GRAVITATIONAL MULTIBODY DYNAMICS: A QUICK-START GUIDE

Course plan:

1) Choosing the right model
2) Going with the flow
3) Periodic motion
4) Stability and orbit manifolds
4) **From CR3BP to ephemeris**

Davide Guzzetti

May-June 2026, NESC Academy Series

AUBURN

Lecture 5:

FROM CR3BP TO EPHEMERIS

Gravitational Multibody Dynamics: a Quick-start Guide

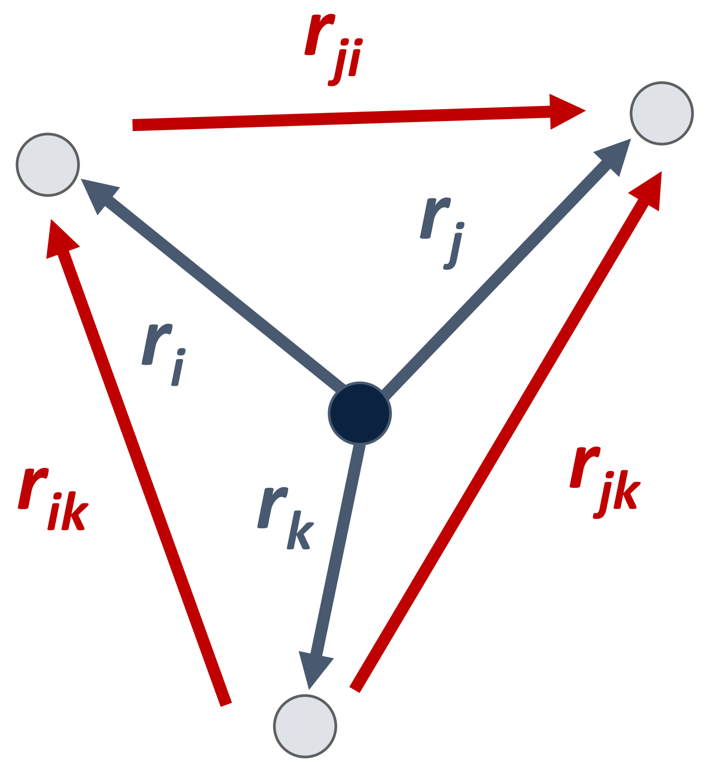
Davide Guzzetti

May-June 2026, NESC Academy Series

AUBURN

TRANSITION TO AN EPHEMERIS MODEL

$$\ddot{\mathbf{r}}_{ik} = -G(m_i + m_k)\frac{\mathbf{r}_{ik}}{\|\mathbf{r}_{ik}\|^3} + G\sum_{j\neq i,k} m_j \left(\frac{\mathbf{r}_j - \mathbf{r}_i}{\|\mathbf{r}_j - \mathbf{r}_i\|^3} - \frac{\mathbf{r}_j - \mathbf{r}_k}{\|\mathbf{r}_j - \mathbf{r}_k\|^3}\right)$$

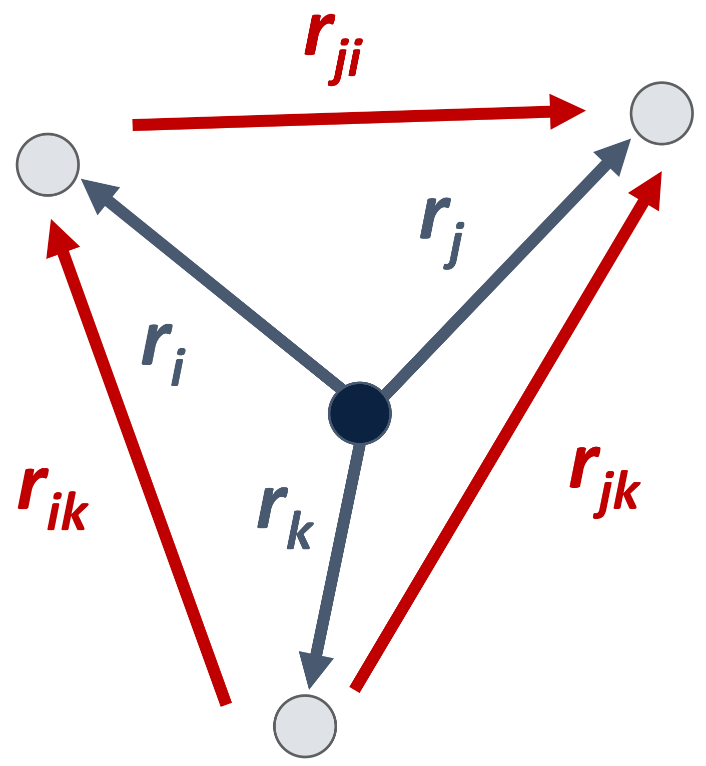


different frame

additional forces

TRANSITION TO AN EPHEMERIS MODEL

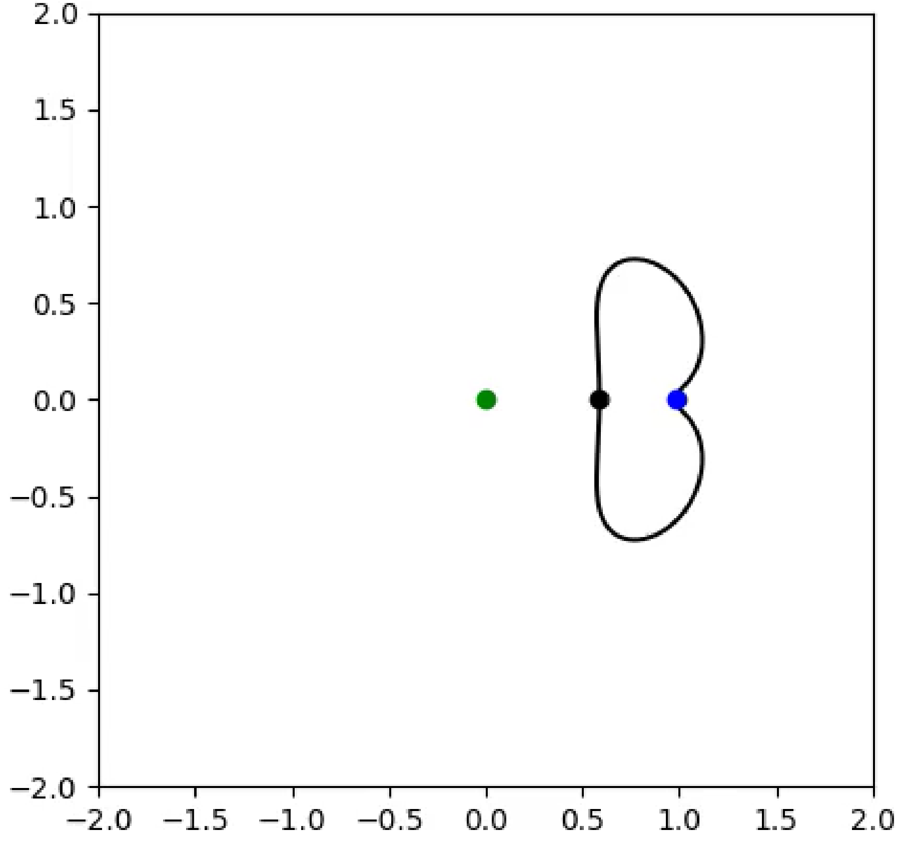
$$\ddot{\mathbf{r}}_{ik} = -G(m_i + m_k)\frac{\mathbf{r}_{ik}}{\|\mathbf{r}_{ik}\|^3} + G\sum_{j\neq i,k} m_j\left(\frac{\mathbf{r}_j - \mathbf{r}_i}{\|\mathbf{r}_j - \mathbf{r}_i\|^3} - \frac{\mathbf{r}_j - \mathbf{r}_k}{\|\mathbf{r}_j - \mathbf{r}_k\|^3}\right)$$

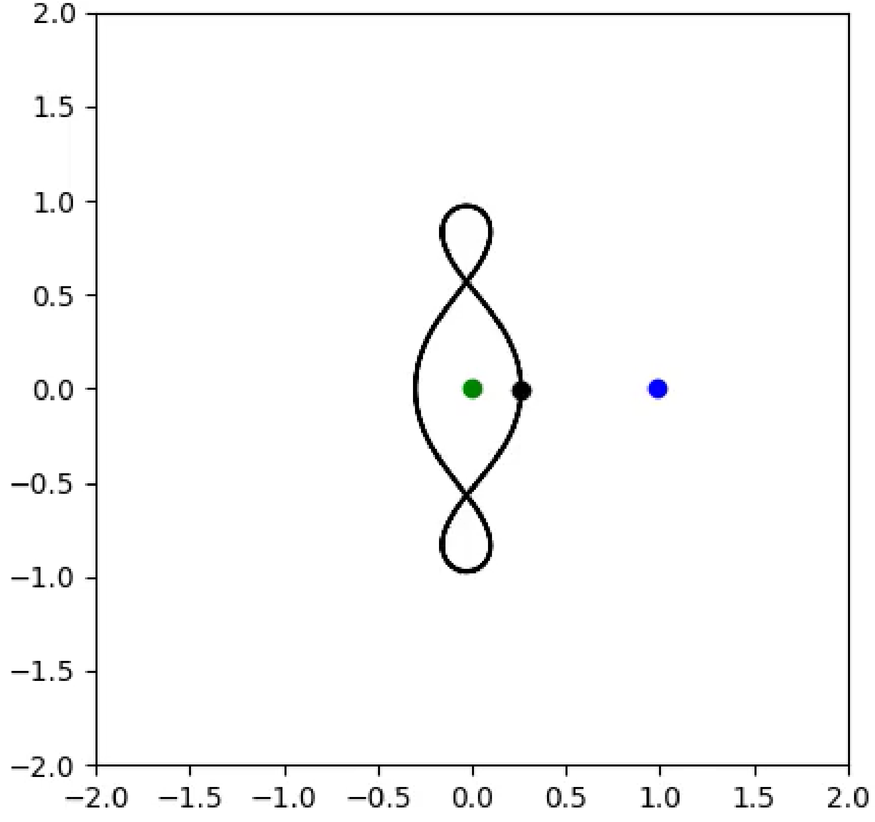


different frame

additional forces

FRAME TRANSFORMATION





- From C3RBP to J2000 coordinates
- From J2000 to CR3BP coordinates

- **From C3RBP to J2000 coordinates**
- From J2000 to CR3BP coordinates

STEP 1: EXTRACT CR3BP STATE

This is a dimensionless state vector in the rotating frame.

$$\mathbf{x}_{\text{CR3BP}} = \begin{bmatrix} x \\ y \\ z \\ \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix}$$

STEP 2: MULTIPLY BY INSTANTANEOUS CHARACTERISTIC UNITS

$$\mathbf{r}_{\text{dim}} = L \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \qquad \mathbf{v}_{\text{dim}} = V \cdot \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix}$$

where $L = r_{21}(t)$ is the instantaneous P1-P2 distance and $V = Ln$ with

$$n = \frac{\|\mathbf{r}_{21} \times \mathbf{v}_{21}\|}{\|\mathbf{r}_{21}\|^2} \approx \sqrt{\frac{G(m_1 + m_2)}{r_{21}^3}}$$

STEP 3: SHIFT ORIGIN FROM BARYCENTER TO PRIMARY (P1)

$$\mathbf{r}_{P1} = \mathbf{r}_{\text{dim}} + \mu \cdot L \cdot \hat{\mathbf{x}}$$

$$\mathbf{v}_{P1} = \mathbf{v}_{\text{dim}}$$

STEP 4: CONSTRUCT INERTIAL ROTATION MATRIX USING SPICE VECTOR

Get the **ephemeris vector of the secondary relative to the primary** from SPICE at time t:

r21(t)=spkezr("SECONDARY", t, "J2000", "NONE", "PRIMARY")

$$\hat{\mathbf{x}} = \frac{\mathbf{r}_{21}(t)}{||\mathbf{r}_{21}(t)||}, \qquad \hat{\mathbf{z}} = \frac{\mathbf{r}_{21}(t) \times \dot{\mathbf{r}}_{21}(t)}{||\mathbf{r}_{21}(t) \times \dot{\mathbf{r}}_{21}(t)||}, \qquad \hat{\mathbf{y}} = \hat{\mathbf{z}} \times \hat{\mathbf{x}}$$

$$R = \begin{bmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \end{bmatrix}$$

STEP 5: TRANSFORM TO J2000 FRAME

$$\mathbf{r}_{\mathrm{J2000}} = R \cdot \mathbf{r}_{P1}$$

$$\mathbf{v}_{\mathrm{J2000}} = R \cdot (\mathbf{v}_{P1} + \boldsymbol{\omega} \times \mathbf{r}_{P1})$$

STEP 5: TRANSFORM TO J2000 FRAME

$$\begin{bmatrix} \mathbf{r}_{J2000} \\ \mathbf{v}_{J2000} \end{bmatrix} = T \begin{bmatrix} \mathbf{r}_{P1} \\ \mathbf{v}_{P1} \end{bmatrix}$$

$$T = \begin{bmatrix} R_{11} & R_{12} & R_{13} & 0 & 0 & 0 \\ R_{21} & R_{22} & R_{23} & 0 & 0 & 0 \\ R_{31} & R_{32} & R_{33} & 0 & 0 & 0 \\ nR_{12} & -nR_{11} & 0 & R_{11} & R_{12} & R_{13} \\ nR_{22} & -nR_{21} & 0 & R_{21} & R_{22} & R_{23} \\ nR_{32} & -nR_{31} & 0 & R_{31} & R_{32} & R_{33} \end{bmatrix}$$

STEP 6: NONDIMENSIONALIZE (OPTIONAL)

$$\tilde{\mathbf{r}}_{\mathrm{J2000}} = \frac{\mathbf{r}_{\mathrm{J2000}}}{L'}, \quad \tilde{\mathbf{v}}_{\mathrm{J2000}} = \frac{\mathbf{v}_{\mathrm{J2000}}}{V'}$$

- From C3RBP to J2000 coordinates
- **From J2000 to CR3BP coordinates**

INVERSE OF THE TRANSFORMATION MATRIX

$$T = \begin{bmatrix} R & 0 \\ R\Omega & R \end{bmatrix}$$

$$T = \begin{bmatrix} A & 0 \\ B & A \end{bmatrix} \quad \Rightarrow \quad T^{-1} = \begin{bmatrix} A^{-1} & 0 \\ -A^{-1}BA^{-1} & A^{-1} \end{bmatrix}$$

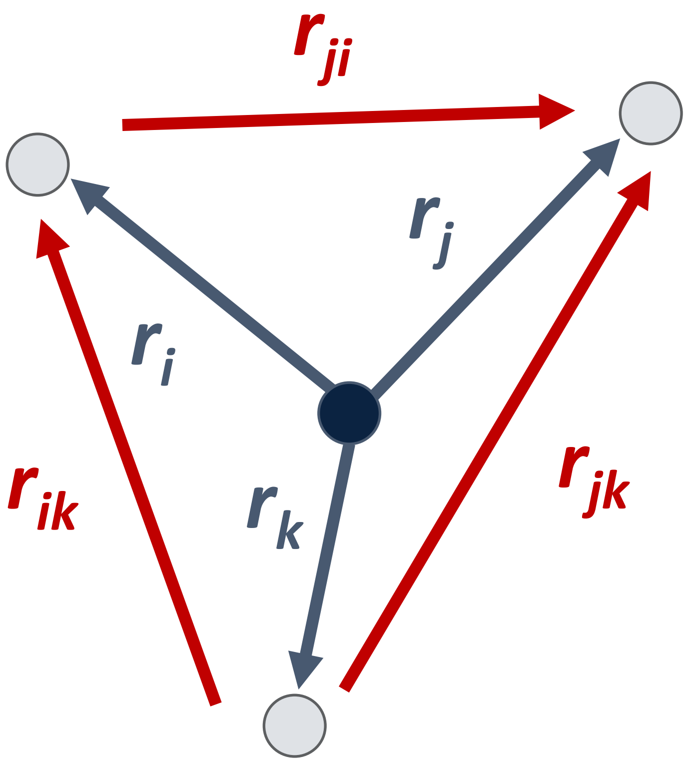
$$T^{-1} = \begin{bmatrix} R^{-1} & 0 \\ -R^{-1}(R\Omega)R^{-1} & R^{-1} \end{bmatrix} = \begin{bmatrix} R^\top & 0 \\ -R^\top(R\Omega)R^\top & R^\top \end{bmatrix} = \begin{bmatrix} R^\top & 0 \\ -\Omega R^\top & R^\top \end{bmatrix}$$

CONVERSION FROM J2000 TO CR3BP STATES

$$\begin{bmatrix} x \\ y \\ z \\ \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} \frac{1}{L} I_{3\times 3} & 0 \\ 0 & \frac{1}{V} I_{3\times 3} \end{bmatrix} \cdot \begin{bmatrix} R^\top & 0 \\ -\Omega R^\top & R^\top \end{bmatrix} \cdot \left(\begin{bmatrix} L' \cdot \tilde{\mathbf{r}}_{\mathrm{J2000}} \\ V' \cdot \tilde{\mathbf{v}}_{\mathrm{J2000}} \end{bmatrix} - \begin{bmatrix} \mathbf{r}_{P1}(t) \\ \mathbf{v}_{P1}(t) \end{bmatrix} \right) - \begin{bmatrix} \mu \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

TRANSITION TO AN EPHEMERIS MODEL

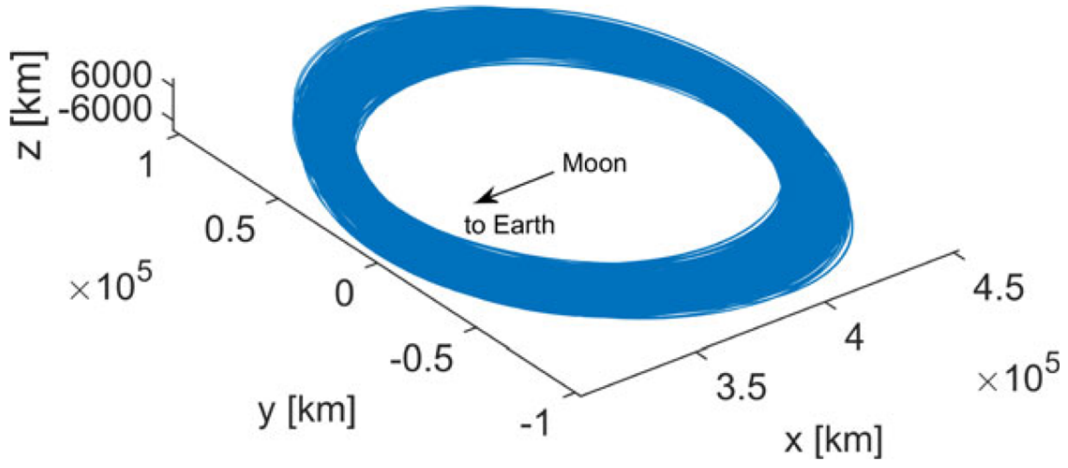
$$\ddot{\mathbf{r}}_{ik} = -G(m_i + m_k)\frac{\mathbf{r}_{ik}}{\|\mathbf{r}_{ik}\|^3} + G\sum_{j\neq i,k} m_j \left(\frac{\mathbf{r}_j - \mathbf{r}_i}{\|\mathbf{r}_j - \mathbf{r}_i\|^3} - \frac{\mathbf{r}_j - \mathbf{r}_k}{\|\mathbf{r}_j - \mathbf{r}_k\|^3}\right)$$



different frame

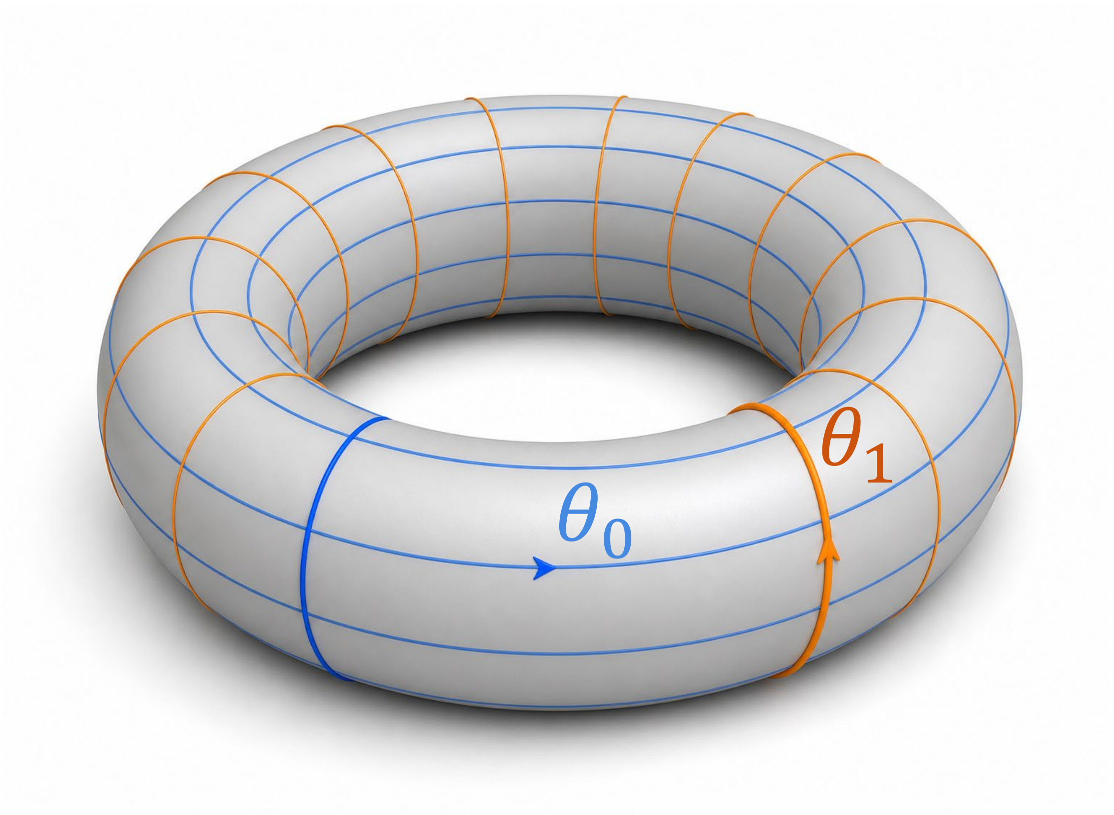
additional forces

QUASI-PERIODIC ORBITS (..OR NEARLY SO)



(a) DRO in RNP frame propagated for 25 years using NAIF SPICE ephemeris data

INVARIANT TORI



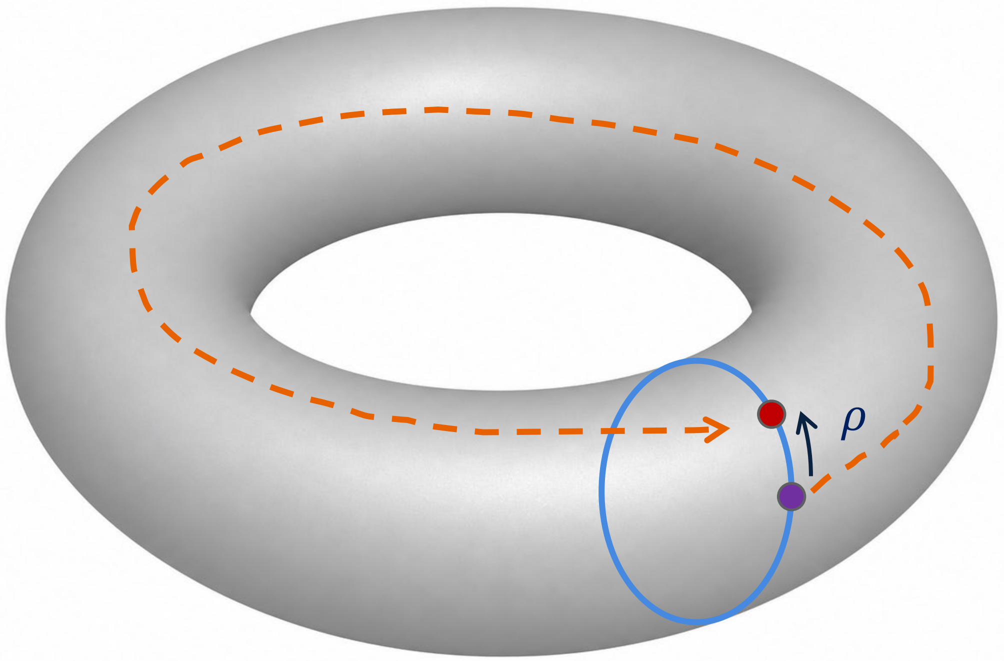
A p-dimensional invariant quasi-periodic torus $\mathfrak{I}$:

- Parameterized by $\boldsymbol{\theta} \in [0,2\pi\,]^p$
- Exists vector field $\boldsymbol{h}$ on torus that induces parallel flow with $\dot{\boldsymbol{\theta}} = \boldsymbol{h}(\boldsymbol{\theta}) = \boldsymbol{\omega}$ where $\boldsymbol{\omega} = [\omega_0, \dots, \omega_{p-1}]$ is constant and is comprised of flow's internal frequencies
- None of the frequencies in $\boldsymbol{\omega}$ are resonant with each other
- Repose EOMs $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, t) \rightarrow \begin{cases} \dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{\theta}) \\ \dot{\boldsymbol{\theta}} = \boldsymbol{\omega} \end{cases}$
- Torus function $\boldsymbol{v} : \mathfrak{I}^{p=2} \rightarrow \mathbb{R}^{n=6}$ must satisfy

$$\sum_{0}^{p-1} \omega_i \frac{\partial \boldsymbol{v}}{\partial \theta_i}(\boldsymbol{\theta}) = f(\boldsymbol{v}(\boldsymbol{\theta}), \boldsymbol{\theta})$$

In pure CR3BP → $p = 2$

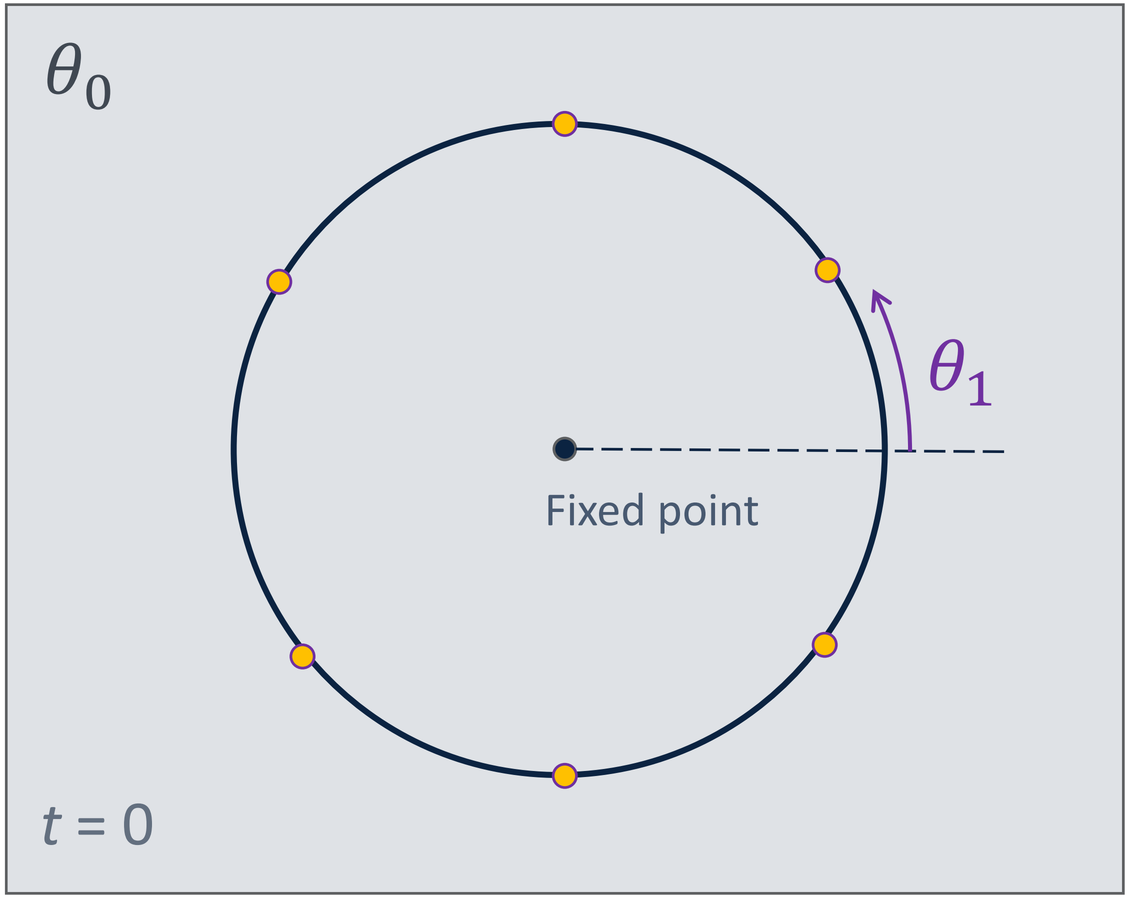
REDUCING DIMENSION WITH STROBOSCOPIC MAP



- Define the torus cross section (**invariant circle**) function $\boldsymbol{u}: \mathfrak{J}^{p-1=1} \rightarrow \mathbb{R}^{n=6}$ such that $\boldsymbol{u}(\theta_1) = \boldsymbol{v}(\theta_0 = \bar{\theta}_0, \theta_1)$ → fix $\theta_0 = \bar{\theta}_0$ (longitudinal angle, associate with periodic orbit)
- Define a **stroboscopic map** with period $T = 2\pi/\omega_0$
- The first map crossings are represented by

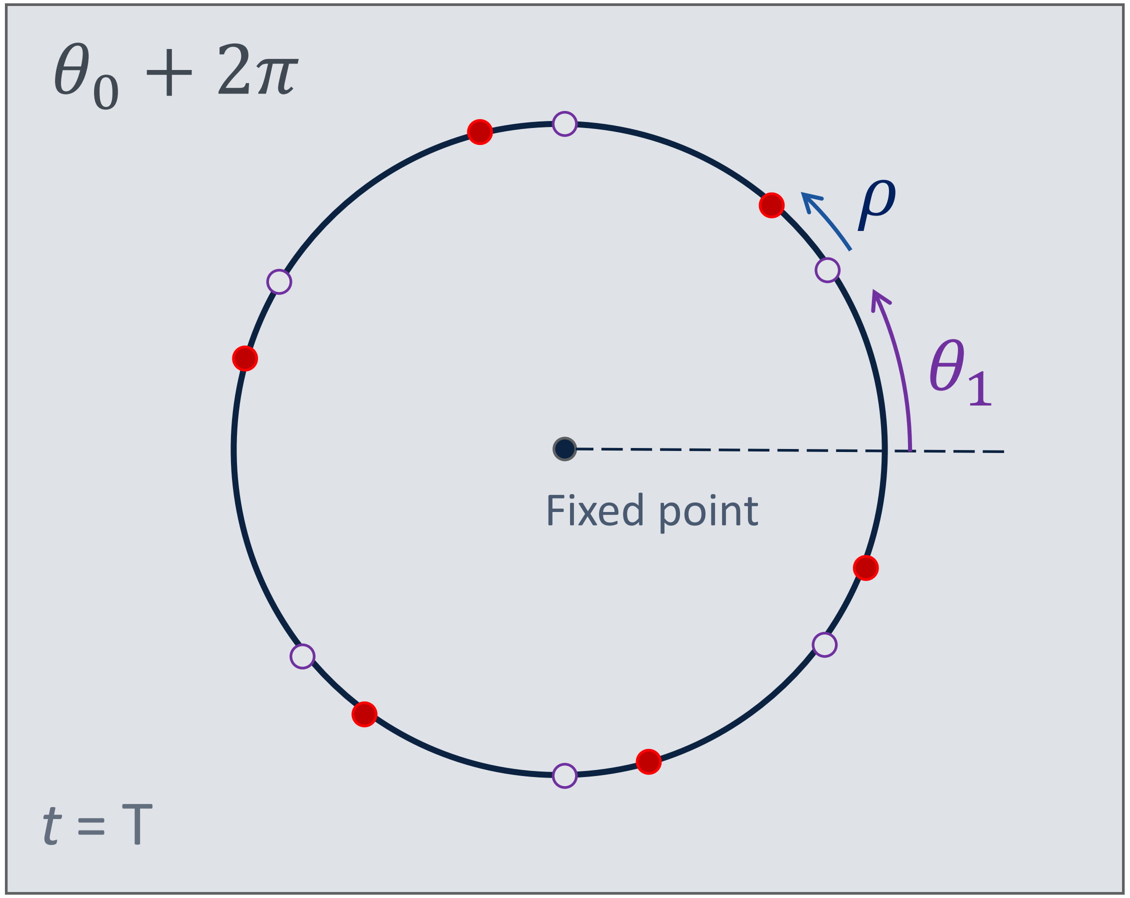
$$\begin{cases} \boldsymbol{x}_{\boldsymbol{m}} = \boldsymbol{F}(\boldsymbol{x}, \theta_1; T, \rho) \\ \theta_{1,m} = \theta_1 + \rho \\ \rho = T\omega_1 = 2\pi\left(\dfrac{\omega_1}{\omega_0}\right) \end{cases}$$

REDUCING DIMENSION WITH STROBOSCOPIC MAP



- Define the torus cross section (**invariant circle**) function $\boldsymbol{u}: \mathfrak{J}^{p-1=1} \rightarrow \mathbb{R}^{n=6}$ such that $\boldsymbol{u}(\theta_1) = \boldsymbol{v}(\theta_0 = \bar{\theta}_0, \theta_1)$ → fix $\theta_0 = \bar{\theta}_0$ (longitudinal angle, associate with periodic orbit)
- Define a **stroboscopic map** with period $T = 2\pi/\omega_0$

REDUCING DIMENSION WITH STROBOSCOPIC MAP



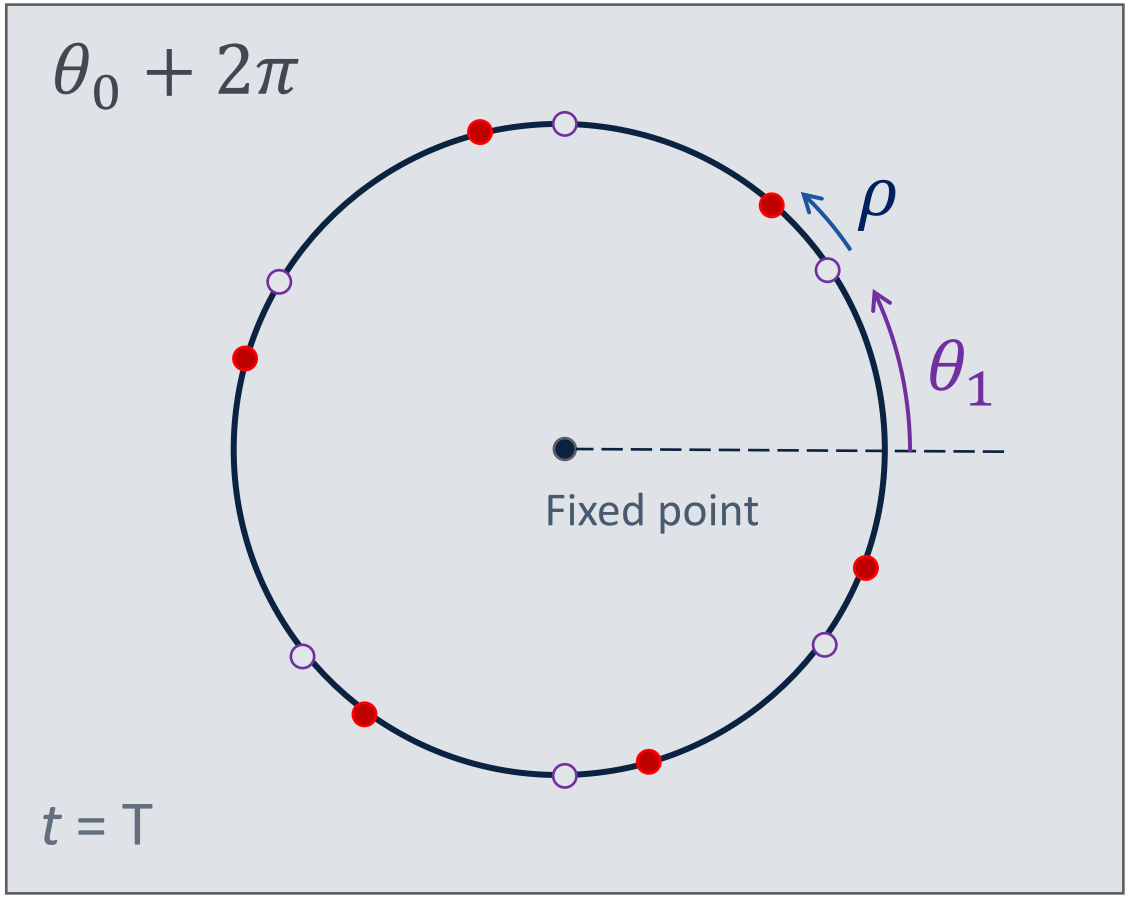
- Define the torus cross section (**invariant circle**) function $\boldsymbol{u}: \mathfrak{I}^{p-1=1} \rightarrow \mathbb{R}^{n=6}$ such that $\boldsymbol{u}(\theta_1) = \boldsymbol{v}(\theta_0 = \bar{\theta}_0, \theta_1)$ → fix $\theta_0 = \bar{\theta}_0$ (longitudinal angle, associate with periodic orbit)
- Define a **stroboscopic map** with period $T = 2\pi/\omega_0$
- The first map crossings are represented by

$$\begin{cases} \boldsymbol{x_m} = \boldsymbol{F}(\boldsymbol{x}, \theta_1; T, \rho) \\ \theta_{1,m} = \theta_1 + \rho \\ \rho = T\omega_1 = 2\pi\left(\frac{\omega_1}{\omega_0}\right) \end{cases}$$

- Subbing into the map $\boldsymbol{x_m} = \boldsymbol{u}(\theta_{1,m}) = \boldsymbol{u}(\theta_1 + \rho) = \boldsymbol{F}(\boldsymbol{u}(\theta_1), \theta_1; T, \rho)$

potentially unknown

INVARIANCE CONDITION



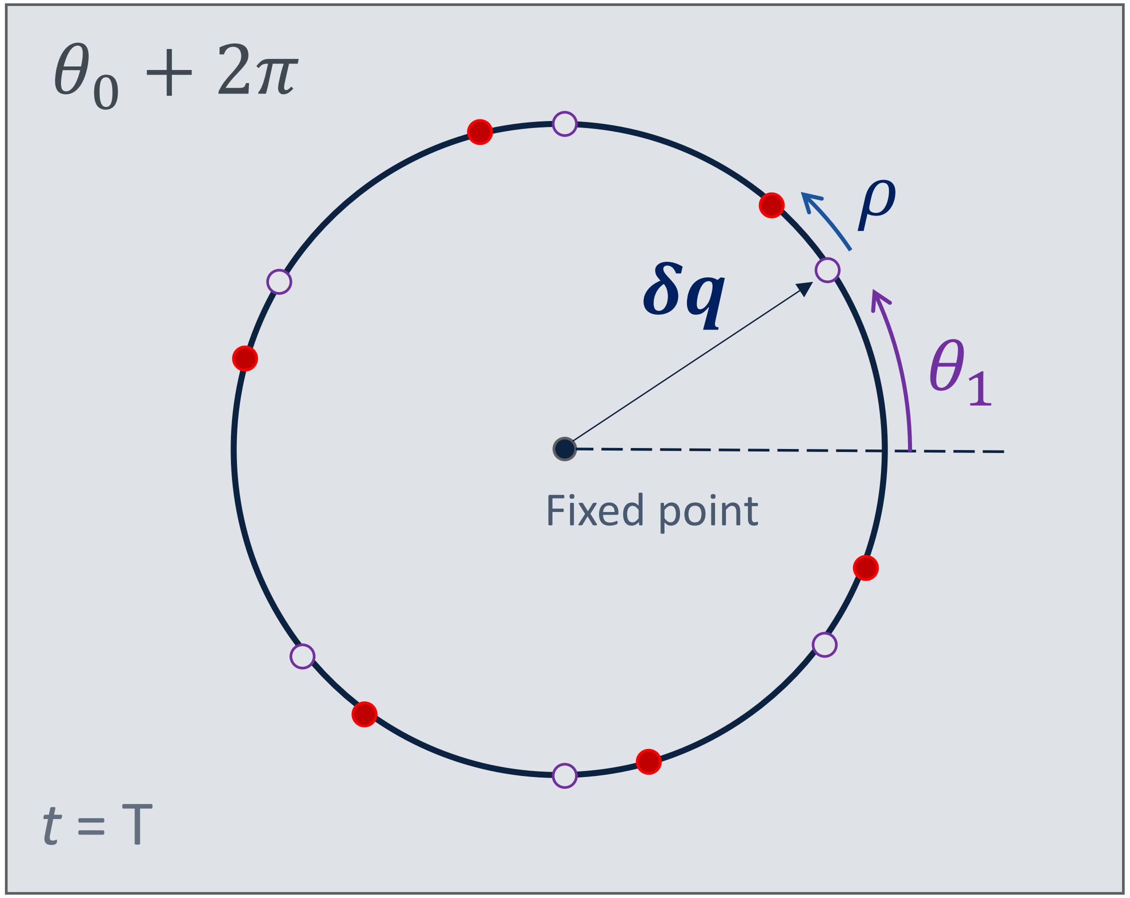
- The goal is to find for a quasi-periodic solution to map $\boldsymbol{F}$
- Define the space of functions $U = \{\boldsymbol{u}: \mathfrak{J}^{p-1=1} \rightarrow \mathbb{R}^{n=6}\}$ to represent invariant circle
- Define the invertible operator $R(-\rho)$ such that $u(\theta_1 - \rho) = R(-\rho)u(\theta_1)$
- Use this operator to define the map $\boldsymbol{G}: U \rightarrow U$

$$\boldsymbol{G}(\boldsymbol{u}(\theta_1)) = R(-\rho)\boldsymbol{F}(\boldsymbol{u}(\theta_1), \theta_1; T, \rho)$$

- Apply the operator to the original map $R(-\rho)\boldsymbol{u}(\theta_1 + \rho) = R(-\rho)\boldsymbol{F}(\boldsymbol{u}(\theta_1), \theta_1; T, \rho)$ to obtain the fixed point, invariance condition

$$\boldsymbol{G}(\boldsymbol{u}(\theta_1)) - \boldsymbol{u}(\theta_1) = \boldsymbol{0}$$

IMPLEMENTATION



- Initial guess → $\boldsymbol{\delta q} = d\,(\mathrm{Re}(\mathbf{v}_c)\cos(\theta_1) - \mathrm{Im}(\mathbf{v}_c)\sin(\theta_1))$
- Map $\boldsymbol{F}$ applied via numerical integration
- First, a single state, located at θ_1 on the invariant curve, is defined using a truncated Fourier series $\boldsymbol{u}(\theta_1) = e^{i\theta_1 \boldsymbol{k}}[C_0]$
- The coefficients $[C_0]$ are derived using a discreate Fourier transform $[C_0] = [D][u]$, where $[u] = [\boldsymbol{u_1}\ \boldsymbol{u_2}\ \boldsymbol{u_3} \ldots \boldsymbol{u_N}]^T$ is the matrix of state vector on the invariant circle
- Finally, $[R(-\rho)] = [D]^{-1}Q(-\rho)[D]$, where $[Q(\rho)] = \mathrm{diag}\left[e^{-i\rho\boldsymbol{k}}\right]$
- Solve the invariance condition for multiple states on the cross section using differential correction