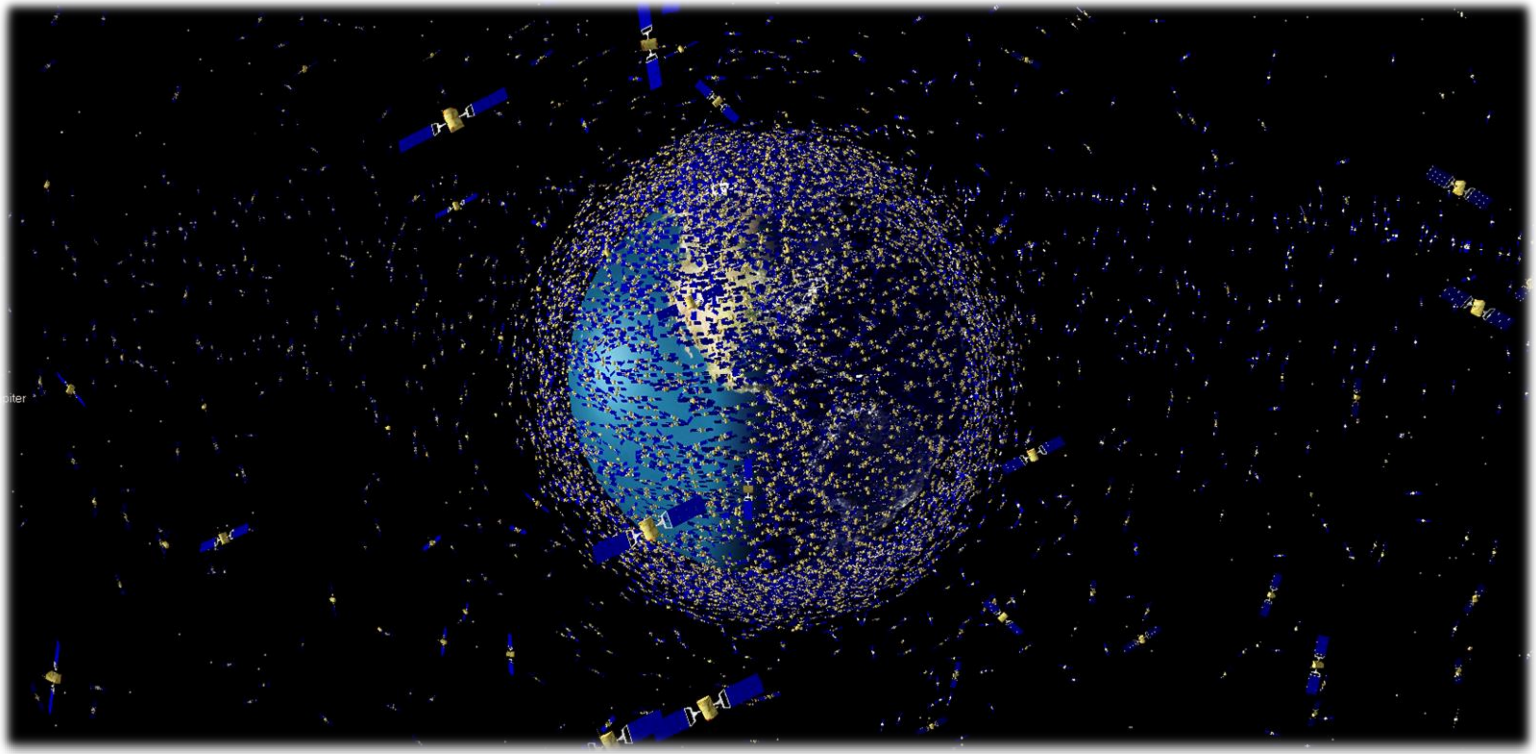
NASA Robotic CARA
Probability of Collision



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CARA Training Materials // 2018 // GSFC

Agenda

- **What is Pc**
- **2-D Pc Method**
  - Assumptions
  - Calculation
- **3-D Method**
- **Positive Definite Covariances**
- **Monte Carlo Calculation**

What is Probability of Collision?

- **Probability of Collision (Pc) represents the likelihood that the range between two objects may become less than the some radius R during the encounter time interval**
  - CARA uses the Hard Body Radius (HBR), which is the sum of the hard body radii of both the primary and secondary object
  - Could be more precisely referred to as probability of hard body incursion
  - Pc uses miss distance, event covariance and hard body radius in its calculation

Pc Calculation:
3-D Situation at Time of Closest Approach (TCA)

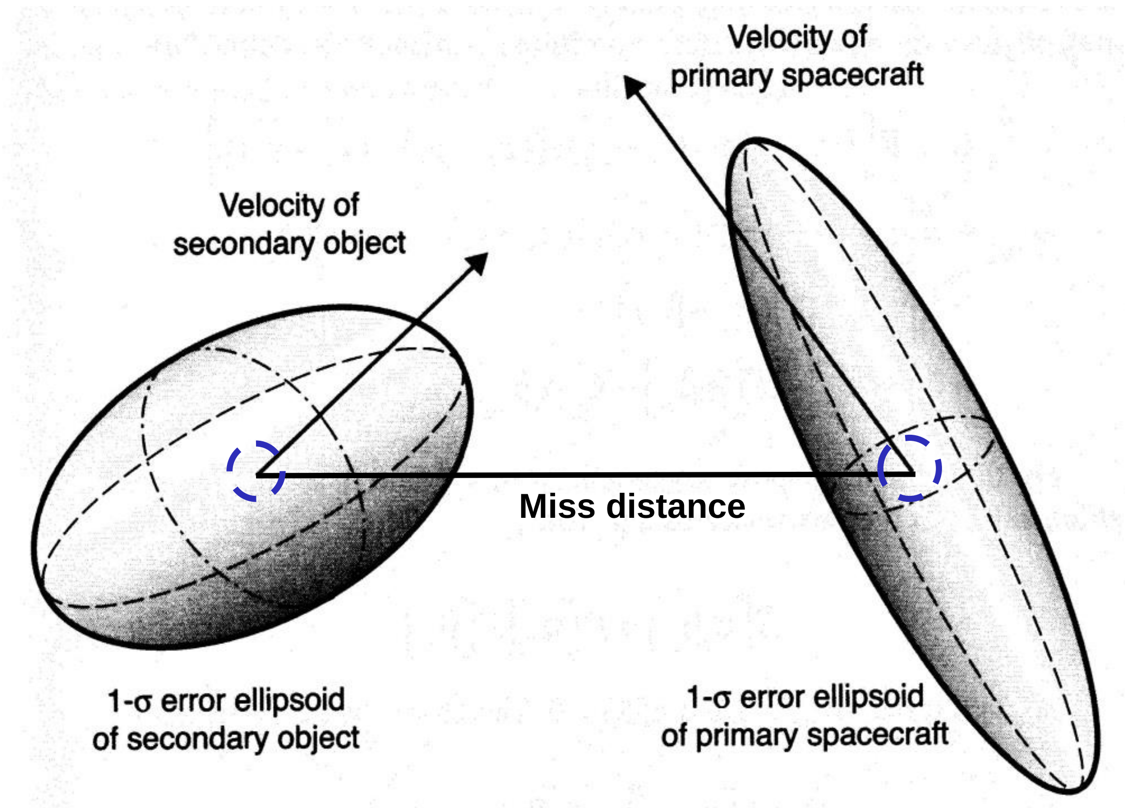


Figure taken from Chan (2008)

Why is Pc Important?

- **Probability of Collision is probably the single most important data point used to evaluate a close approach event and make a recommendation**
- **The Probability of Collision calculation considers the miss distance as well as the uncertainties of the event**
  - Therefore, you can say that miss distance is always considered when evaluating based on Pc
  - However Pc does NOT embody all risk for an event, no single factor does
  - Evaluating other data such as OD quality is still critical
- **Pc vs Miss Distance**
  - Not all small miss distance events are alike
  - “Small Miss” can be arbitrary, how is this value defined? What is small?
- **While CARA does not have a threshold for maneuvering based on miss distance, we do filter the initial screening data based on miss components (Radial, In-track, Cross-Track) in order to process a manageable amount of data**

Probability of Collision (P_c) Methodologies

- **2-D Probability of Collision**
  - Presently most common in industry
  - Assumes rectilinear relative motion with constant relative velocity between primary and secondary objects in the encounter period
  - Assumes covariance constant over encounter period
  - Encounter period must therefore be very short
- **3-D Probability of Collision**
  - Attempt to eliminate 2-D assumptions
  - Slightly slower calculation than 2-D Pc
- **Monte Carlo Probability of Collision**
  - P_c calculation independent of most (sometimes all) assumptions
  - Requires the most computational resources/time

2-D PC METHOD

2-D Pc

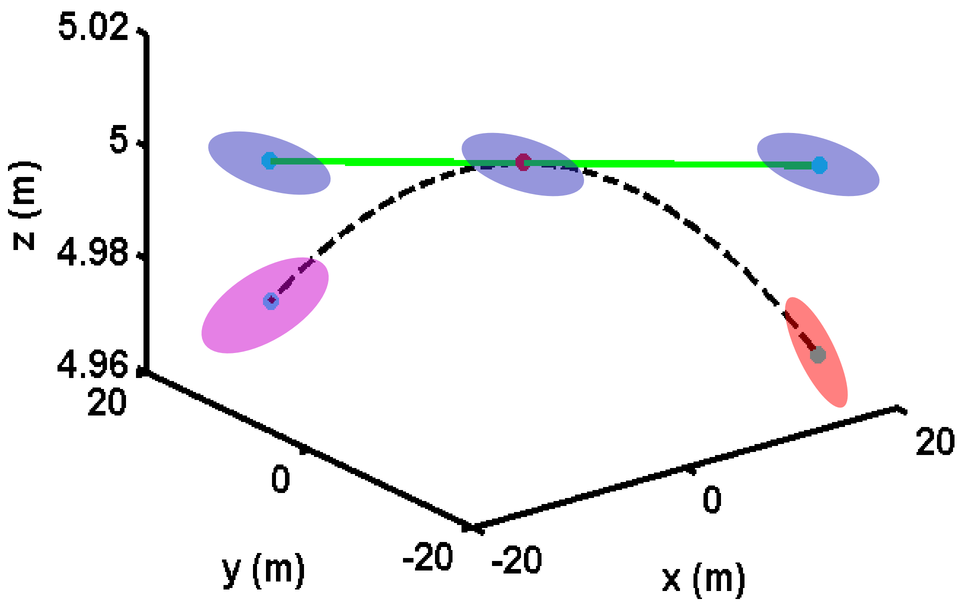
- **Most common method used by CARA for Pc calculation**
- **Less computationally expensive than 3-D Pc or Monte Carlo Pc calculation**
- **Requires several simplifying assumptions**
  - Assumptions explained on following slides

2-D Pc Calculation: Simplifying Assumptions

- **2-D assumption is only valid for:**
  - Conjunctions which are hyperkinetic—duration is thus very short
  - Motion is rectilinear in encounter region
  - Combined covariance does not change during encounter and can be held static for calculation purposes
  - Error volumes (position random variables about the mean) are uncorrelated
- **2-D Pc will most likely underrepresent the risk:**
  - Low-velocity encounters
  - Non-rectilinear motions
    - Conjunction can "persist" for a long time with a non-zero probability
  - Certain high-velocity conjunctions with very long, thin covariances
    - Small changes in the covariance's orientation can change Pc substantially
  - May apply to other edge cases

Encounter Region: Actual 3-D Situation

- **2-D simplification assumptions during encounter**
  - Presumes trajectory straight (green)
  - Presumes covariances static (blue)
- **Actual situation**
  - Trajectories are curvilinear (black)
  - Covariances vary in size and orientation throughout the encounter (pink, orange)



High Level 2-D Pc Calculation

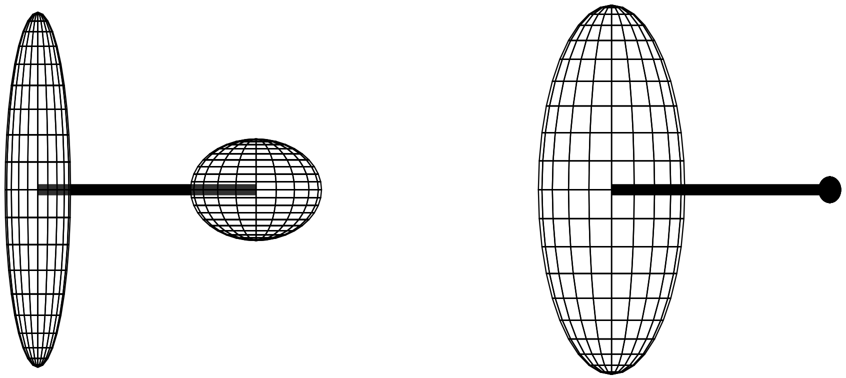
- **Combine covariances in consistent reference frame – typically inertial**
- **Combine the two objects’ hard body radii**
- **Define conjunction plane**
- **Project relative state and combined uncertainty into conjunction plane**
- **Determine the portion of the position uncertainty that falls within the combined HBR area**
- **Output is the Pc**

Combining Error Volumes

- **Relative position error can be expressed as the additive combination of the two satellite position covariances (proof given in Chan 2008)**
  - Presumes state errors are uncorrelated (more on this later)
  - $C_a + C_b = C_c$
  - Covariance combination must take place in a common coordinate frame
  - CARA software transforms the position and velocity state vectors for both the primary and secondary objects from RIC to ECI J2000
- **Through this combination, all relative error can be positioned at one of the two satellite positions**
  - Secondary satellite is typically used as location for joint state estimate error ellipsoid

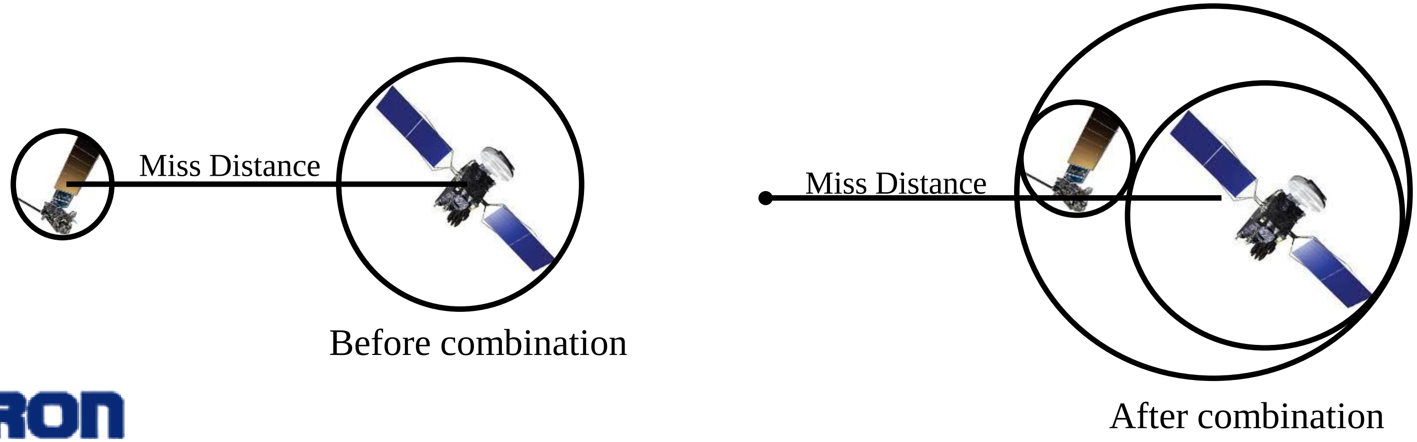
2-D (and 3-D) Pc Calculation: Combining Error Volumes—Crude Schematic

- **Left is situation before combination of covariances**
  - Secondary (left) and primary (right) each have proper covariance matrices
- **Right is situation after combination**
  - All relative error placed about one end of the relative position vector



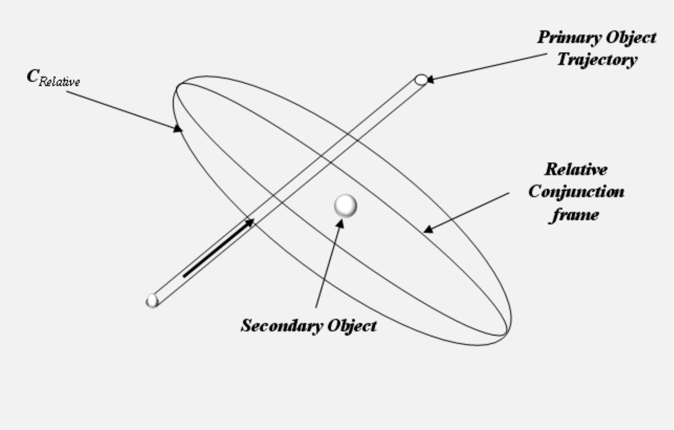
2-D and 3-D Pc Calculation: Combining HBRs

- **Spatial composition of satellite typically represented by circumscribing sphere**
- **Assume that 2-D physical size of both objects can be transferred to one of the two objects by defining a “supersphere” that encompasses both smaller spheres – called the Hard Body Radius**
  - Similar in concept to procedure for combining uncertainty volumes
  - Conservative approach that could bear refinement, but standard practice
- **Collision can take place if miss distance is less than the HBR**
  - For purposes of Pc calculation, this is considered a collision



2-D Pc Calculation: Simplifying Assumptions—Crude Schematic

- **Combined covariance centered at secondary position (with respect to primary) at TCA**
- **Primary path shown as straight "soda straw" in direction of relative velocity**
- **If soda straw can be considered infinitely straight and long, then integration of variable in that direction approaches unity**
  - Overall Pc will be 1 * the Pc calculated in a plane perpendicular to the relative velocity vector
  - Thus can reduce dimensionality of problem from 3 to 2 dimensions and work with projection into "conjunction plane"



2-D Pc Calculation: Conjunction Plane—Crude Schematic

- Combined covariance projected into plane normal to the relative velocity vector and placed at origin
- Axes rotated to place primary on x-axis at (nominal miss distance, 0)
- Projected HBR given as circle of radius equal to sum of both spacecraft circumscribing radii (projected supersphere)
- Z-axis perpendicular to x-axis in conjunction plane

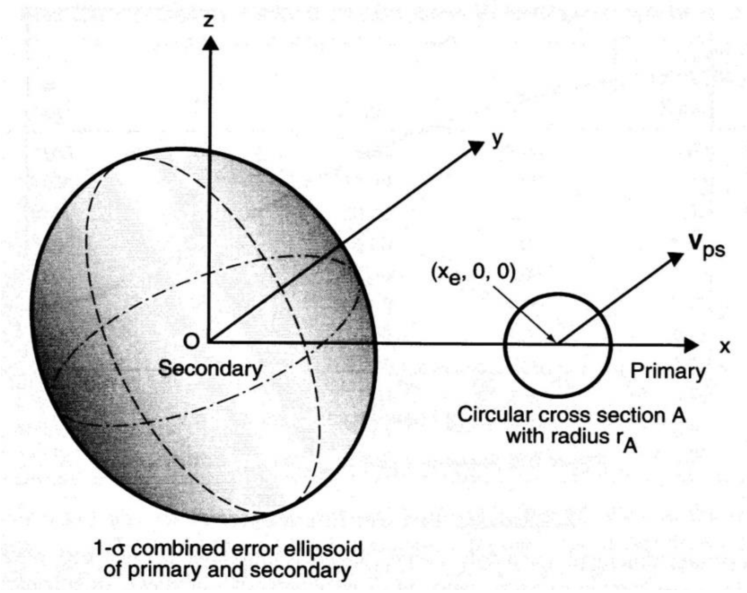
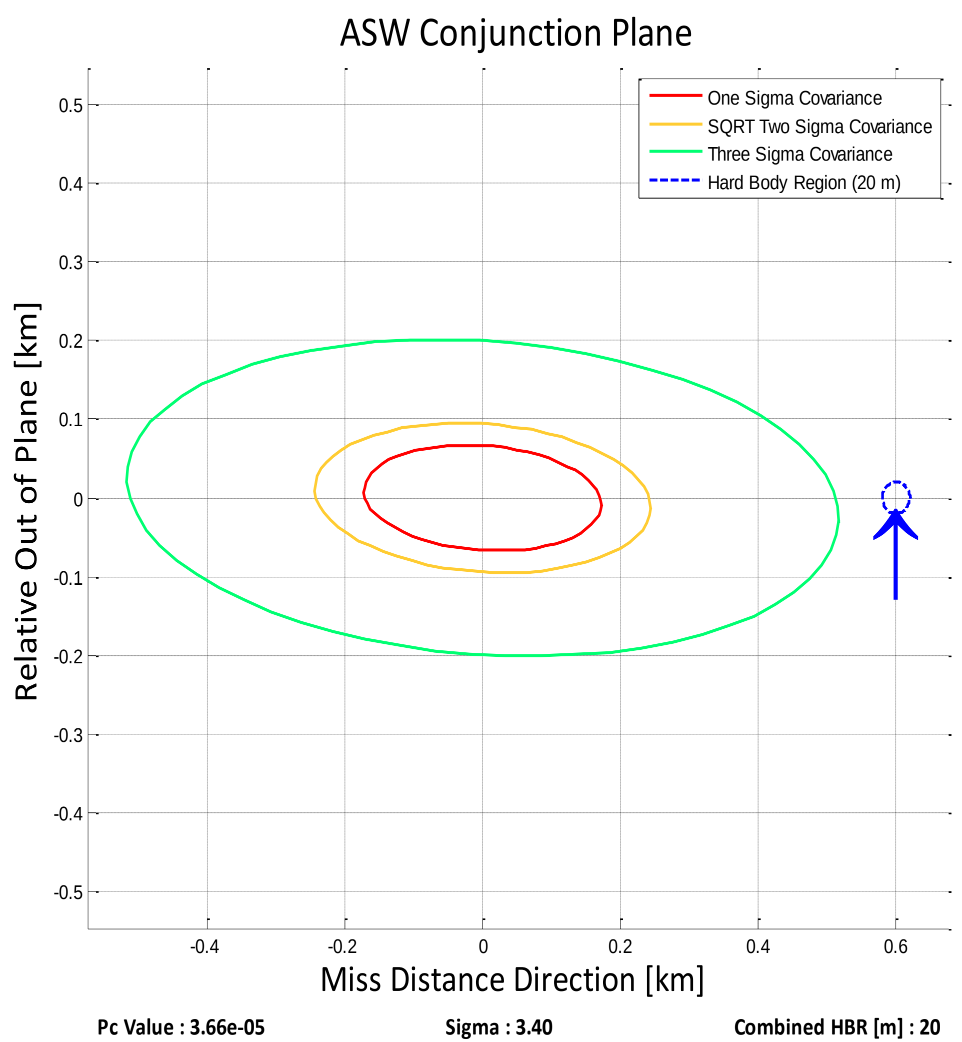


Figure taken from Chan (2008)

2-D Conjunction Plane Plot

- This is what the 2-D conjunction plane looks like in operations
- The 1, Sqrt(2), and 3 sigma ellipses are the projection of the combined covariance ellipse
- The Hard Body Region is the projection of the cylinder into 2-D space
- The secondary is centered at 0
- The distance from the origin to the Center of the HBR is the relative position, or the miss distance



2-D Pc Process

- **Computing the 2-D integral to calculate Pc**

$$P_c = \frac{1}{2\pi\sqrt{|C|}} \iint_A e^{-\frac{1}{2}\rho_{conj}^T C_{comb}^{-1} \rho_{conj}} dx\, dz\,.$$

Where the integration limits for x, z are:

$$-\sqrt{HBR^2 - z^2} \leq x \leq \sqrt{HBR^2 - z^2}$$
$$-HBR \leq z \leq HBR$$

- **The next slide consider different integration methods that can be employed to solve this problem**
  - In general, the 2-D Pc calculation will be solved numerically

2-D P_c Integration Methods

Method	Description	Conclusion
Foster*	Evaluates P_c by numerically solving the 2-D integral	Slowest, but can be sped up by increasing the step size for many cases without adversely affecting accuracy
Chan	Reformulates the problem into that of a 1-D Rician distribution using the concept of equivalent areas	By far the fastest but also the most restrictive due to relative object size limitations, and shown to be notably less accurate in certain cases
Patera	Based on a 1-D pdf and is formulated in the form of a line or contour integral	Produces good results, especially his most recent object-oriented formulation
Alfano	Based on a 1-D pdf expressed as two error functions (erf) and one exponential term	Performs well because it determines the number of integration steps on a case-by-case basis

* This is the current approach used internally by CARA. Matlab is used for numerical integration of the 2-D integral.

When is 2-D Pc not Adequate?

- **Inadequately researched question in CA**
  - CARA presently pursuing question—more on this later
- **Simplifications for 2-D Pc expected not to inhere when conjunction duration distended**
  - When the two objects remain in proximity for an extended period, relative motion is not rectilinear and position covariances not invariant
- **What is an "extended" period?**
  - Relative velocity < 10 m/s (Frigm and Rohrbaugh, 2008)
    - But study used synthetic test cases and may not be determinative
  - "Encounter Length" > 500 sec
    - Time for primary to transverse n-sigma version of combined covariance
    - Reasonable suggestion, but again based on synthetic cases
  - In both situations, truth criterion used probably questionable

3-D PC METHOD

3-D Pc Calculation: Plain Language Explanation

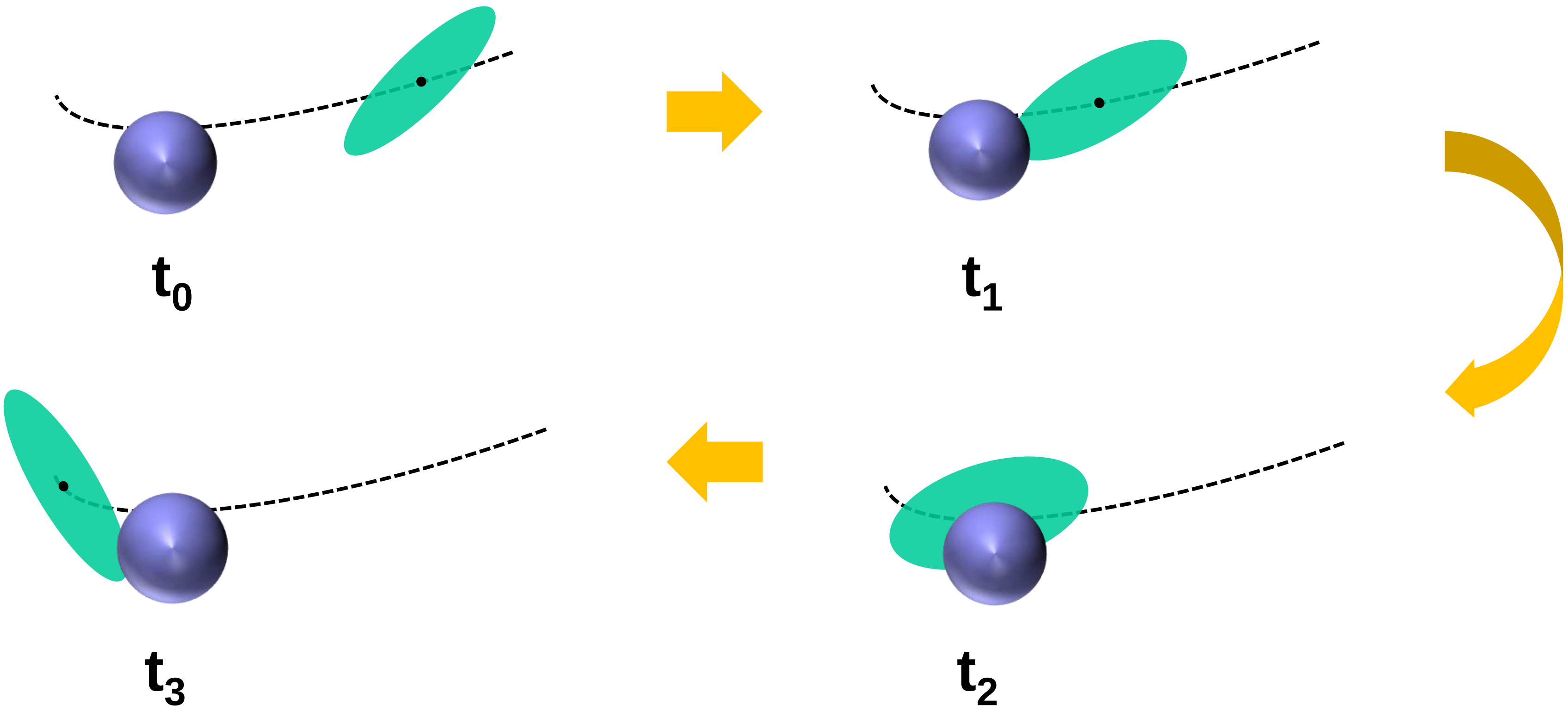
- **Begin problem set-up in manner similar to that for 2-D Pc**
  - Combine uncertainty volumes and place at secondary end of relative position vector
  - Combine HBR values into single sphere and place at primary end of relative position vector
- **However, do not limit investigation to a single instant of time or perform a dimensional reduction**
  - Consider HBR sphere about the primary
  - Identify a time period to investigate
  - At each instant during that time period, determine the portion of the combined uncertainty (placed about secondary) that intersects the surface of the HBR sphere
    - This is the instantaneous rate of Pc change, or “Pc Rate”
  - A time integral of this Pc Rate quantity produces the total Pc value
- **Hope is this will produce accurate Pc when 2-D Pc conditions not met**

3-D Pc Calculation: Component Parts

- **Need to be able to evaluate first derivative of Pc at any given moment in the encounter**
- **Need to be able to sum up all of these evaluations to get a Pc for the entire encounter**
  - To do this, need estimate of encounter duration
- **Need to allow trajectories to follow their appropriate path**
  - Eliminate rectilinear assumption
- **Need to allow covariances to change naturally, in shape and orientation, throughout the encounter**
  - Eliminate static covariance assumption
- **Should consider velocity uncertainties**
  - Use entire 6 x 6 covariance rather than limit oneself to the 3 x 3

3-D Pc Pictorial Progression

- Blue sphere is primary (as size of HBR); green ellipsoid is combined covariance (1-σ); black path is relative trajectory



3-D Pc Calculation Methodology

- **Methodology worked out by V.T. Coppola (2012)**
  - Expanded by DeMars *et al.* (2014), who discuss the "probability rate," dPc/dt
  - Probability rate is the instantaneous "rate of incursion" of uncertainty PDF into HBR sphere calculated by the surface area integral

$$\frac{dP_c}{dt} = F(t) = \oint_{4\pi} I\left(\hat{\mathbf{r}}, t\right) d^2\hat{\mathbf{r}}$$

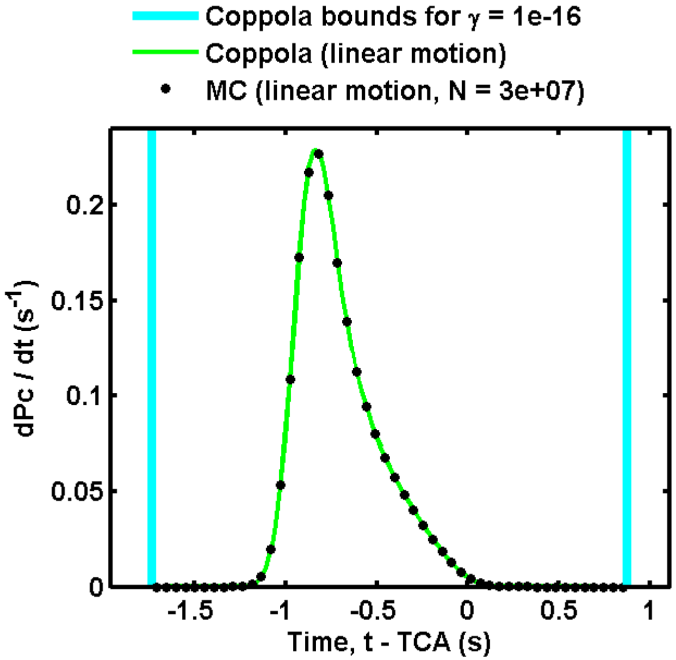
  - Approach greatly aided by extremely fast method of integrating over the unit sphere called Lebedev Quadrature (Lebedev 1999)
- **Pc for encounter a 1-D time integral of probability rate**

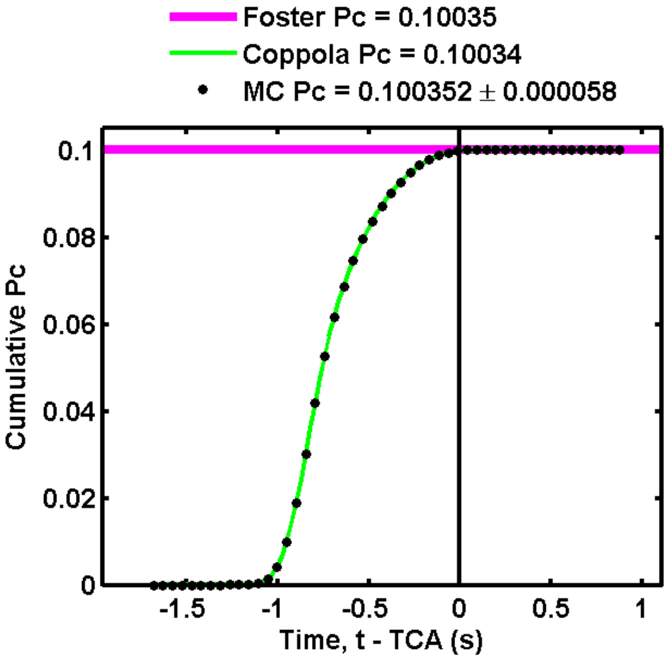
$$P_c = P_0 + \int_{t_0}^{t_0+T} \left(\frac{dP_c}{dt} \right) dt$$

  - Integration bounds can usually be chosen to drive P_0 essentially to zero

Omitron

Pc Rate Plot and Coppola Bounds



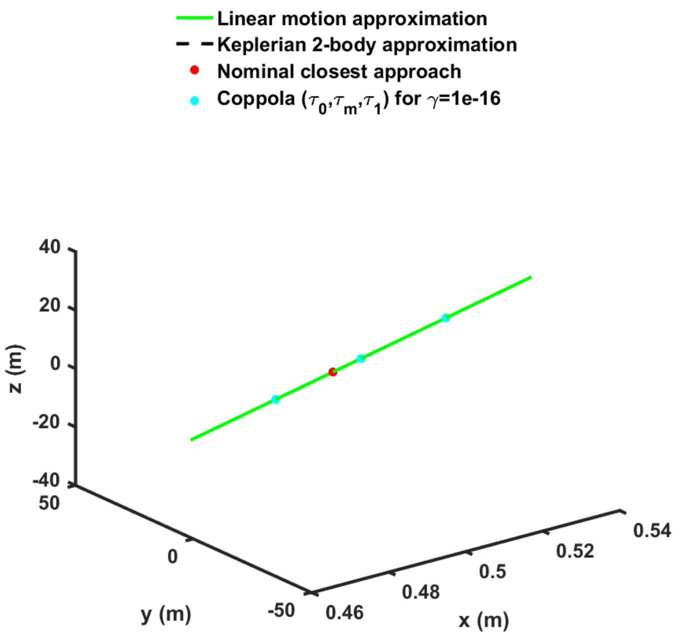


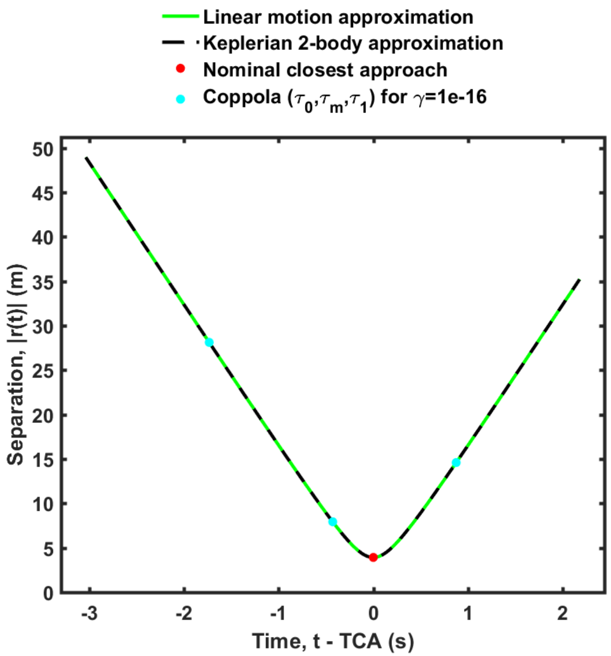
- **Plot shows "Pc Rate" (density incursion rate) as a function of time from TCA**
- **A single, hyperkinetic event will often have a Pc Rate plot that looks like this**
- **Note that point of highest risk not at TCA**
- **"Coppola Bounds" are his estimate of the appropriate size of integration region**

- **Pc CDF (Cumulative Density Function) plot shows accumulated Pc along integration time-span**
- **2-D Pc calculation has horizontal line CDF**
- **If 2-D assumptions valid, 3-D curve will converge to 2-D value**

Plots for Understanding 3-D Pc: Relative Motion Plot—Linear Case

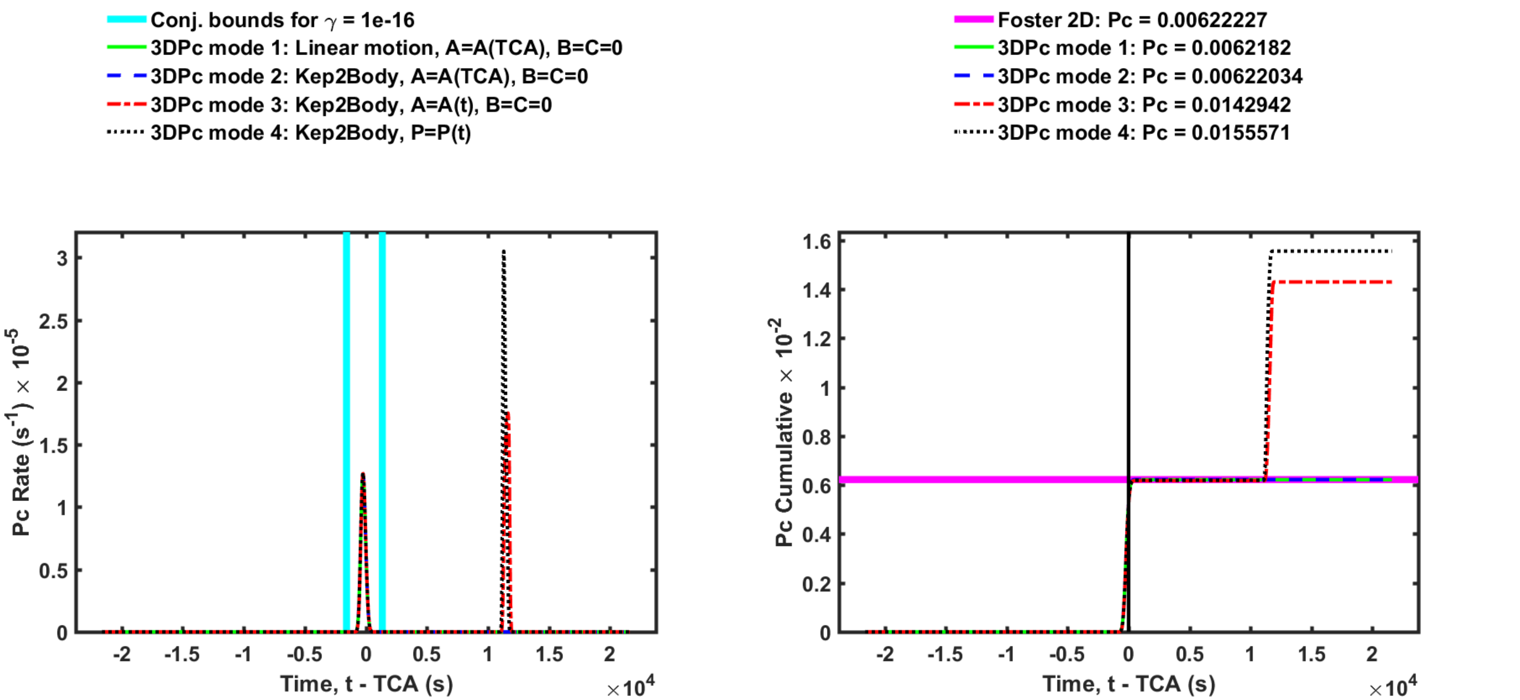
- **Left plot shows relative motion in encounter region**
  - Here lines are straight and overlap each other; linear assumption valid
- **Right plot shows overall separation distance vs time**
  - Linear and 2-body lines overlap and reach a single minimum; linear assumption valid





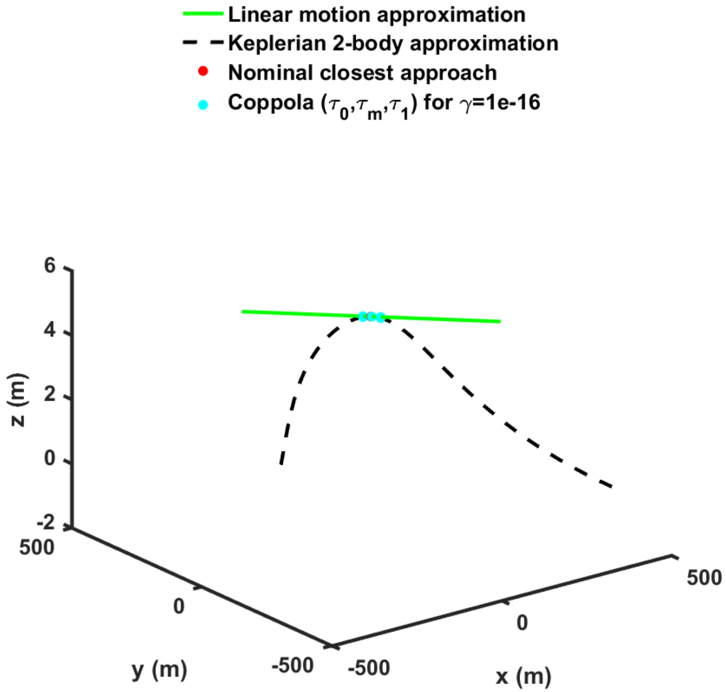
Plots for Understanding 3-D Pc: Pc Rate and Pc Accumulation—Curvilinear Case

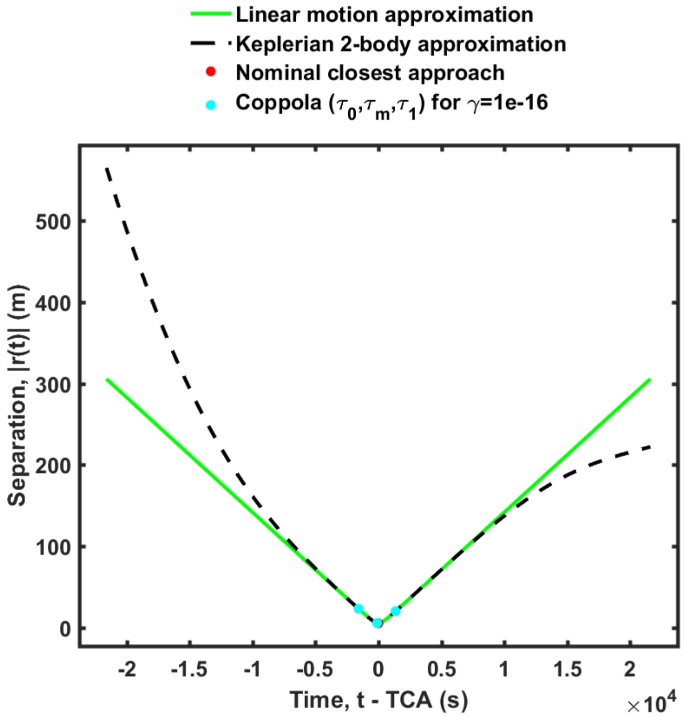
- **Left plot shows Pc rate (1st derivative of Pc) vs time**
  - Two peaks, one outside of nominal integration bounds—linear assumption poor
- **Right plot shows integrated Pc rate**
  - Sectioned Pc accumulation that does not agree with 2-D value



Plots for Understanding 3-D Pc: Relative Motion Plot—Curvilinear Case

- **Left plot shows relative motion in encounter region**
  - Curves differ substantially; linear assumption not valid
- **Right plot shows overall separation distance vs time**
  - Single minimum, but lines differ substantially; different Pcs will result from each





Problems with 3-D Pc Calculation

- **3-D Pc theory requires separation of position and velocity portions of covariance**
  - Usually means Cartesian, rather than equinoctial, representation required
- **Velocity portion of covariance in Cartesian coordinates appears to introduce representation problems**
  - For certain events, when velocity portion of covariance included, 3-D Pc produces wildly incorrect values (as verified by equinoctial Monte Carlo)
  - When velocity portion of covariance zeroed out, proper Pc produced
  - No good explanation developed for root cause of problem
- **CARA operational version of 3-D Pc operates with zeroed velocity portions of covariance**
  - Still produces improved calculation occasionally, but far less frequently and meaningfully than originally thought

POSITIVE DEFINITE COVARIANCES

Positive Definite Covariances: Plain-Language Explanation (1 of 2)

- **Covariance matrices define an error ellipsoid about a satellite's nominal position or velocity**
- **A positive definite matrix will define an ellipsoid that has positive values for the ellipsoid axes and is thus physically meaningful**
  - Pc calculations with non-positive-definite covariances questionable
- **Theory indicates that all covariance matrices generated by the OD process should be positive definite**
  - However, truncation and roundoff error (and very occasionally observability conditions) can prevent this in a given covariance matrix

Positive Definite Covariances: Plain-Language Explanation (2 of 2)

- **Problem is accentuated with larger covariance matrices, especially those that mix units and have large condition numbers**
  - This takes place in move from 2-D to 3-D Pc, in which not just the upper 3 x 3 portion but the entire 6 x 6 state covariance matrix is required
- **Problem can also arise when interpolating covariances**
  - This problem exists on the ASW system, and implementation of their interpolation approach has introduced instances of the problem
- **CARA has developed remediation techniques employed by automation on a case by case basis**
  - Forces matrix's (small) negative eigenvalues to reasonable positive values
    - Easier for position matrices, for which lower limit for position errors can be established
  - Pc result generally insensitive to small corrections of this nature

MONTE CARLO METHODS

Monte Carlo in State Error Estimation

- **A covariance matrix characterizes the expected errors in the state**
  - Gives variances of each parameter and their cross-correlations
- **By performing random draws from the covariance and adding these to the state, can produce a family of reasonable states for the object**
- **If the covariance already exists at the time of interest, performing these draws can produce the family of states at that time**
  - Requires "realism" of covariance at that time
- **If used with an epoch covariance, if each state in this family propagated forward, can produce a reasonable set of future states as well**
  - This avoids "covariance realism" errors encountered when trying to propagate the covariance itself forward in time

Monte Carlo Pc Calculation: Plain Language Explanation

- **Perform random draws on the primary and secondary covariances and add to nominal state estimates**
  - This produces a family of X states for the primary and secondary object
- **Propagate each pair of states forward and find TCA for that pair**
  - E.g., propagate and find TCA for X(1) for primary and X(1) for secondary, X(2) for primary and X(2) for secondary, &c.
- **If the miss distance at TCA < HBR, this pair represents a “hit”**
- **# of hits / X = Pc**
- **This procedure can be followed from epoch or from the nominal TCA**
  - If from epoch, called “brute force Monte Carlo”
  - If from mean TCA, called “Monte Carlo from TCA” or “CDM Mode Monte Carlo”

Cartesian vs Element Representations of States

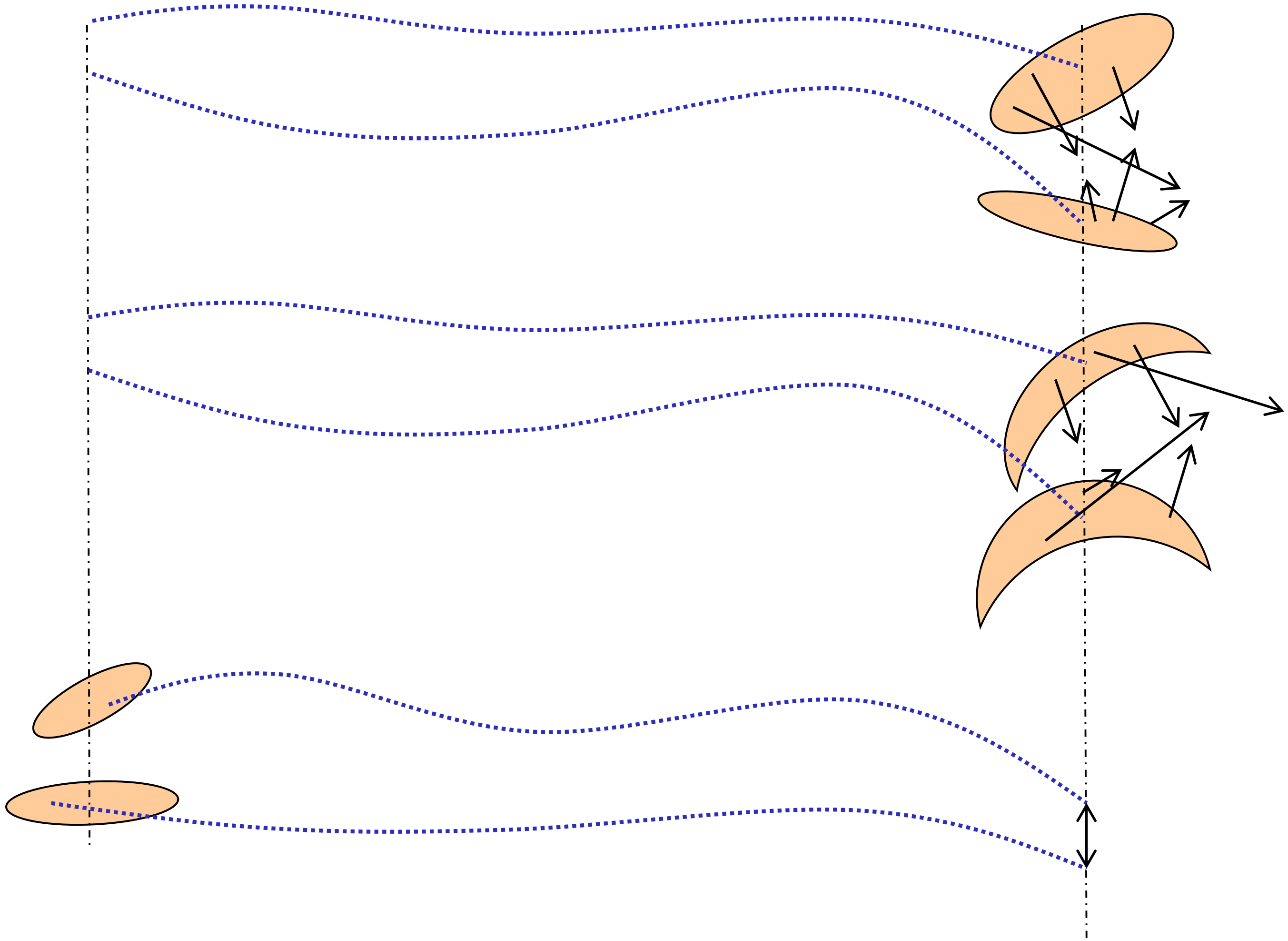
- **Three-dimensional states require six parameters to specify fully**
  - Seven if one wishes to include the time for which that state is representative
- **Cartesian representations specify 3-D position and 3-D velocity**
  - Use an orthogonal "Cartesian coordinate system"
  - Straightforward to visualize, and needed for certain force model applications
- **Element representations specify states in terms of geometry**
  - Keplerian elements, which can experience singularities
  - Equinoctial elements, which are transformed versions of traditional elements in order to eliminate most singularities
- **Because orbits are curvilinear and not rectilinear, element representations perform better under linearization**
  - Element-formulated covariances give better covariance realism
- **Working in element representations therefore desirable**
- **Different Monte Carlo results obtained depending on whether sampling performed in Cartesian or equinoctial frame**

Different Monte Carlo Types: Cartoon Schematic

Level II: propagate covariances to TCA; generate MC samples in Cartesian space and find TCA between pairs

Level III: propagate covariances to TCA; generate MC samples in element space and find TCA between pairs

Level IV: Generate samples at epoch; propagate every pair of samples forward to its proper TCA



Different Monte Carlo Types: Tabular Comparison

Level	Description	Commentary
I	• Linear propagation of position covariances to nominal TCA • Position error draws taken from propagated covariance • Pairs of perturbed states "propagated" to TCA with rectilinear motion (constant state velocity)	Essentially reproduces 2-D Pc Can be used to examine distributions of component errors (but analytical solutions for these distributions exist)
II	• Linear propagation of covariances to nominal TCA • Position and velocity error draws taken from propagated covariance • Pairs of perturbed states propagated to TCA with two-body propagator	Considers full state error in executing perturbations and performs actual propagation to find point-pair TCAs
IIIa	• Linear propagation of position covariances to nominal TCA, but natively in element (equinoctial) space • Six-element state perturbations taken from equinoctial covariances; propagated natively in element space to find TCA for each pair	Works natively in curvilinear (element) space; covariances should be more accurately representative of real state error distributions; does not consider correlation between covariances
IIIb	• Linear propagation of covariances to nominal TCA • Covariance converted from Cartesian to equinoctial elements • Large number of error samples taken from equinoctial covariance and back-converted (non-linearly) to Cartesian framework • Back-converted error samples used to create state perturbations; MC at this point follows Level II approach	Published study (Sabol 2010) showed that this resampling approach renders very similar results to an equinocitally-native method (*i.e.,* Level IIIa)
IV	• Primary and secondary states perturbed at epoch • Each pair of perturbations propagated forward, will full non-linear dynamics, to its proper TCA • ASW propagator used, along with JBH09 atmospheric density model • Execution times can be extremely long	Sometimes called "brute force MC," it is the gold standard for evaluating actual conjunction Pc; no known limitations Produces reliable comparison results for actual conjunctions only if ASW propagator and atmospheric model used

Monte Carlo Pc Calculation: Confidence Intervals and Sample Reuse (1 of 2)

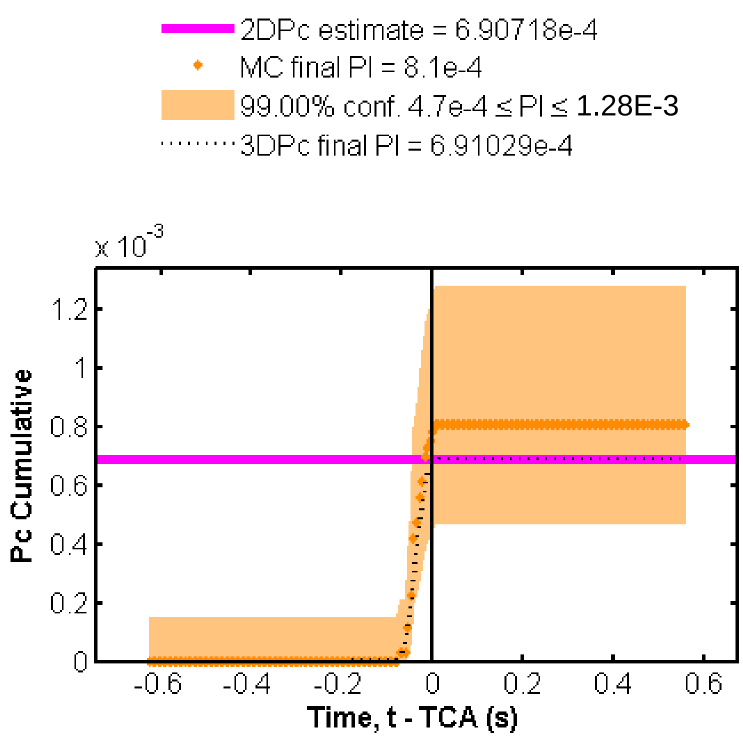
- **Monte Carlo methods should eventually converge on correct Pc**
  - Question is whether enough trials were run to presume convergence
- **Methods exist for calculating confidence interval for solution**
  - Usually give confidence range and Pc value boundaries
    - *e.g.,* Pc estimate is 1E-04, with 95% confidence that true Pc between 5E-05 and 2E-04
    - To narrow (or improve) the confidence interval, increase the number of trials
  - Because MC for Pc is a binary exercise (each trial either is or is not a hit), results conform to a binomial distribution
  - Multiple approaches for estimating binomial confidence intervals
    - CARA uses Clopper-Pearson approach (embedded in MATLAB "binofit" function)
- **However, these methods work only if MC samples are not reused!**

Monte Carlo Pc Calculation: Confidence Intervals and Sample Reuse (2 of 2)

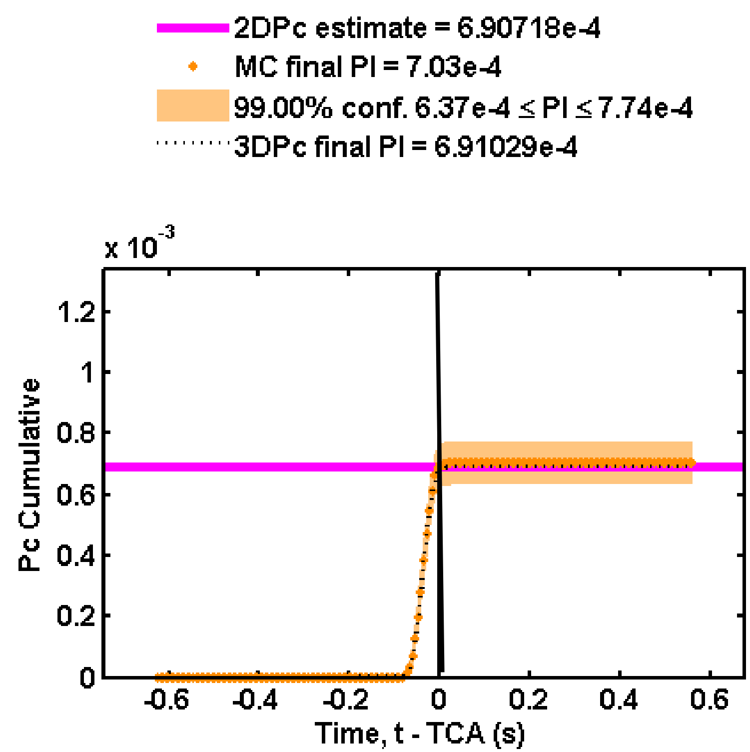
- **Computationally expensive to propagate large numbers of trajectories forward to TCA**
- **Practitioners tempted to reuse each propagated sample for primary by comparing to each propagated sample for secondary**
  - Single-comparison method: p(1) → s(1), p(2) → s(2), … p(n) → s(n)
  - Multiple-comparison method: p(1) → s(1), p(1) → s(2), … p(1) → s(n), p(2) → s(1), p(2) → s(2), … p(2) → s(n), &c.
  - If N samples produced for each, this produces N^2 trials
  - Seems like easy and efficient way to obtain more trails with same propagation
- **However, confidence interval calculations invalid with this approach**
  - Demonstrated by R. Carpenter in 2016 paper
  - MC with this approach may converge on the correct answer, but not possible to know whether enough samples run to trust any given result
- **MC methods for operational use need to have reliable accompanying confidence intervals**

Sample Monte Carlo Output

- **These two Monte Carlo runs used a different number of trials**
  - The plot at right using more trials also has a smaller confidence interval
  - Both suggest that the true Pc is somewhat higher than the 2-D Pc



3.6E4 Trials



1.01E6 Trials

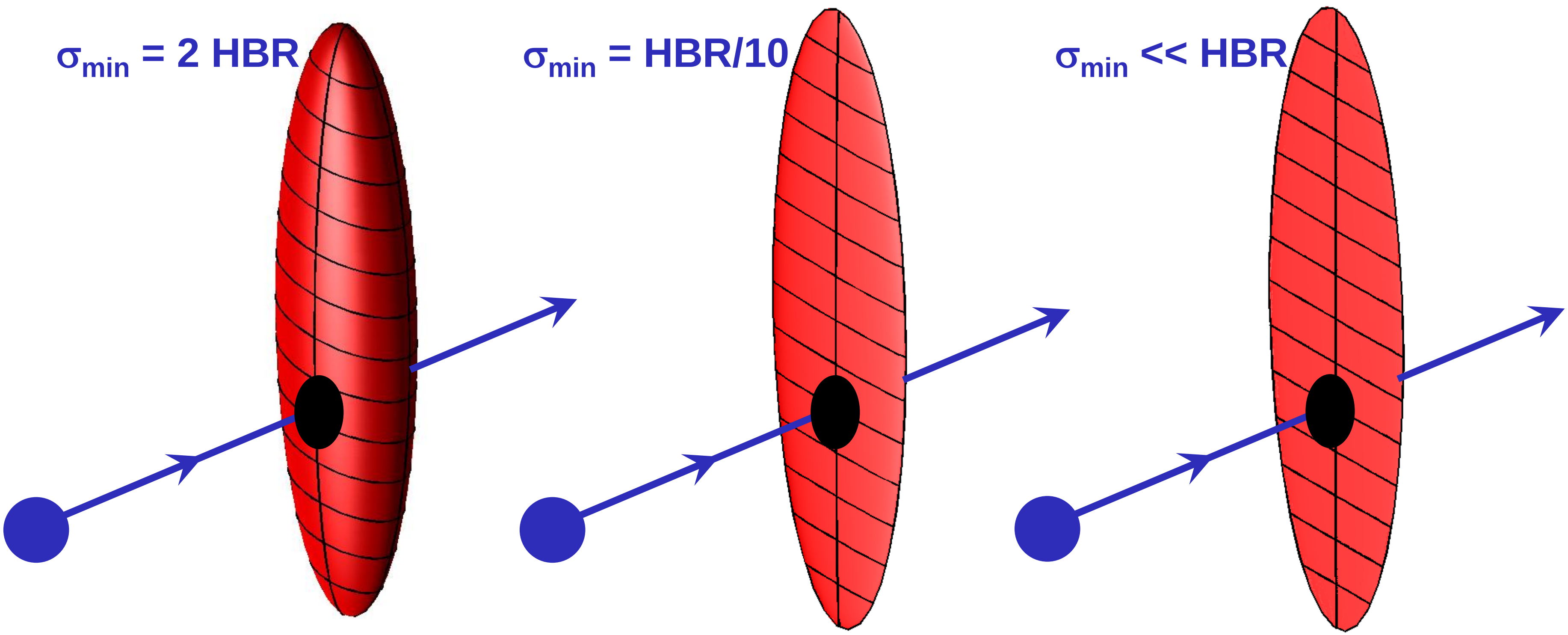
Summary

- **What is Pc**
- **2-D Pc Method**
  - Assumptions
  - Calculation
- **3-D Method**
- **Positive Definite Covariances**
- **Monte Carlo Calculation**

BACK UP

Collision Probability as a Function of σ_{min}

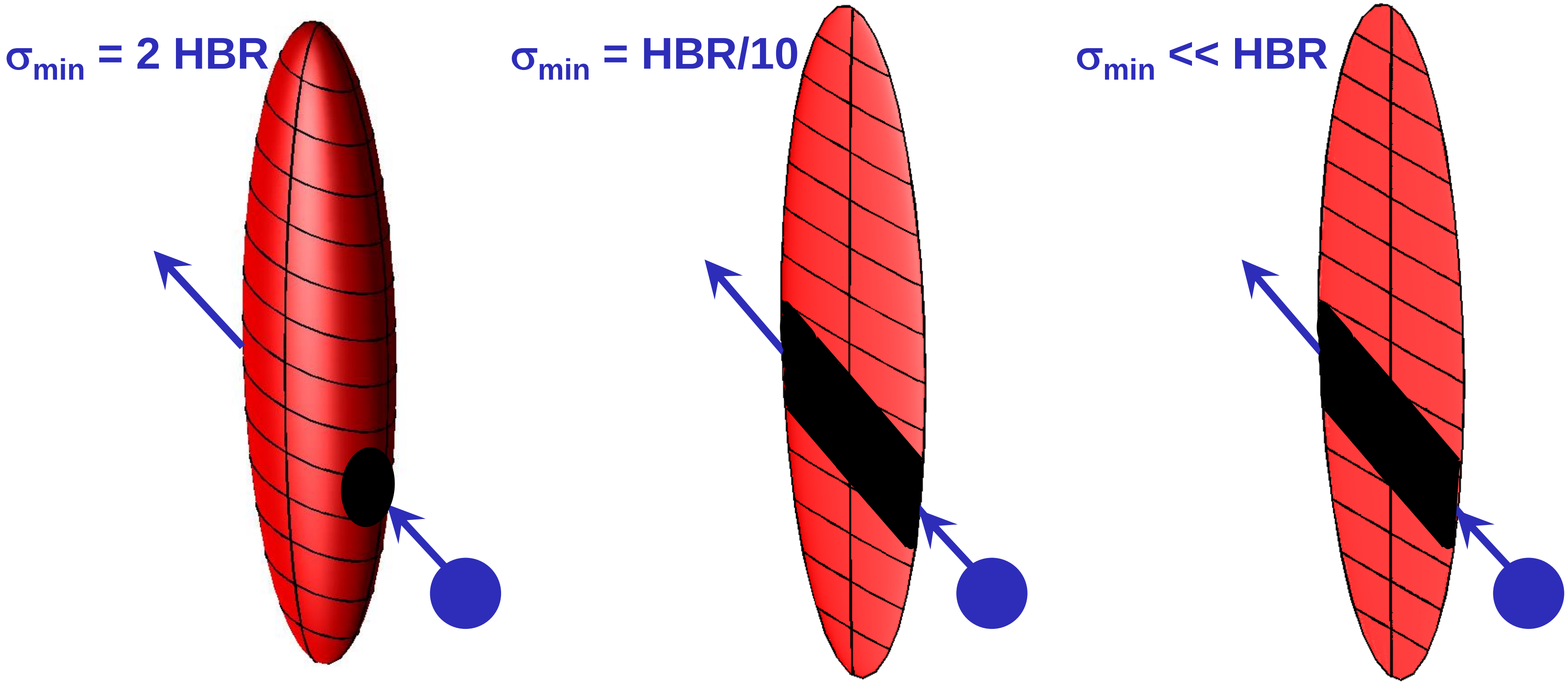
The collision probability represents an integral of the relative position PDF over the volume carved out along the path of the collision sphere



These three conjunctions produce similar Pc values (because they've carved out similar fractions of the PDF)

Collision Probability Visualization: Pc as a Function of σ_{min}

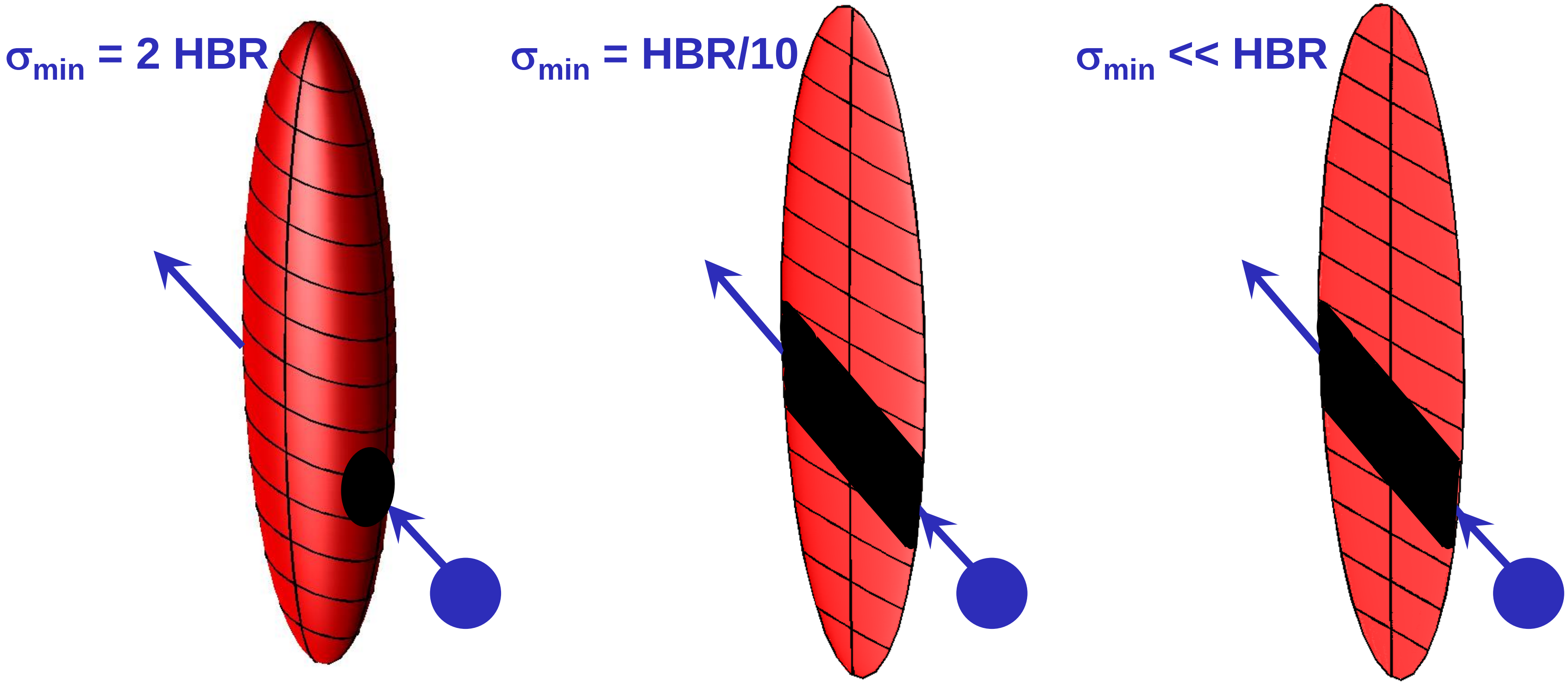
The collision probability represents an integral of the relative position PDF over the volume carved out along the path of the collision sphere



The two conjunctions on the right will produce similar Pc values
The one on the left will produce a smaller Pc value

Collision Probability Visualization: Pc as a Function of σ_{min}

The collision probability represents an integral of the relative position PDF over the volume carved out along the path of the collision sphere



Pc values are insensitive to the σ_{min} value whenever $0 < \sigma_{min} \ll$ HBR
This can be used to set a sensible eigenvalue clipping level

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