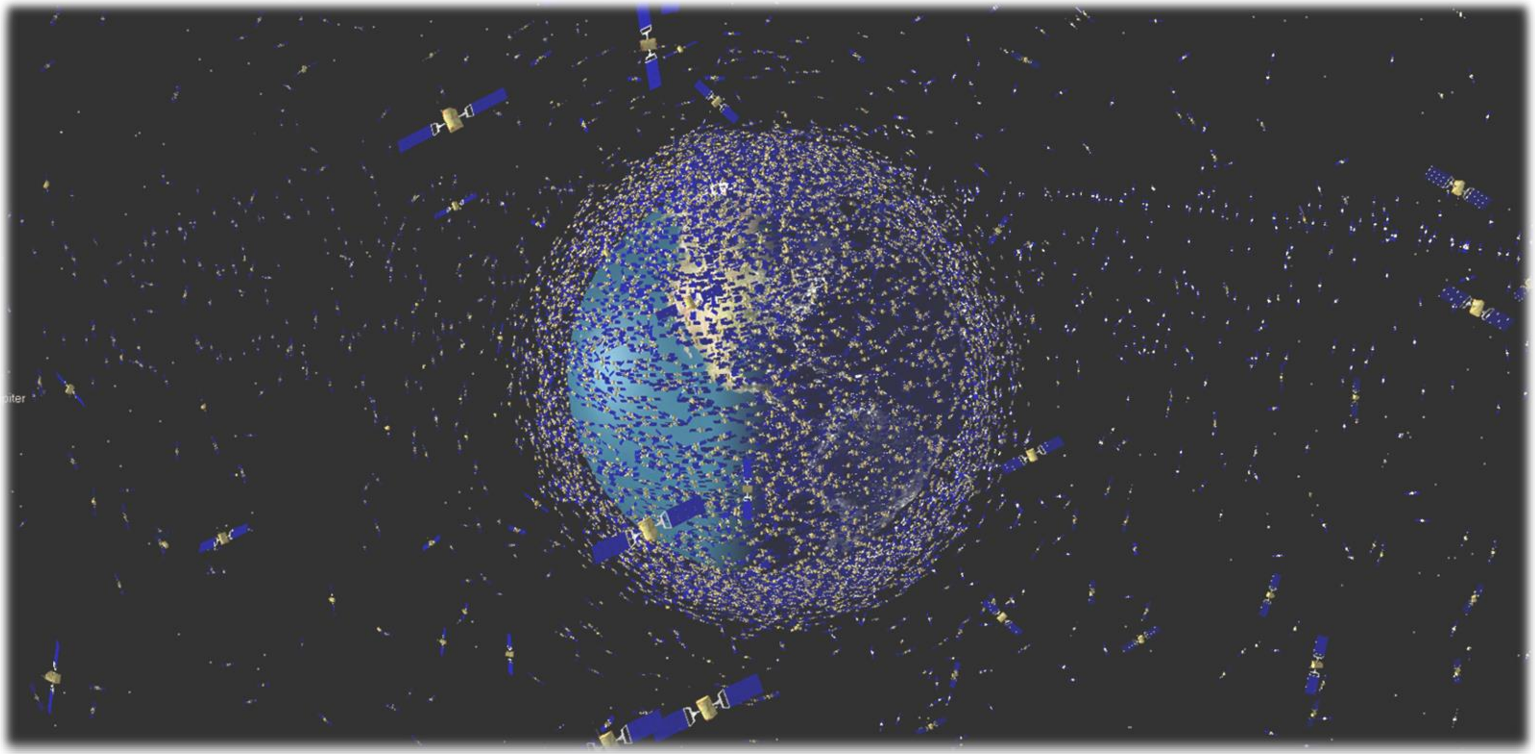
NASA Robotic CARA
Satellite State Estimate Covariance



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Agenda

- Covariance basics
- Use of covariance in probability of collision (Pc) calculation
- Covariance generation and propagation methods
- Covariance tuning
- Covariance realism improvement methods
- Covariance theory compatibility

OD Solutions

- **Purpose of OD**
  - Generate estimate of the object's state at a given time (called the *epoch time*)
  - Generate additional parameters and constructs to allow object's future states to be predicted (accomplished through orbit *propagation*)
  - Generate a statement of the estimation error, both at epoch and for any predicted state (usually accomplished by means of a *covariance matrix*)
- **Error types**
  - OD approaches (either batch or sequential estimation) presume that they solve for all significant systematic errors
  - Remaining solution error is thus presumed to be random (Gaussian) error
  - Sometimes estimates of this error can be intentionally inflated to try to improve the fidelity of the error modeling
  - Nonetheless, presumed to be Gaussian in form and unbiased
    - Issue of non-Gaussian error volumes addressed later in package

OD Parameters Generated by ASW Solutions

- **Solved for: State parameters**
  - Six parameters needed to determine 3-d state fully
  - Cartesian: three position and three velocity parameters in orthogonal system
  - Element: six orbital elements that describe the geometry of, and object location within, the orbit
- **Solved for: Non-conservative force parameters**
  - Ballistic coefficient (C_DA/m); describes susceptibility of spacecraft state to atmospheric drag
  - Solar radiation pressure (SRP) coefficient (C_RA/m); describes susceptibility of spacecraft state to radiation momentum from sun
- **Considered: ballistic coefficient and SRP consider parameter**
  - Not solved for but “considered” as part of the solution
  - Derived from information outside of the OD itself
  - Discussed later

OD Uncertainty Modeling

- **Characterizes the overall uncertainty of the OD epoch and/or propagated state**
  - Uncertainty of each estimated parameter and their interactions
- **This is a characterization of a multivariate statistical distribution**
- **In general, need more than two moments to describe a distribution**
  - Mean, variance, skewness, and kurtosis often used
  - Requires higher-order tensors to do this for a multivariate distribution
- **Assumptions about error distribution can simplify situation substantially**
  - Presuming the solution is unbiased places the mean error values at zero
  - Presuming the error distribution is Gaussian eliminates the need for the third and fourth moments (skewness and kurtosis)
  - Error distribution can thus be expressed by means of variances of each solved-for component and their cross-correlations
  - Thus, error can be fully represented by means of a covariance matrix

Covariance Matrix Construction: Symbolic Example

- **Three estimated parameters (a, b, and c)**
- **Variances of each along diagonal**
- **Off-diagonal terms the product of two standard deviations and the correlation coefficient (ρ); matrix is symmetric**

	a	**b**	**c**	…
a	σ_a^2	$\rho_{ab}\sigma_a\sigma_b$	$\rho_{ac}\sigma_a\sigma_c$	…
b	$\rho_{ab}\sigma_a\sigma_b$	σ_b^2	$\rho_{bc}\sigma_a\sigma_c$	…
c	$\rho_{ac}\sigma_a\sigma_c$	$\rho_{bc}\sigma_a\sigma_c$	σ_c^2	…
…	…	…	…	…

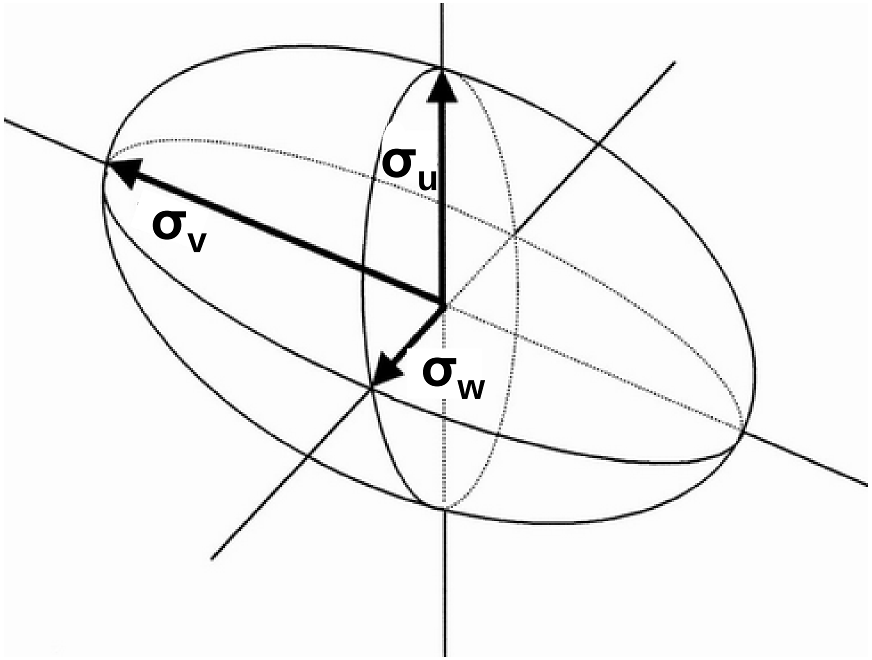
Example Covariance from CDM

- **8 x 8 matrix typical of most ASW updates**
  - Some orbit regimes not suited to solution for both drag and SRP; these covariances 7 x 7
- **Mix of different units often creates poorly conditioned matrices**
  - Condition number of matrix at right is 9.8E+11—terrible!
- **Often better numerically (and more intuitive) to separate matrix into sections**
- **First 3 x 3 portion (amber) is *position* covariance—often considered separately**

	U (m)	V (m)	W (m)	Udot (m/s)	Vdot (m/s)	Wdot (m/s)	B (m2/kg)	AGOM (m2/kg)
U	6.84E+01	-2.73E+02	6.38E+00	2.76E-01	-7.14E-02	8.75E-03	-3.83E-02	-3.83E-02
V	-2.73E+02	1.10E+05	3.23E+01	-1.17E+02	-8.99E-02	2.51E-02	-1.28E-01	-1.28E-01
W	6.38E+00	3.23E+01	4.47E+00	-3.26E-02	-6.83E-03	1.81E-03	-3.73E-03	-3.73E-03
Udot	2.76E-01	-1.17E+02	-3.26E-02	1.24E-01	1.10E-04	-2.47E-05	1.46E-04	1.46E-04
Vdot	-7.14E-02	-8.99E-02	-6.83E-03	1.10E-04	7.57E-05	-9.39E-06	4.10E-05	4.10E-05
Wdot	8.75E-03	2.51E-02	1.81E-03	-2.47E-05	-9.39E-06	2.06E-05	-4.39E-06	-4.39E-06
B	-5.07E-03	1.30E+00	4.34E-05	-1.38E-03	7.97E-07	7.26E-07	1.64E-05	-6.28E-07
AGOM	-3.83E-02	-1.28E-01	-3.73E-03	1.46E-04	4.10E-05	-4.39E-06	-6.28E-07	2.31E-05

Position Covariance Ellipse

- **Position covariance defines an "error ellipsoid"**
  - Placed at predicted satellite position
  - Square root of variance in each direction defines each semi-major axis (UVW system used here)
  - Off-diagonal terms rotate the ellipse from the nominal position shown
- **Ellipse of a certain "sigma" value contains a given percentage of the expected data points**
  - 1-σ: 19.9%
  - 2-σ: 73.9%
  - 3-σ: 97.1%
  - Note how much lower these are than the univariate normal percentage points



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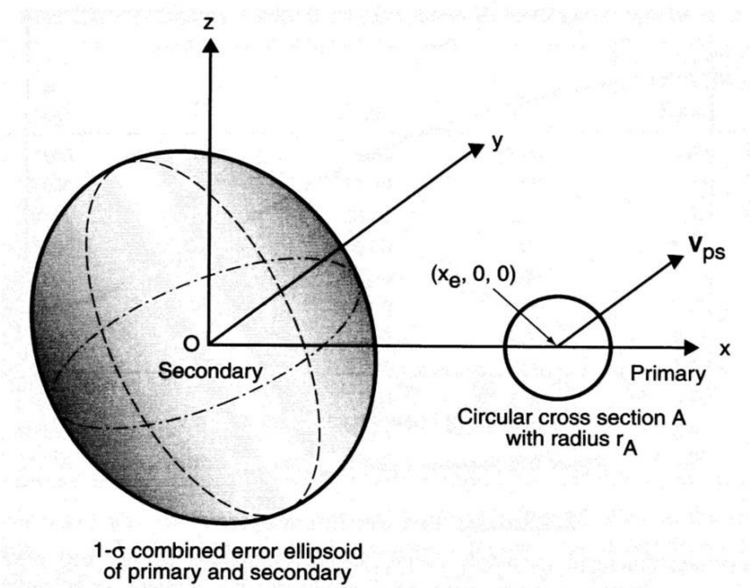
Covariance in Calculation of 2D Probability of Collision (Pc)

- Primary and secondary covariances combined and projected into 2D *conjunction plane* (plane perpendicular to velocity vector)
- Primary placed on x-axis at (miss distance, 0) and represented by circle of radius equal to sum of both spacecraft circumscribing radii
- Z-axis perpendicular to x-axis in conjunction plane
- Pc is portion of combined error ellipsoid that falls within the hard-body radius circle

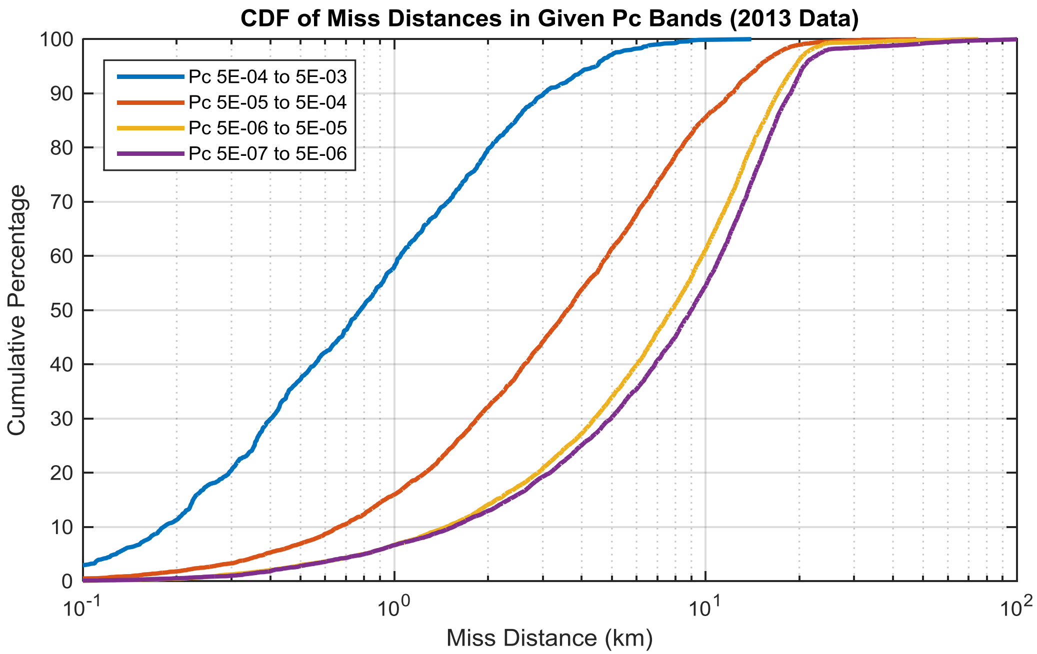
$$P_C = \frac{1}{\sqrt{(2\pi)^2 \left|C^*\right|}} \iint_A \exp\left(-\frac{1}{2} \vec{\rho}^T C^{*-1} \vec{\rho}\right) dXdZ$$

Covariance essential to Pc calculation, which is the most important factor in collision risk assessment



Pc vs Miss Distance Calculations

- **Pc is the best single-parameter encapsulation of the risk**
- **Without Pc, have only the miss distance**
- **Correlation between miss distance and Pc very poor**
  - Four Pc bands shown below; correlation with miss distance poor in all cases
- **Important to have Pc, and covariance necessary for its calculation**



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Batch Epoch Covariance Generation

- **Batch least-squares update (ASW method) uses the following minimization equation**
  - $dx = (A^TWA)^{-1}A^TWb$
    - dx is the vector of corrections to the state estimate
    - A is the time-enabled partial derivative matrix, used to map the residuals into state-space
    - W is the "weighting" matrix that provides relative weights of observation quality (usually 1/σ, where σ is the standard deviation generated by the sensor calibration process)
    - b is the vector of residuals (observations – predictions from existing state estimate)
- **Covariance is the collected term $(A^TWA)^{-1}$**
  - **A** the product of two partial derivative matrices:
    - $A = \frac{\partial(obs)}{\partial X_0} = \frac{\partial(obs)}{\partial X}\frac{\partial X}{\partial X_0}$
    - First term: partial derivatives of observations with respect to state at obs time
    - Second term: partial derivatives of state at obs time with respect to epoch state

Batch Epoch Covariance Generation

- **Formulated this way, this covariance matrix is called an *a priori* covariance**
  - **A** does not contain actual residuals, only transformational partial derivatives
  - So $(A^TWA)^{-1}$ is a function only of the amount of tracking, times of tracks, and sensor calibration relative weights among those tracks
    - Not a function of the actual residuals from the correction
    - Rather, an *a priori* estimate of the errors that should be experienced given the tracking density, times, and relative accuracies
- **Limitations of *a priori* covariance**
  - Does not account well for unmodeled errors, such as transient atmospheric density prediction errors
    - Because not examining actual fit residuals
  - W-matrix only as good as sensor calibration process
    - Principal weakness of present process

Covariance Propagation Method 1: Full Monte Carlo

- **Creates *n* state (position and velocity) perturbations at epoch**
  - Covariance at epoch describes uncertainty of state at epoch
  - Can use this to create set of *n* possible realizations of the epoch state, conforming to the distribution parameters specified by the covariance
- **Propagates each of these forward to the time of interest**
  - Use the full non-linear dynamics of the propagator
  - Thus produce *n* states at TCA (for CA application)
- **Summarizes set of *n* states statistically**
  - Usually empirically, through non-parametric techniques (*e.g.*, percentiles, empirical distribution functions)

Covariance Propagation Method 2: Linear Mapping

- **Non-linear dynamics of orbit propagation can be linearized**
  - These linear approximations valid for "short" periods about epoch state
- **State transition matrix (Φ) the encapsulation of this linearization**
  - Can be used for state propagations (but often is not)
  - Can also be used for propagation of covariance $[\Phi(t,t_o)*C(t_o)*\Phi^T(t,t_o)]$
- **Covariance propagation can also be augmented via the addition of "process noise"**
  - Process noise matrix (Q) formulated, which specifies acceleration uncertainty in each coordinate principal direction
    - Intent is to compensate for unmodeled and inadequately-modeled perturbations
    - Can potentially remediate some of the limitations introduced by the linearization
  - Process noise matrix propagated through use of a process noise transition matrix, in a manner similar to state transition: $[\Gamma(t,t_o)*Q(t_o)* \Gamma^T(t,t_o)]$

Covariance Propagation Method Comparison

Method	Advantages	Disadvantages
Monte Carlo	• Most accurate method of characterizing uncertainty, as there are no inherent simplifying assumptions or activities (such as linearization)	• Very large number of samples required to characterize tails of distribution • Far more computationally intensive than other methods
Linear Mapping	• Much faster and less computationally intensive than other methods • Process noise provides mechanism for covariance tuning/realism adjustments	• Least accurate, especially for long propagations • Imposes *a priori* statistical structure upon uncertainty volume • Use of process noise requires careful tuning process

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Covariance Tuning

- **For CA, position covariance needs to be a realistic representation of the state uncertainty volume at the propagation point of interest**
- **Two aspects to this requirement**
  - Does the position error volume conform to a trivariate Gaussian distribution?
  - If so, is it of the proper dimensions and orientation?

Non-Gaussian Error Volumes

- **As propagation times increase, linearized dynamics put under more and more strain**
  - Divergence between rectilinear linearization and curvilinear orbital framework
- **Framing error volume in element space, regardless of how covariance is propagated, very strongly preserves Gaussian nature**
  - Demonstrated in Sabol (2010) paper
- **Non-Gaussian issue of significance relatively infrequently, especially for high-Pc events**
  - Shown in Ghrist and Plakalovic (2012)
- **However, does occur occasionally and can significantly alter Pc**
  - Hall *et al.* (2018)
  - Can remediate situation by performing Monte Carlo in element space
    - Monte Carlo from TCA works quite well
  - Emerging research (hope is this fall) will give criteria for when non-Gaussian issues arise with 2-D Pc

Covariance Tuning: Covariance Realism Evaluation Method

- **Presume a precision observation available for a satellite**
  - reference orbit, or observation such as definitive solution
- **Position differences between predicted ephemeris and precision position are dU, dV, and dW**
  - Can be collected into vector ε
- **Mahalanobis distance ($\varepsilon * C^{-1} * \varepsilon^T$) represents the ratio of the difference to the covariance's prediction**
  - For a trivariate distribution, expected value is 3
- **A group of such calculations should conform to a chi-squared distribution with three degrees of freedom**
  - The method of distribution testing of groups of such calculations is used to determine if covariance properly sized

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Covariance Tuning: Covariance Irrealism Remediation

- **Three general methods for remediation**
  - Scale factor computation and application
  - Process noise definition and application
  - Consider parameter application

Scale Factor Calculation

- **Used presently by JSC as part of ISS CA**
- **Mahalanobis distance calculation histories used to calculate scale factors to make covariances more realistic**
  - Can be satellite-by-satellite or for entire orbit regimes
  - Usually for a particular propagation state (e.g., 72-hour propagation)
- **Iterative (or visual) techniques used to calculate covariance scale factors that will make Mahalanobis test results match expectations**
  - Can be single scale factor by which entire matrix is pre- and post-multiplied (k*C*k)
  - Can be a scale factor for each of the coordinate axes (K*C*K, where K is a diagonal matrix of three axis-specific scale factors)
- **Factors produced by off-line analysis and then applied to each covariance in conjunctions**
  - Can be function of orbital parameters and propagation time, or simply calculated for all objects for typical propagation state of final CA decision

Process Noise

- **Most commonly used with Kalman filter techniques, although can be applied to any covariance propagation**
- **Iterative approach to defining acceleration variances**
  - Originally produced as technique for compensating for inadequate geopotential solution when processing restrictions limited geopotential fidelity
  - Now employed more as a general method to compensate for covariance inadequacies
- **Variances used to define Q matrix, which is propagated through process noise state transition matrix Γ**
  - Added to propagated covariance
- **If done correctly, construct should work for all propagation states**
  - Advantage over scale factor approach, for which separate scale factor needed for each (significantly) different propagation state

ASW Covariance Propagation and Consider Parameters

- **Covariance in VCM is unaltered covariance**
- **When propagating VCM covariance, the propagator**
  - Scales the covariance by the weighted RMS if it is greater than unity
    - C* = C * WRMS ^ 2
    - Was an early attempt to improve covariance realism; not clear this is still a good idea
  - Applies the consider parameter to the ballistic coefficient variance
    - C*(7,7) = C*(7,7) + Cpd ^ 2
    - More later on how this value is determined
  - May apply a consider parameter to the solar radiation pressure variance
    - C*(8,8) = C*(8,8) + Cps ^ 2 (would be position (9,9) on VCM)
    - Presently not used (Cps set to 0)
  - Propagates the altered covariance using linearized dynamics
    - $\Phi * C^{**} * \Phi^{T}$
  - Converts propagated matrix from equinoctial to Cartesian coordinates

Altered Covariance Positions

- **Ballistic coefficient consider parameter (DCP) applied to ballistic coefficient variance (orange)**
- **If used, solar radiation pressure consider parameter applied to solar radiation pressure variance (purple)**

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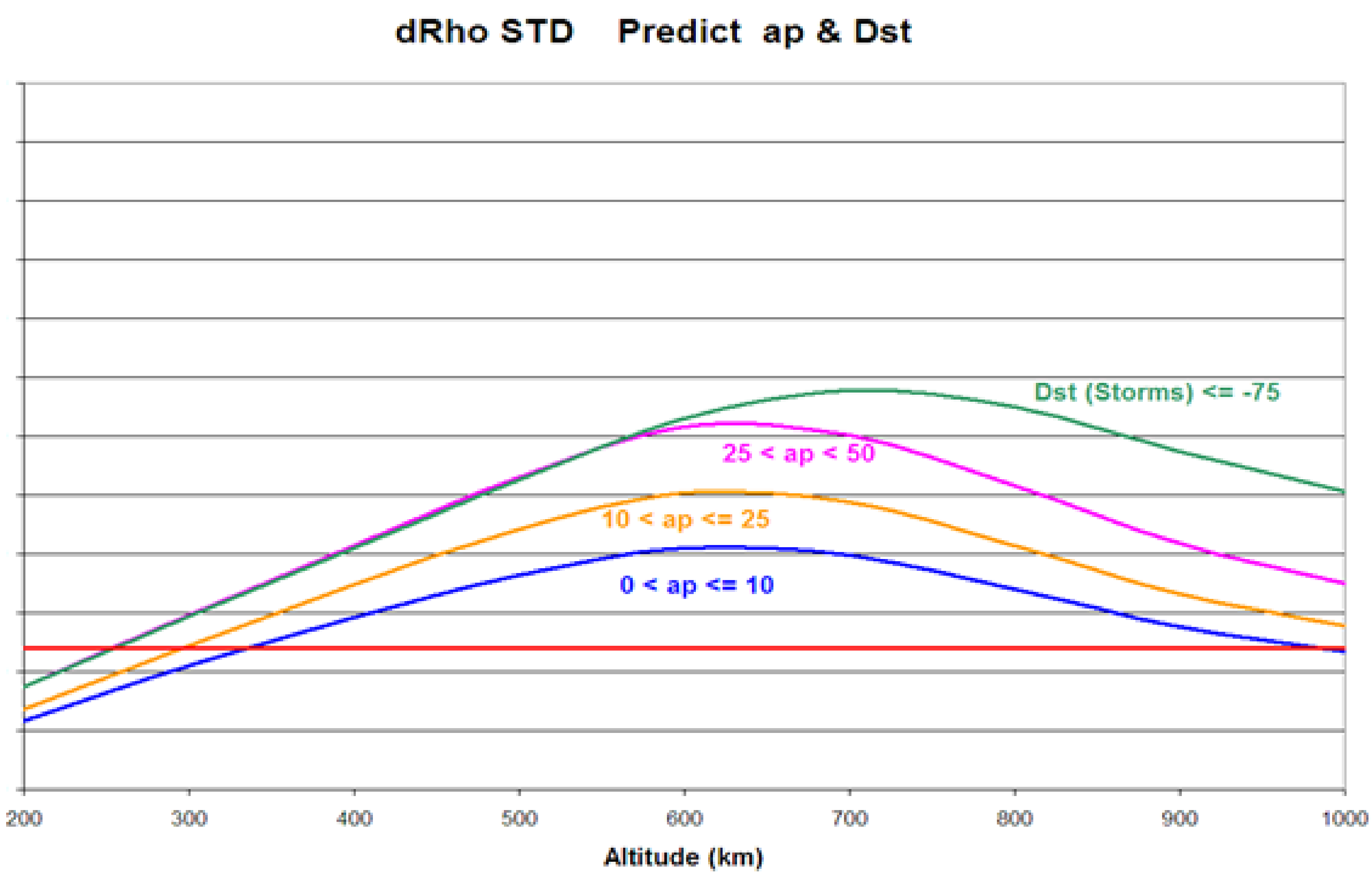
W	6.38E+00	3.23E+01	4.47E+00	-3.26E-02	-6.83E-03	1.81E-03	-3.73E-03	-3.73E-03
Udot	2.76E-01	-1.17E+02	-3.26E-02	1.24E-01	1.10E-04	-2.47E-05	1.46E-04	1.46E-04
Vdot	-7.14E-02	-8.99E-02	-6.83E-03	1.10E-04	7.57E-05	-9.39E-06	4.10E-05	4.10E-05
Wdot	8.75E-03	2.51E-02	1.81E-03	-2.47E-05	-9.39E-06	2.06E-05	-4.39E-06	-4.39E-06
B	-5.07E-03	1.30E+00	4.34E-05	-1.38E-03	7.97E-07	7.26E-07	1.64E-05	-6.28E-07
AGOM	-3.83E-02	-1.28E-01	-3.73E-03	1.46E-04	4.10E-05	-4.39E-06	-6.28E-07	2.31E-05

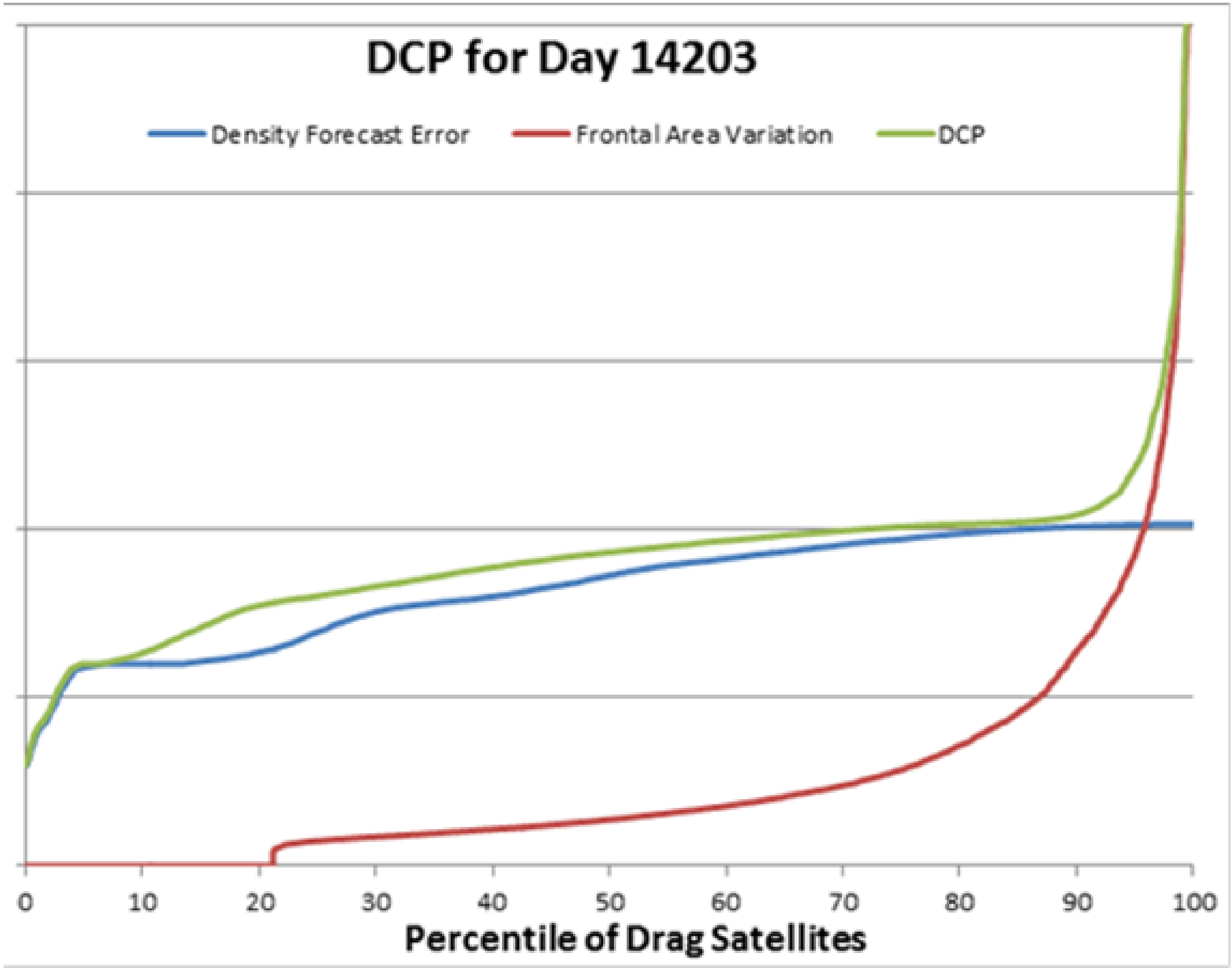
Dynamic Consider Parameter (DCP)

- **Specifies global error in the atmospheric density forecast**
  - Parameterizes percent RMS error in terms of
    - Satellite height (perigee altitude)
    - Geomagnetic activity (a_p and Dst)
  - Density forecast error combination of solar/geomagnetic indices and DCA
    - Directly compared numerous forecast densities to actual density
    - Discretized heights from 200 km to 1000 km averaged over lat / lon
  - Sampled high, low, and medium solar cycles (2001, 2005, 2013)
    - Found most variation parameterizable via a_p conditions (versus F_{10})
    - Functions optimized for 3-day predictions—this is the tuning point!
- **Determines satellite-specific frontal area variation in prediction**
  - Quantifies ballistic coefficient RMS error through satellite histories
    - Looks back in time up to a year in most cases (upfront preprocessing)
    - Ascertains error / target for 3-day predictions (accounting for time-lags)
- **Combine the two uncertainty components to obtain DCP value**
  - Additive in variance sense as the root sum of squares

B Consider Parameter Values



DCP Components RMS Uncertainty



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Covariance Theory Compatibility

- **Batch covariance is governed by the amount and quality of tracking data used in the OD**
- **Propagated covariance is a product of the particular propagation technique used and tuning applied**
  – Tuning itself a function of the adequacy of the OD force modeling
- **Thus, important that covariance be generated from same OD basis that produced the state estimate**
- **This is not possible for O/O ephemerides that lack a covariance**
  – Forced to use O/O state estimate and ASW covariance (or a synthesized covariance when no ASW covariance exists)
  – Such a covariance a questionable representation of O/O ephemeris error
    - $\sigma_{O/O}^2 = \sigma_{ASW}^2 + \sigma_{Diff}^2$
    - The difference variance is unknown, so using an ASW covariance with an O/O ephemeris understates the uncertainty but by an unknown amount