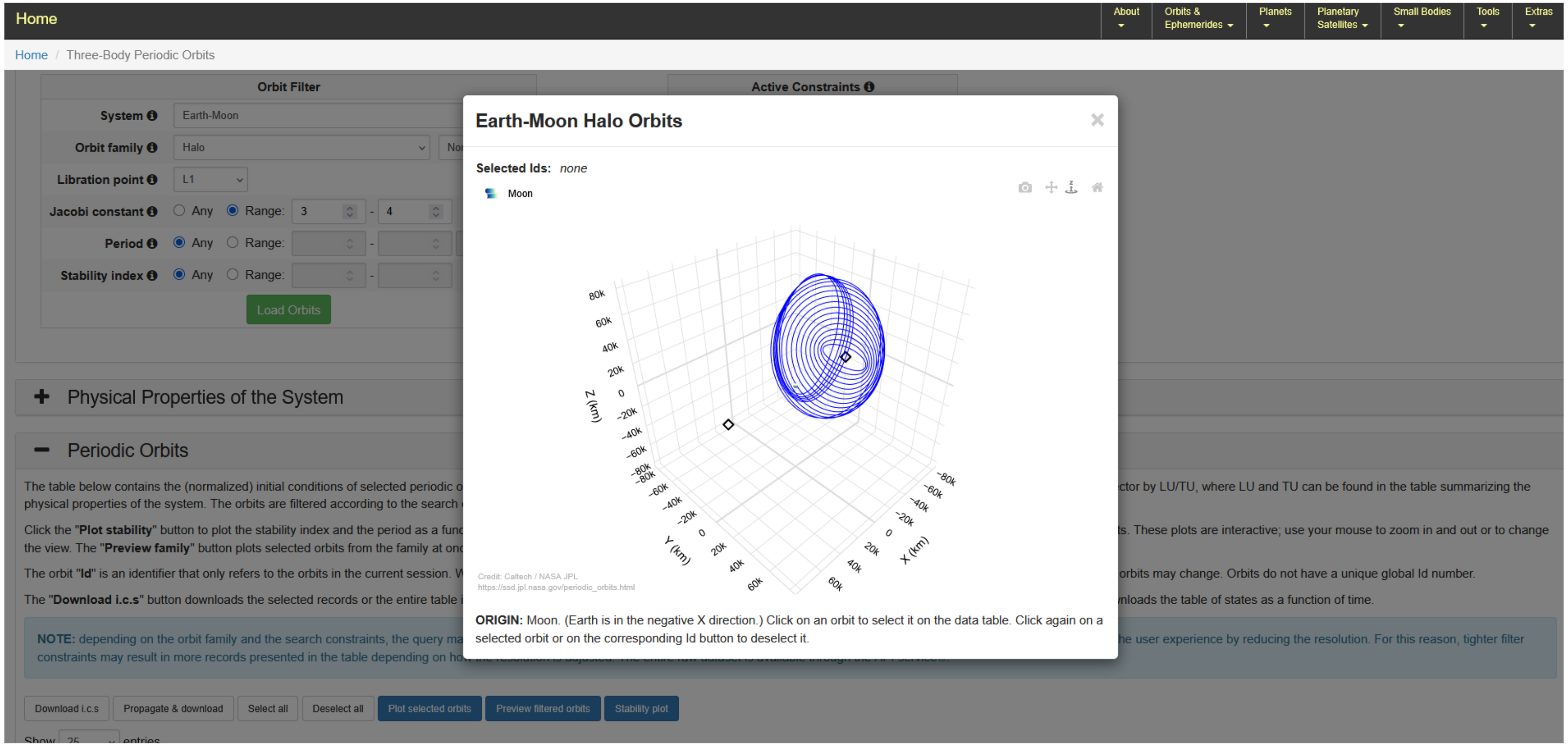
TYPE
Interactive Multimedia, Websites

Are you interested in trajectory design in cislunar space? Learn how trajectory design works in the Earth-Moon system and multi-body gravity fields in this series of free, online modules. This self-paced course is designed for participants who have taken an undergraduate orbital mechanics class but have little or no experience with gravitational multi-body dynamics.

Click here for more information and to register.

RESOURCES

https://ssd.jpl.nasa.gov/tools/periodic_orbits.html#/periodic



RESOURCES

https://github.com/the-aerospace-corporation/CislunarSystemVRSimulator