GRAVITATIONAL MULTIBODY DYNAMICS: A QUICK-START GUIDE

Course plan:

1) **Choosing the right model**
2) **Going with the flow**
3) **Periodic motion**
4) **Stability and orbit manifolds**
5) **From CR3BP to ephemeris**

Davide Guzzetti

May-June 2026, NESC Academy Series

AUBURN

Lecture 4:

STABILITY AND ORBIT MANIFOLDS

Gravitational Multibody Dynamics: a Quick-start Guide

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A PERIODIC ORBIT AS A FIXED POINT OF THE FLOW MAP

Draw schematics

A PERIODIC ORBIT AS A FIXED POINT OF THE FLOW MAP

The flow map advances an initial state

$$\boldsymbol{\varphi}_t : \mathbf{x}_0 \mapsto \mathbf{x}(t; \mathbf{x}_0)$$

A periodic orbit is a fixed point of the time-T map

$$\boldsymbol{\varphi}_T(\mathbf{x}_0) = \mathbf{x}_0$$

Let Sigma be a codimension-one surface transverse to the flow

$$\Sigma = \{\mathbf{x} : h(\mathbf{x}) = 0\} \quad \text{and} \quad \nabla h(\mathbf{x})^T \mathbf{f}(\mathbf{x}) \neq 0$$

The Poincare' map advances a point on the section to its next return to the section

$$P : \Sigma \to \Sigma \qquad P(\mathbf{x}_k) = \boldsymbol{\varphi}_{\tau(\mathbf{x}_k)}(\mathbf{x}_k)$$

A periodic orbit that intersects the section once per period becomes a fixed point of the return map. If it intersects m times before repeating, it is a fixed point of the m-th iterate

$$P(\mathbf{x}^*) = \mathbf{x}^* \quad \text{or} \quad P^m(\mathbf{x}^*) = \mathbf{x}^*$$

VARIATIONAL DYNAMICS ALONG A PERIODIC ORBIT

Let the reference solution be periodic with period T, add a small perturbation

$$\mathbf{x}(t) = \mathbf{x}^*(t) + \delta\mathbf{x}(t) \qquad \mathbf{x}^*(t+T) = \mathbf{x}^*(t)$$

Linearize

$$\delta\dot{\mathbf{x}} = A(t)\delta\mathbf{x} \qquad A(t) = D\mathbf{f}(\mathbf{x}^*(t))$$

Because the reference orbit is periodic, the Jacobian matrix along the orbit is also periodic

$$A(t+T) = A(t)$$

The STM over one period is the derivative of the time-T flow map at the orbit point

$$D\boldsymbol{\varphi}_T(\mathbf{x}_0^*) = \Phi(T,0)$$

$$\dot{\Phi}(t,0) = A(t)\Phi(t,0), \qquad \Phi(0,0) = I$$

With the return map written in the same coordinates, this is the derivative DP at the fixed point

$$DP(\mathbf{x}_0^*) = D\boldsymbol{\varphi}_T(\mathbf{x}_0^*) = \Phi(T,0)$$

THE MONODROMY MATRIX

The monodromy matrix is the STM evaluated over one full period

$$M = \Phi(T, 0)$$

The monodromy matrix is the derivative of the period map at a periodic-orbit fixed point

$$DP(\mathbf{x}_0^*) = M = D\boldsymbol{\varphi}_T(\mathbf{x}_0^*) = \Phi(T, 0)$$

It maps perturbations from one revolution to the next

$$\delta\mathbf{x}_{k+1} = M\delta\mathbf{x}_k$$

Repeated application describes the perturbation evolution over multiple periods

$$\delta\mathbf{x}_k = M^k\delta\mathbf{x}_0$$

Its eigenvalues are characteristic multipliers

$$M\mathbf{w}_i = \rho_i\mathbf{w}_i$$

Their magnitudes determine linear growth, decay, or neutral behavior over one period

$$|\rho_i| < 1 \quad |\rho_i| > 1 \quad |\rho_i| = 1$$

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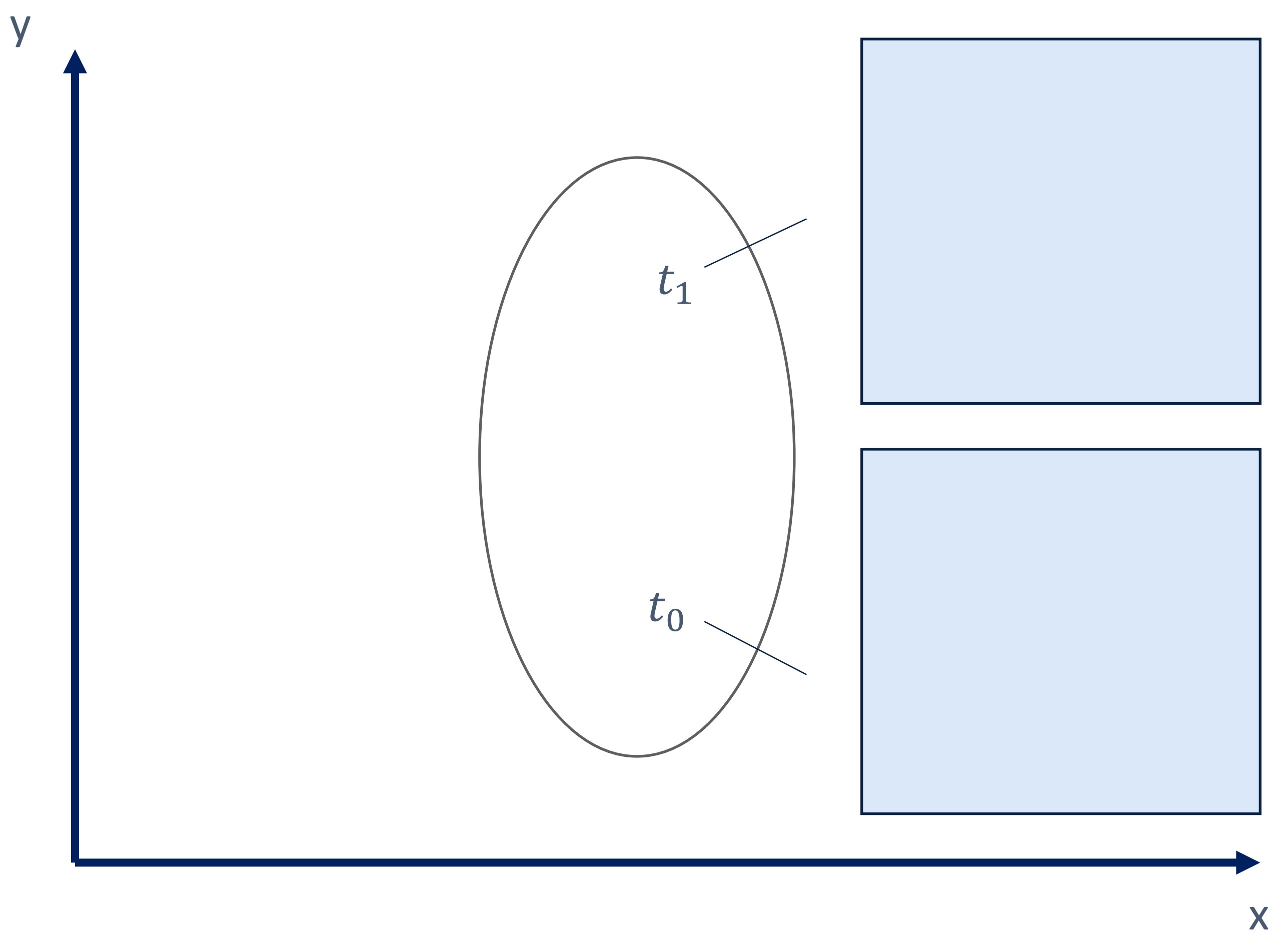
$|\rho_i| < 1$

$|\rho_i| = 1$

$|\rho_i| > 1$

EIGENSTRUCTURE ALONG AN ORBIT



$$M_1 \boldsymbol{w_1} = \rho_1 \boldsymbol{w_1}$$

$$M_0 \boldsymbol{w_0} = \rho_0 \boldsymbol{w_0}$$

STABILITY CLASSIFICATION

Blank space to sample Agar diagrams

1) $|\rho_i| < 1 \;\; \forall i$ → linearly asymptotically stable
2) $|\rho_i| \leq 1 \;\; \forall i, \exists j: |\rho_j| = 1$, ρ_j *is semisimple* → marginally stable
3) $\exists i: \; |\rho_i| > 1$ → linearly unstable

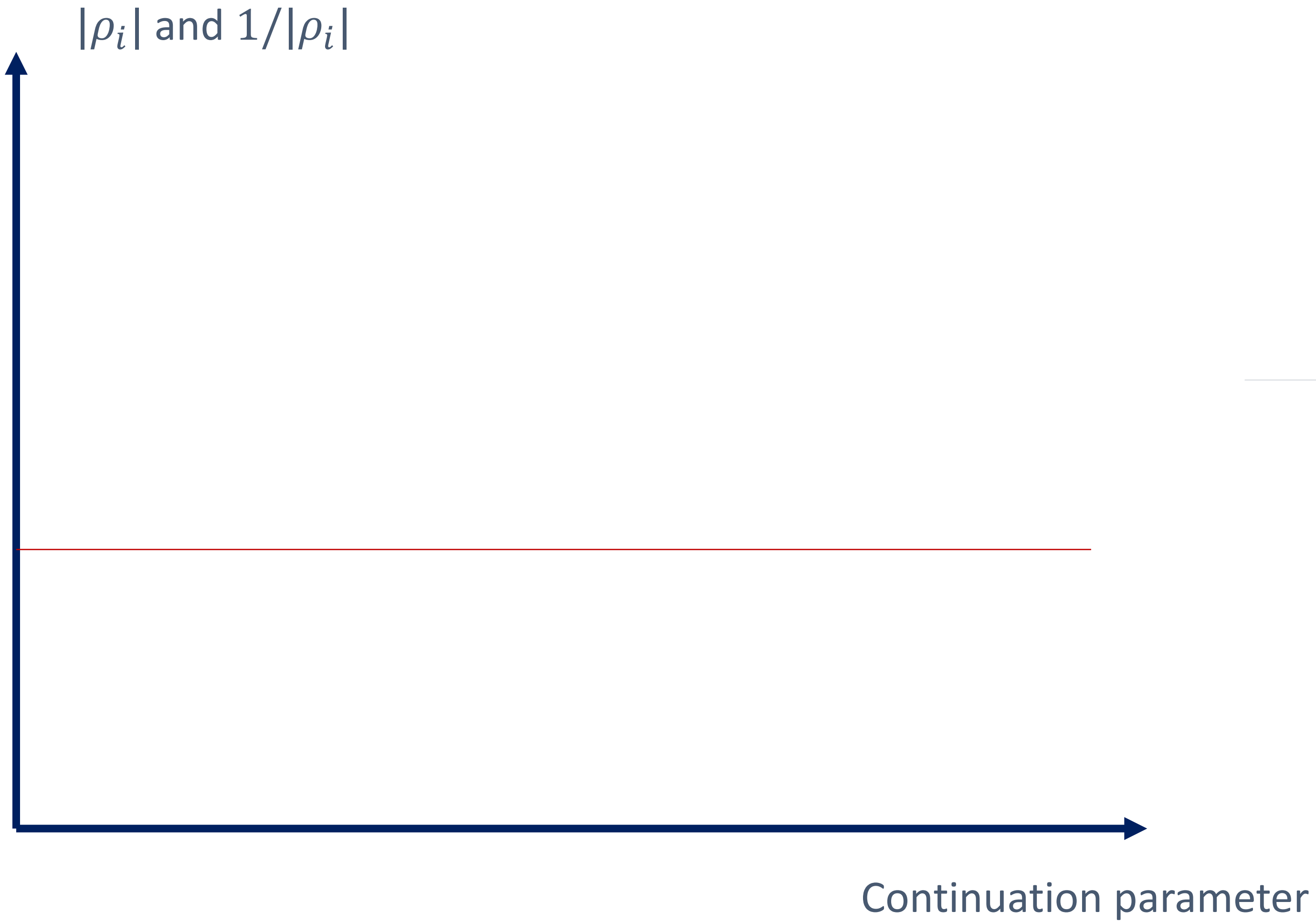
Notable fixed points

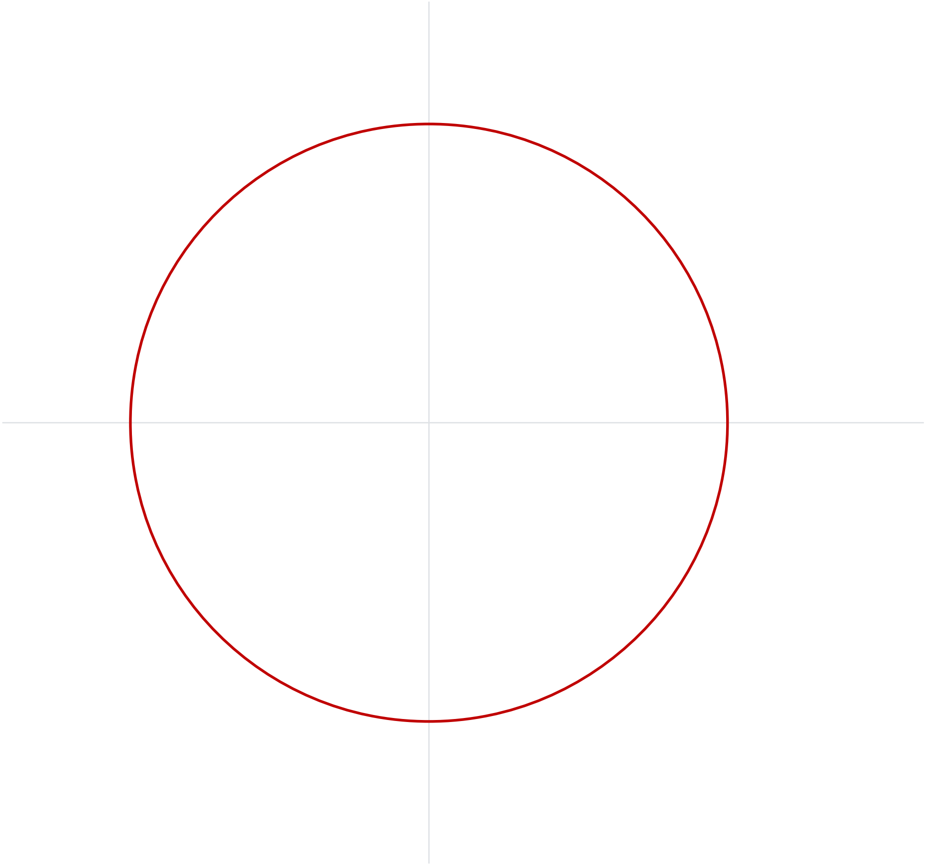
1) $\exists i, j: |\rho_i| > 1$ and $|\rho_j| < 1$ → saddle point
2) $\forall i \; |\rho_i| \; \neq 1$ → hyperbolic point (no neutral directions)

NOTABLE PROPERTY IN CR3BP

- Always a pair of unitary eigenvalues
- Always in reciprocal pairs $\left(\rho_i, \frac{1}{\rho_i}\right)$
- Complex eigenvalues always in complex conjugate pairs $(a + ib, a - ib)$

EIGENSTRUCTURE AND ORBIT BIFURCATIONS





LOCAL MANIFOLDS

Its eigenvalues are characteristic multipliers

$$M\mathbf{w}_i = \rho_i \mathbf{w}_i$$

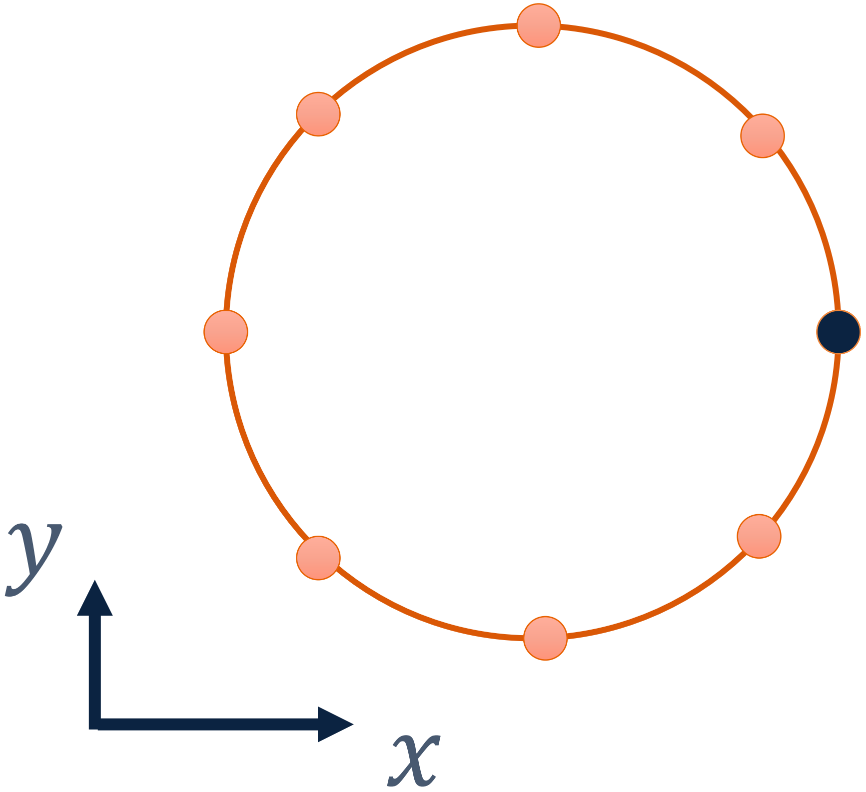
Their magnitudes determine linear growth, decay, or neutral behavior over one period

$|\rho_i| < 1$

$|\rho_i| = 1$

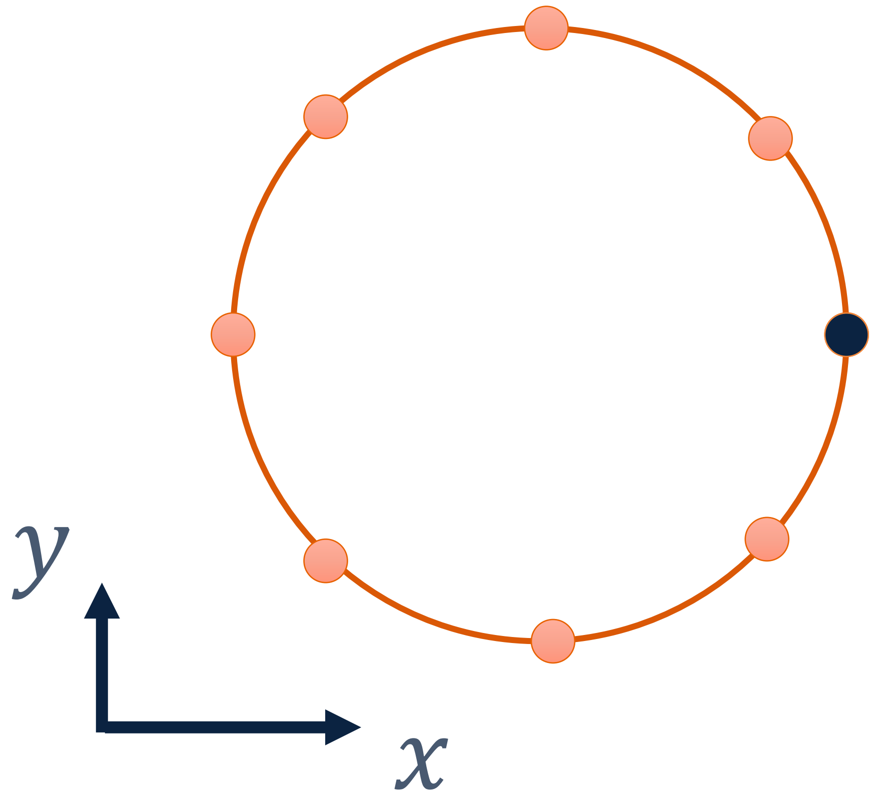
$|\rho_i| > 1$

COMPUTATION OF GLOBAL STABLE/UNSTABLE MANIFOLDS



$$\bar{x}^* = [x, y, z, \dot{x}, \dot{y}, \dot{z}]$$

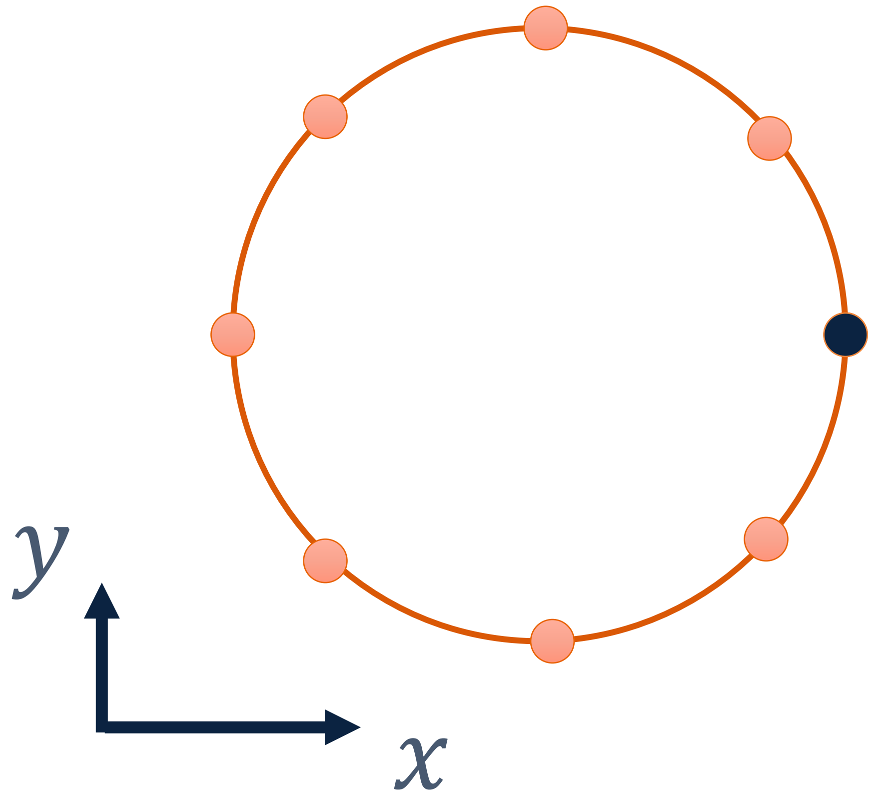
COMPUTATION OF STABLE/UNSTABLE MANIFOLDS



$\bar{x}^* = [x, y, z, \dot{x}, \dot{y}, \dot{z}]$

Compute $[DP]$

COMPUTATION OF STABLE/UNSTABLE MANIFOLDS



$$\bar{x}^* = [x, y, z, \dot{x}, \dot{y}, \dot{z}]$$

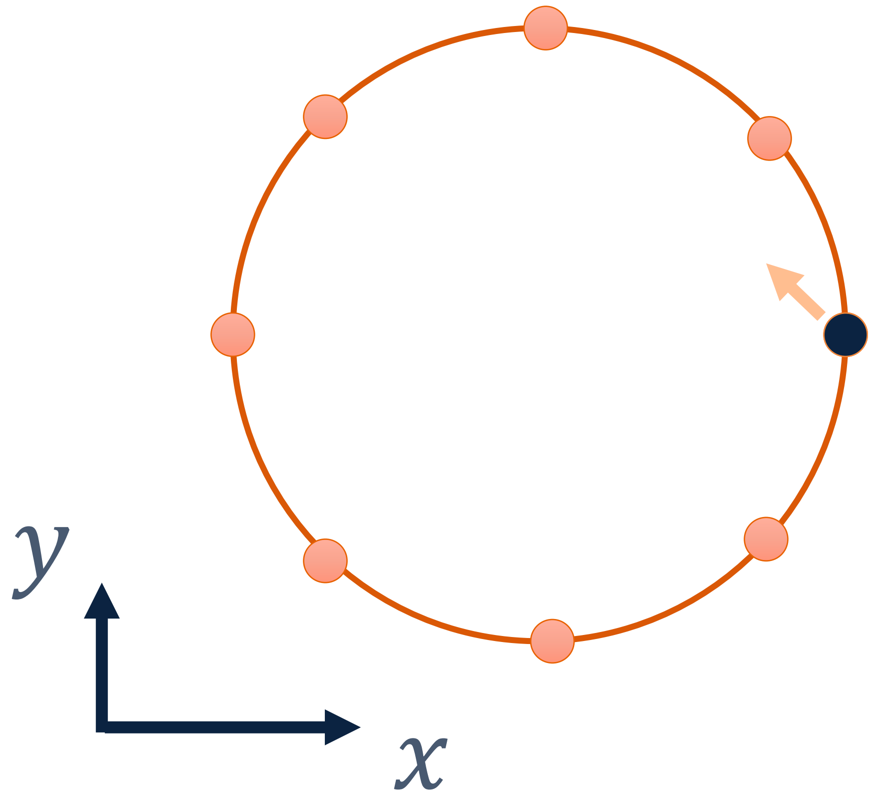
Compute $[DP]$

Compute eigenvalues and eigenvectors

$\boldsymbol{w}_s$: $|\rho_s| < 1$

$\boldsymbol{w}_u$: $|\rho_u| > 1$

COMPUTATION OF STABLE/UNSTABLE MANIFOLDS



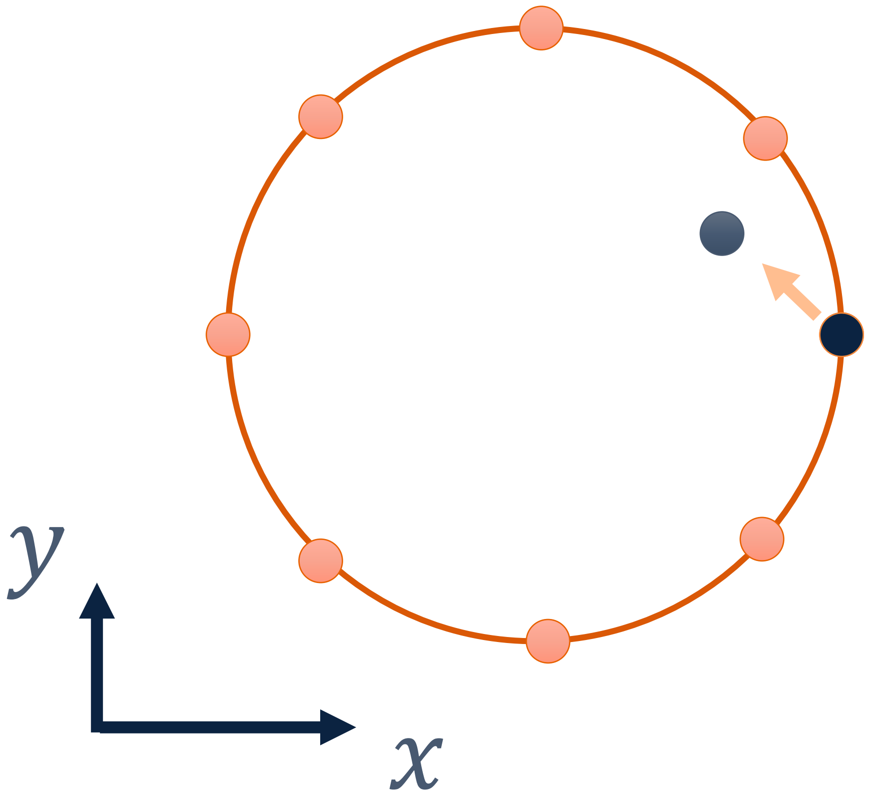
$$\bar{x}^* = [x, y, z, \dot{x}, \dot{y}, \dot{z}]$$

$$\boldsymbol{w}_u: \quad |\rho_u| > 1$$

$$\boldsymbol{w}_u = [x_u, y_u, z_u, \dot{x}_u, \dot{y}_u, \dot{z}_u]$$

$$\widehat{w}_u = \frac{\boldsymbol{w}_u}{\sqrt{x_u^2 + y_u^2 + z_u^2}}$$

COMPUTATION OF STABLE/UNSTABLE MANIFOLDS



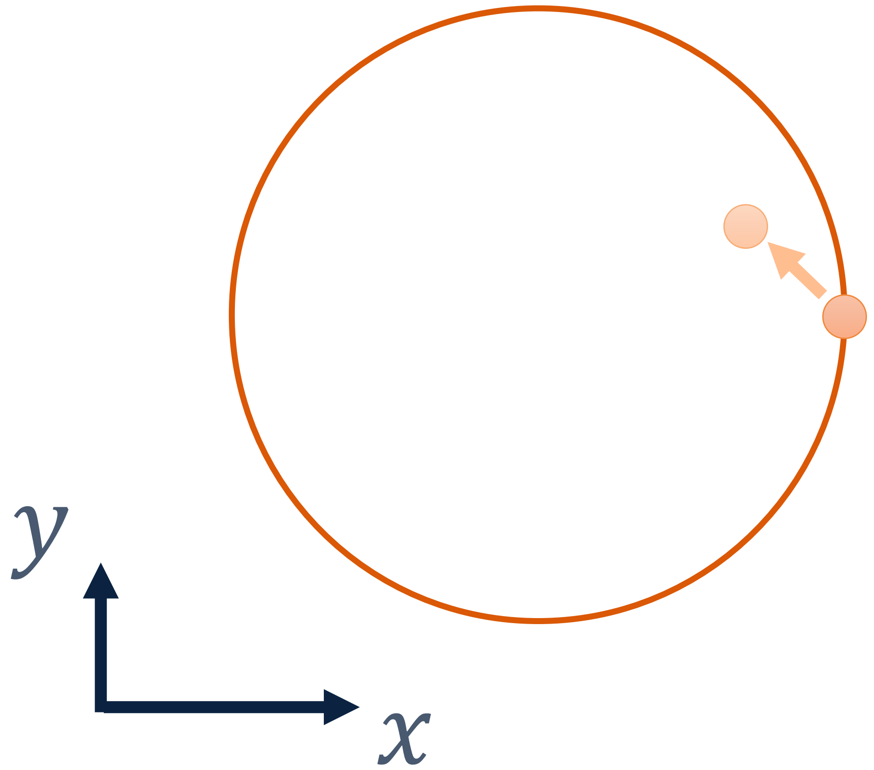
$$\bar{x}^* = [x, y, z, \dot{x}, \dot{y}, \dot{z}]$$

$$\boldsymbol{w}_u: \quad |\rho_u| > 1$$

$$\widehat{w}_u = \frac{\boldsymbol{w}_u}{\sqrt{x_u^2 + y_u^2 + z_u^2}}$$

$$\bar{x}_u = \bar{x}^* \pm d\widehat{w}_u$$

COMPUTATION OF STABLE/UNSTABLE MANIFOLDS



$$\bar{x}^* = [x, y, z, \dot{x}, \dot{y}, \dot{z}]$$

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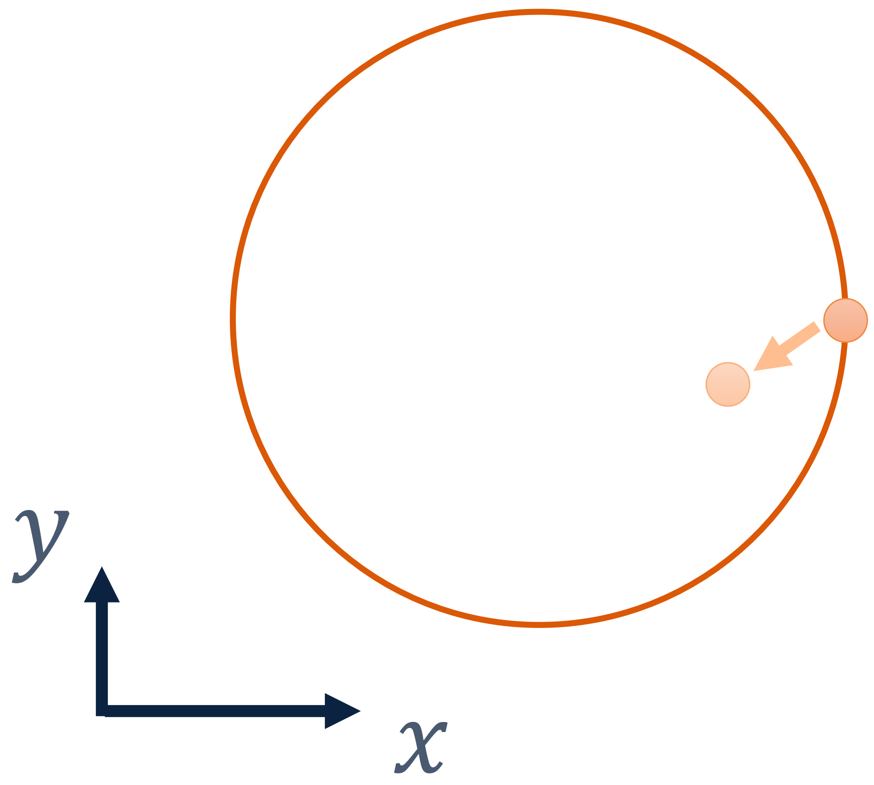
$$\bar{x}_u = \bar{x}^* \pm d\widehat{w}_u$$

Propagate forward [0 t]

COMPUTATION OF STABLE/UNSTABLE MANIFOLDS



$$\bar{x}^* = [x, y, z, \dot{x}, \dot{y}, \dot{z}]$$

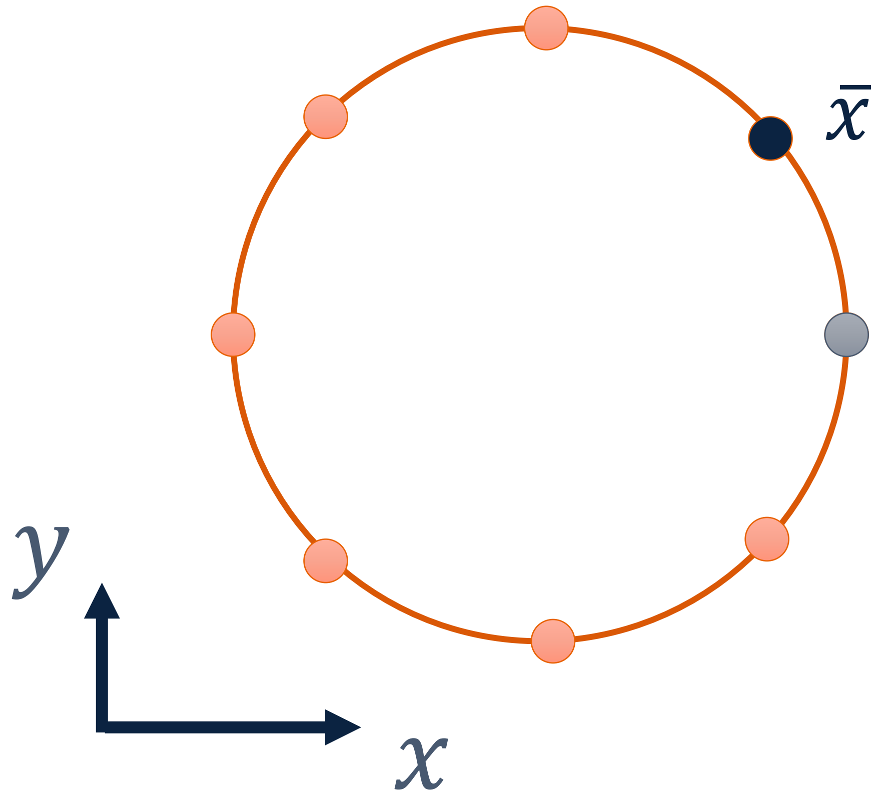
$$\boldsymbol{w}_s: \quad |\rho_s| < 1$$

$$\widehat{w}_s = \frac{\boldsymbol{w}_s}{\sqrt{x_s^2 + y_s^2 + z_s^2}}$$

$$\bar{x}_s = \bar{x}^* \pm d\widehat{w}_s$$

Propagate backward [0 -t]

COMPUTATION OF STABLE/UNSTABLE MANIFOLDS

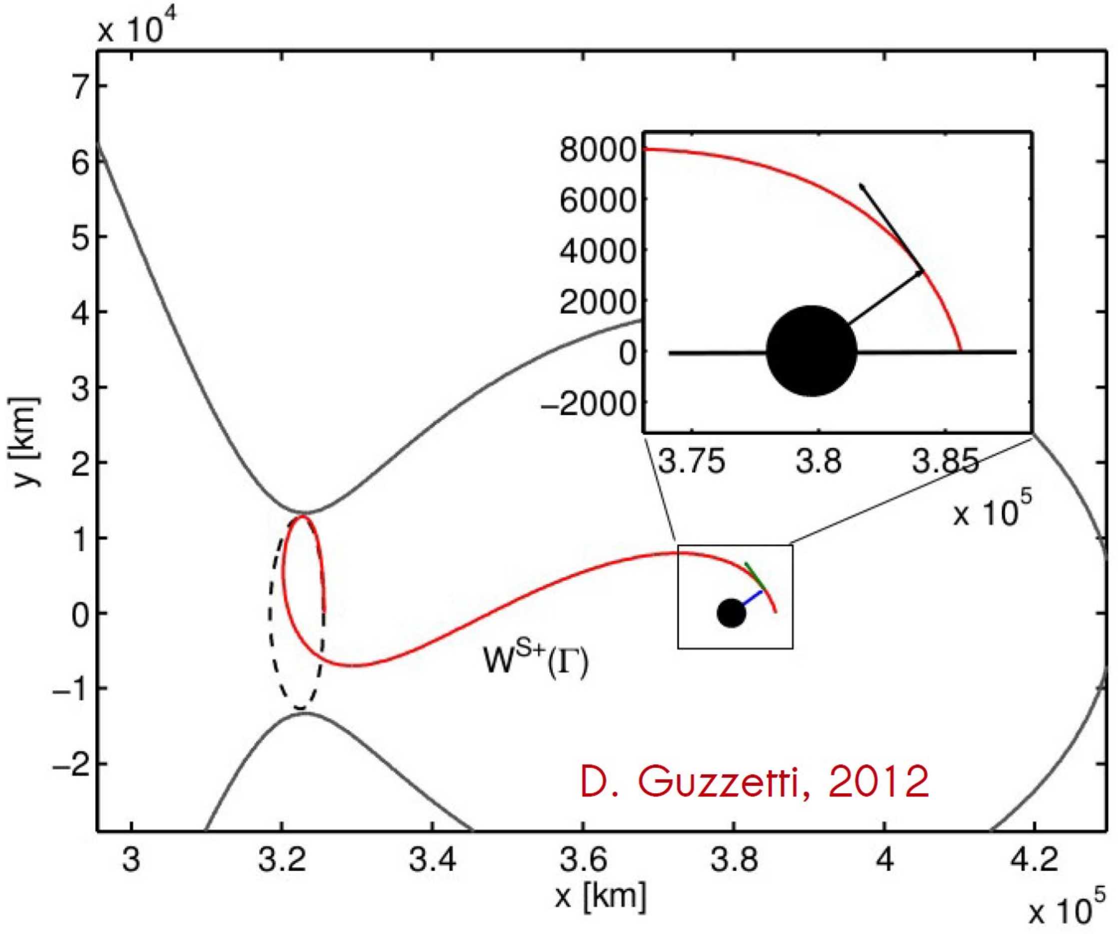


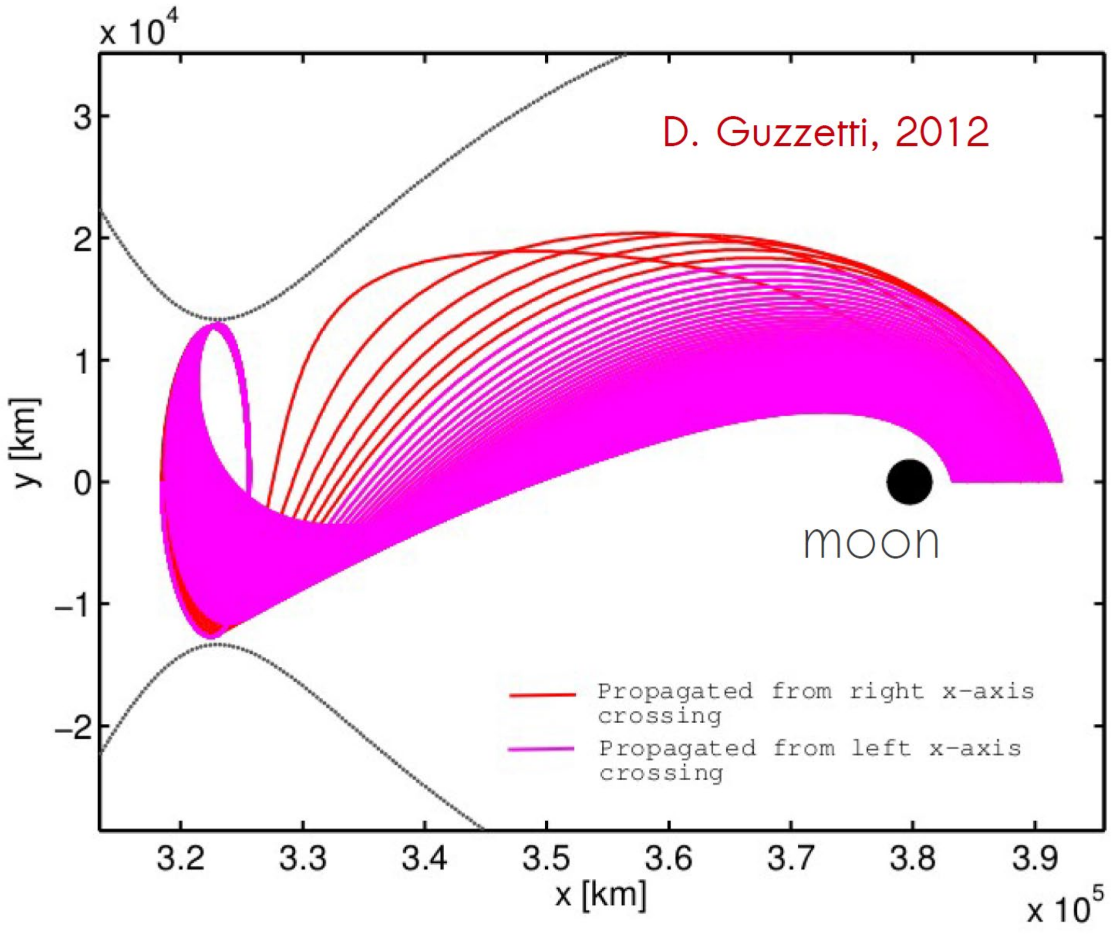
$$\bar{x}^* = [x, y, z, \dot{x}, \dot{y}, \dot{z}]$$

$$\bar{x}^* = [x, y, z, \dot{x}, \dot{y}, \dot{z}]$$

repeat

GLOBAL INVARIANT ORBIT MANIFOLDS





STABILITY INDEX

For each reciprocal multiplier pair, $\rho_i, \qquad \rho_i^{-1}$

define a stability index as the half-sum of the pair

$$a_i = \frac{1}{2}\left(\rho_i + \rho_i^{-1}\right)$$

For an elliptic unit-circle pair (exclude +1 and -1)

$$\rho_i = e^{i\theta} \qquad \rho_i^{-1} = e^{-i\theta}$$

$$a_i = \cos\theta \qquad |a_i| < 1$$

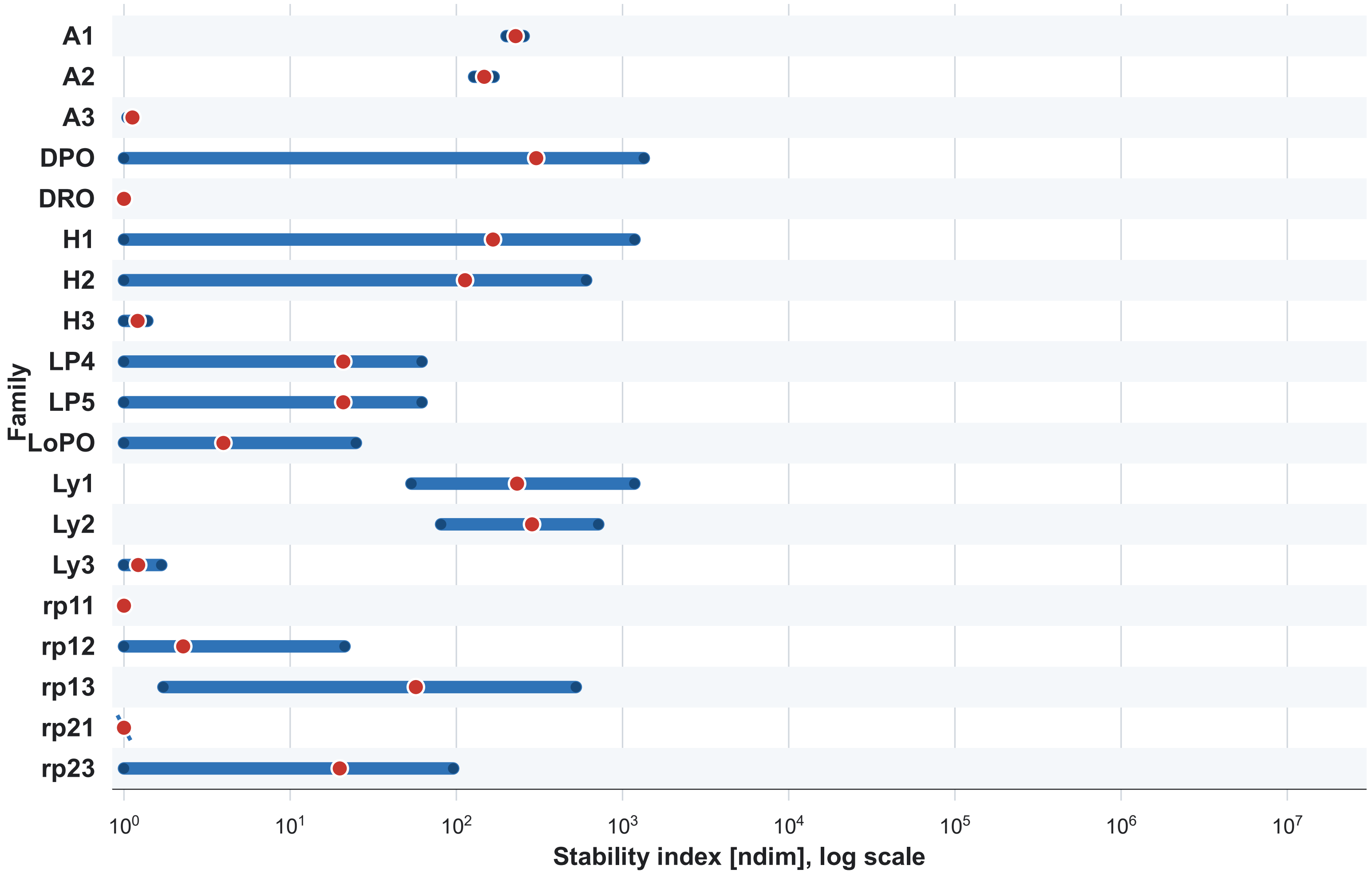
Sometime appears as computed from the eigenvalue of maximum modulus

$$s_{max} = \frac{1}{2}\left(|\rho_{max}| + |\rho_{max}^{-1}|\right)$$

This quantity is always at least one and equals one for all unit-circle pairs

$$s_{max} \geq 1, \quad s_{max} = 1 \; if \; |\rho_{max}| = 1$$

STABILITY ACROSS ORBIT FAMILIES



$$s_{\max} = \frac{1}{2}\left(|\lambda_{\max}| + |\lambda_{\max}^{-1}|\right)$$

Guzzetti, D., Bosanac, N., Folta, D. and Howell, K. "A Framework for Efficient Trajectory Comparisons in the Earth-Moon Design Space," AIAA/AAS Astrodynamics Specialist Conference, San Diego, CA, August 2014.

A READING

VOL. 7, NO. 6, JUNE 1969 AIAA JOURNAL

Stability of Periodic Orbits in the Elliptic, Restricted Three-Body Problem

R. Broucke*
Jet Propulsion Laboratory, Pasadena, Calif.

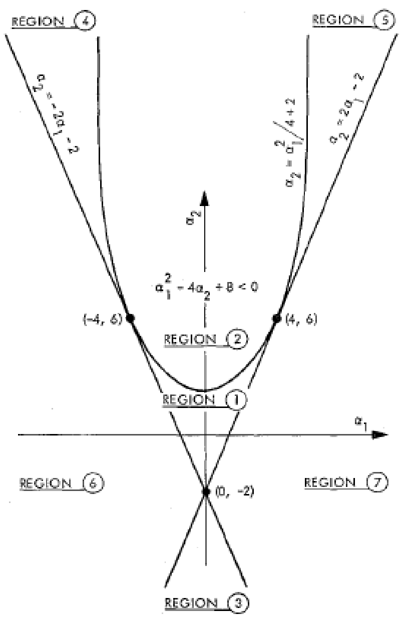


Fig. 1 Seven stability regions.

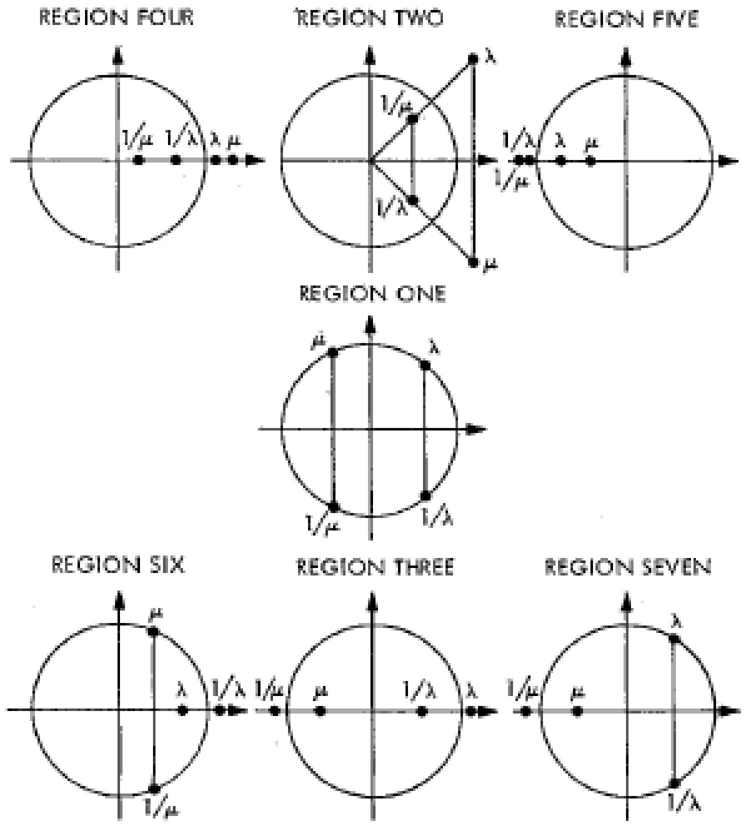


Fig. 2 Configuration of roots for each stability region.