Robust Stability: From Disk Margin to Neural Network Analysis

Peter Seiler

NASA GNC Webcast

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UNIVERSITY OF MICHIGAN

Outline

- An Introduction to Disk Margins
  - Motivation for Disk Margins
  - Disk Margins for SISO Systems
  - Multi-Loop Margins

- Stability Margins for Non-Classical Controllers
  - Linear time-varying systems (along trajectories)
  - Neural Network Controllers

References for Additional Reading

- An Introduction to Disk Margins
  - Seiler, Packard, Gahinet, An Introduction to Disk Margins, arXiv, 2020.
  - Douglas, Robust Control, Part 2: Understanding Disk Margin, https://www.youtube.com/watch?v=XazdN6eZF80
  - Matlab: `diskmargin, diskmarginplot, umargin`

- Stability Margins for Non-Classical Controllers
  - Seiler, Moore, Meissen, Arcak and Packard, Finite Horizon Robustness Analysis of LTV Systems, Automatica, 2019.
  - Yin, Seiler, Arcak, Stability Analysis using Quadratic Constraints for Systems with Neural Network Controllers, arXiv, 2020.

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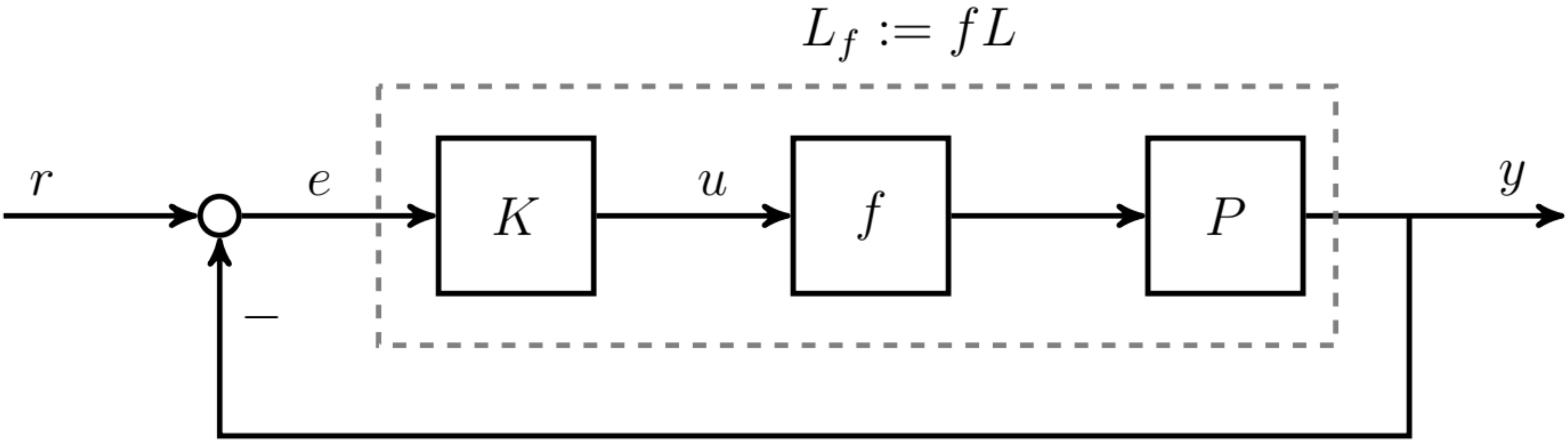
Classical Margins

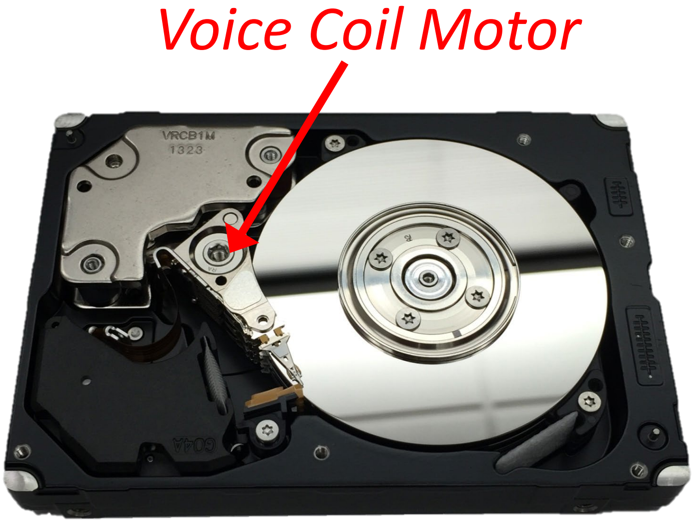
- Assume the feedback loop is nominally stable (f=1)
- Stability Margins:
  - Gain: Insert real gain variation f=g. How much gain variation away g=1 before instability occurs?
  - Phase: Insert phase variation f=$e^{j\phi}$. How much phase variation away ϕ=0 before instability occurs?



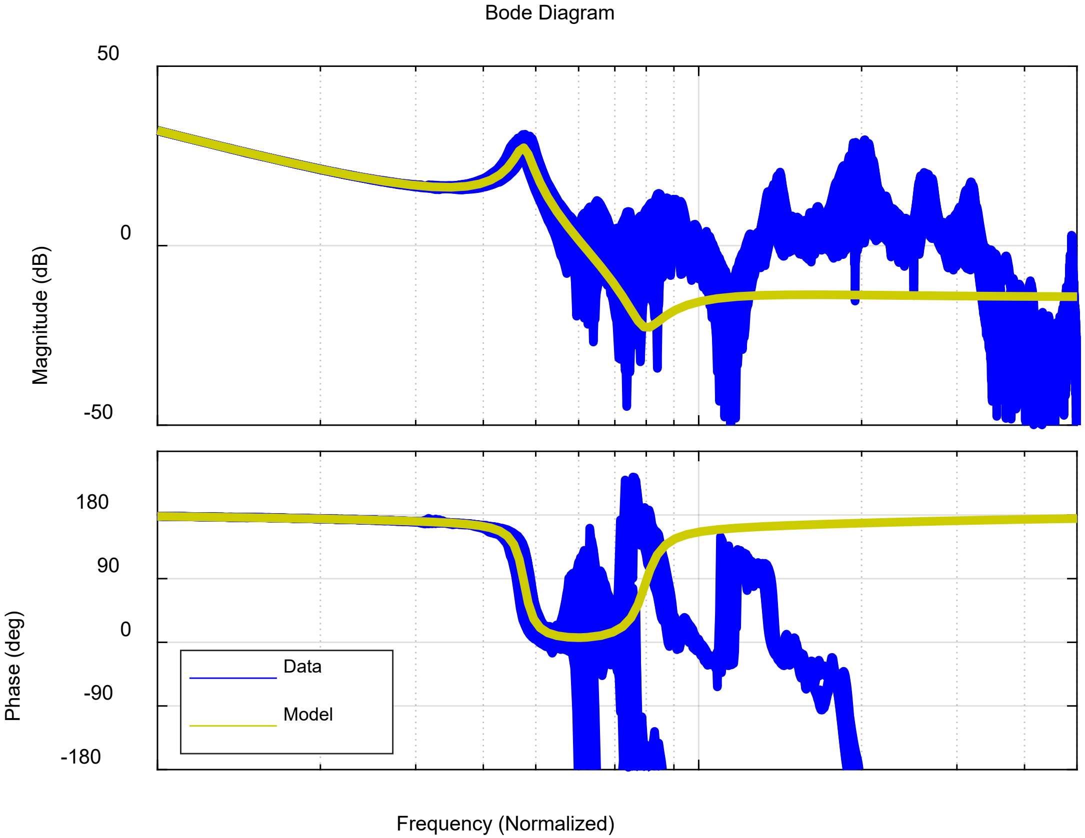
Hard Disk Drives

Example of an electromechanical system with flex. modes.

Use a simplified model (yellow) for control design but there is a lot of part-to-part unmodeled dynamics.

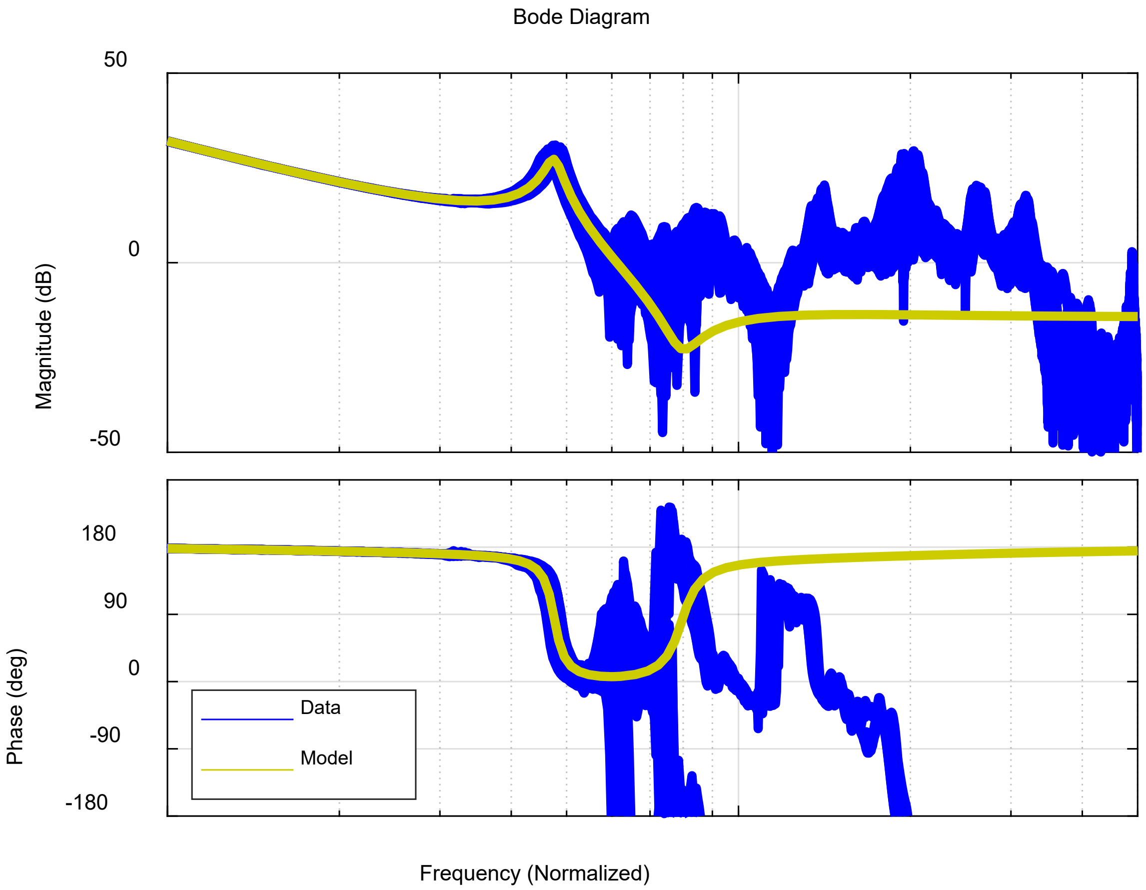


Experimental frequency responses (blue) and simplified model (black).



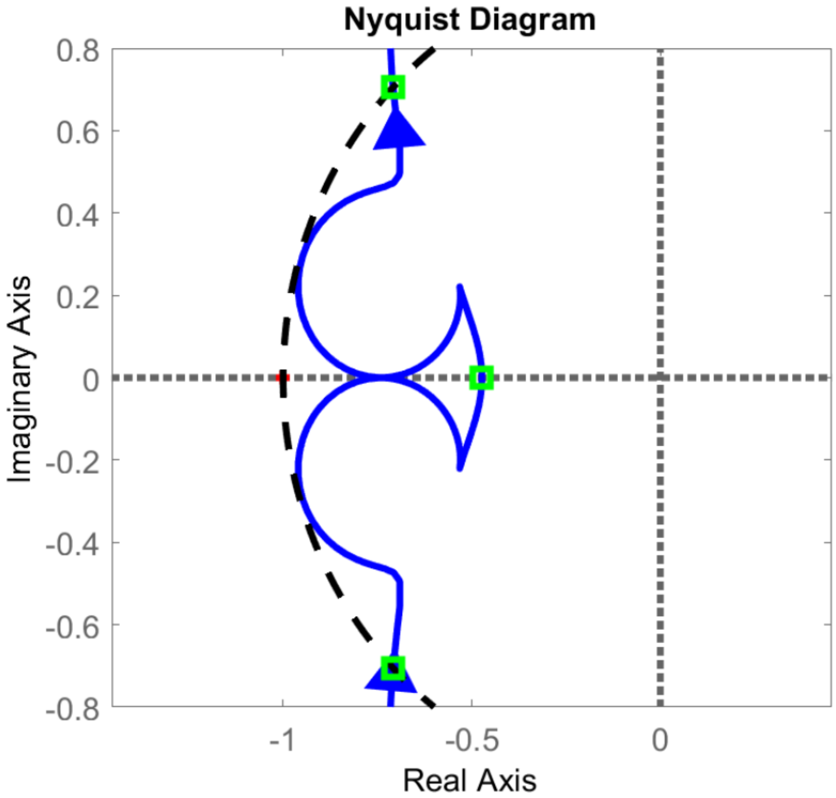
Limitations of Classical Margins

1. Real systems differ from their mathematical models in both magnitude and phase.



Limitations of Classical Margins

1. Real systems differ from their mathematical models in both magnitude and phase.
2. Small plant perturbations may cause robustness issues even if the system has large gain/phase margins.



Limitations of Classical Margins

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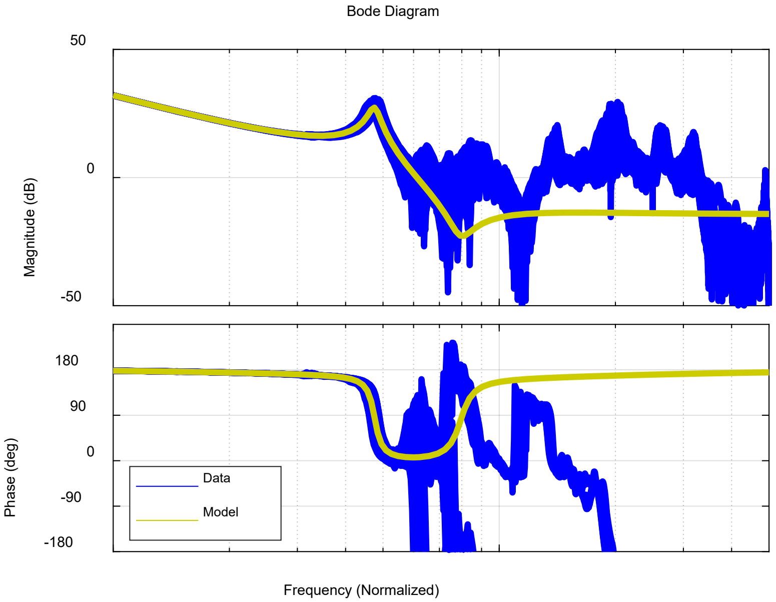
3. Margin requirements must account for the increase in model uncertainty at higher frequencies.



Limitations of Classical Margins

1. Real systems differ from their mathematical models in both magnitude and phase.
2. Small plant perturbations may cause robustness issues even if the system has large gain/phase margins.
3. Margin requirements must account for the increase in model uncertainty at higher frequencies.
4. There are alternative robustness margins that provide more useful extensions to MIMO systems.

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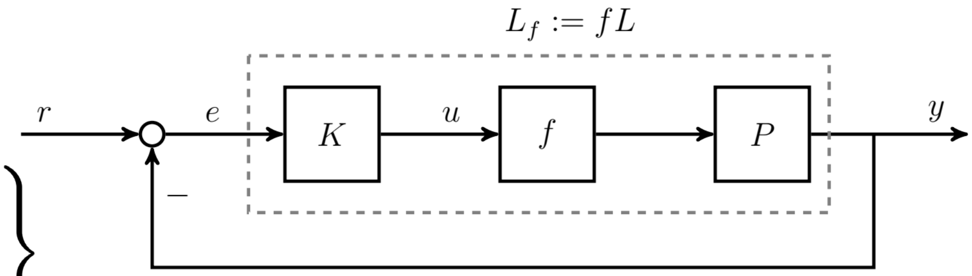
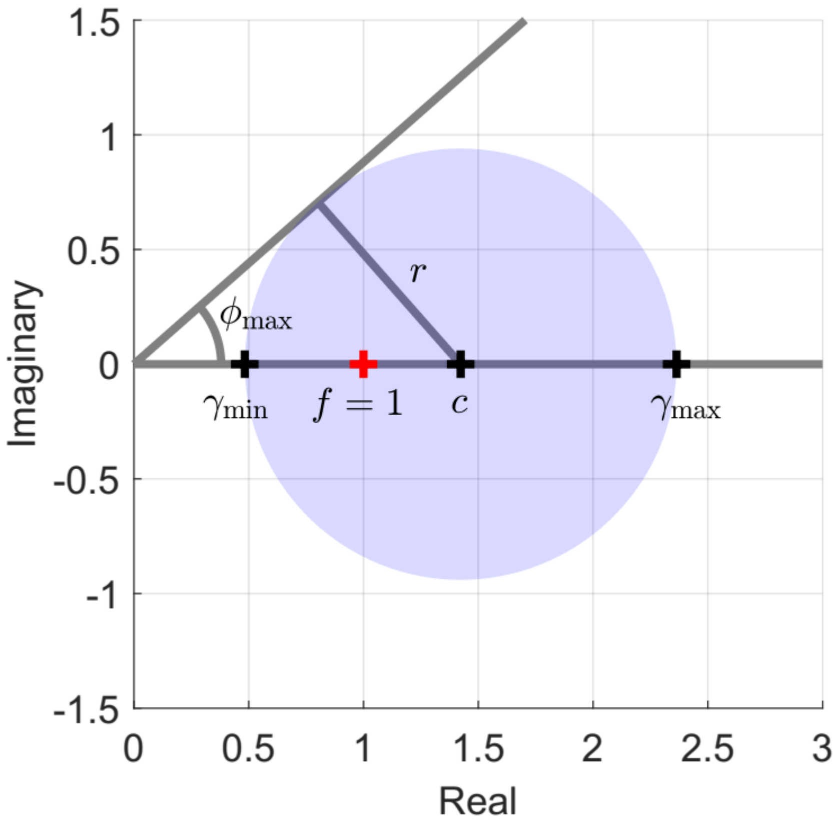
Modeling Gain and Phase Variations



Symmetric Disks

$$D(\alpha) = \left\{ \frac{1+\frac{\delta}{2}}{1-\frac{\delta}{2}} \; : \; \delta \in \mathbb{C} \;\&\; |\delta| < \alpha \right\}$$

- Disk of gain and phase variations around *f=1.*



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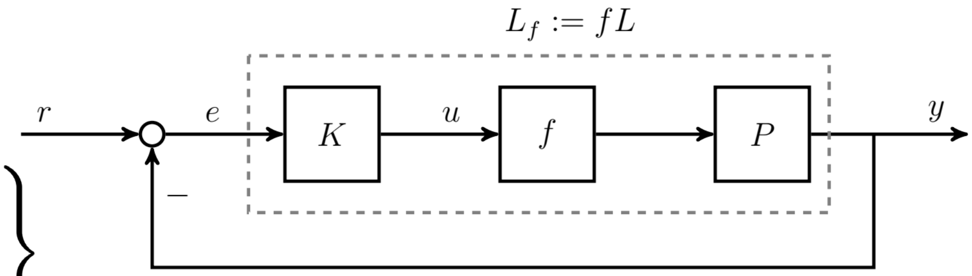
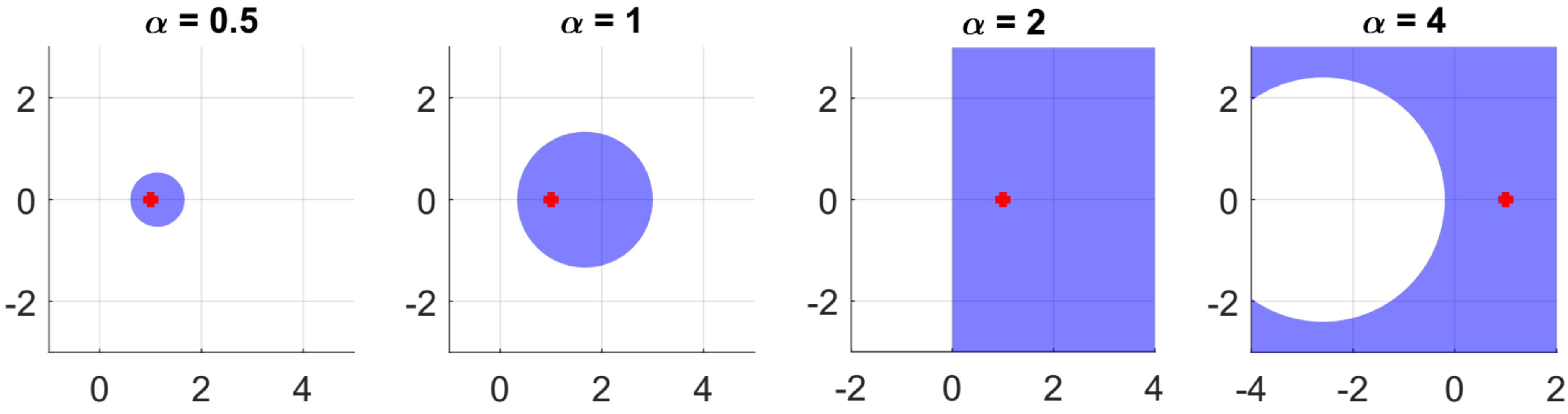


- Disk of gain and phase variations around *f=1.*
- Disk grows with increasing α



Modeling Gain and Phase Variations

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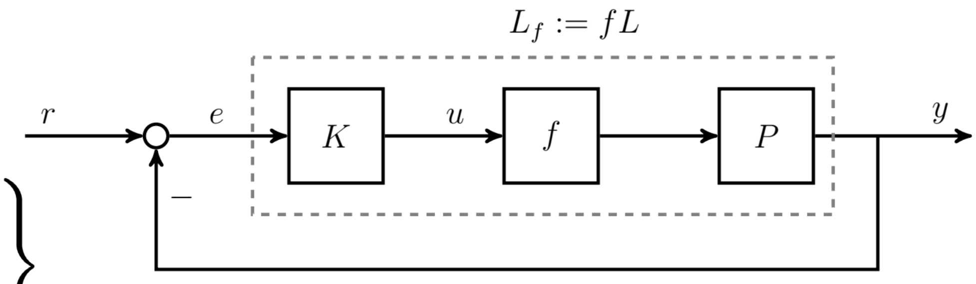
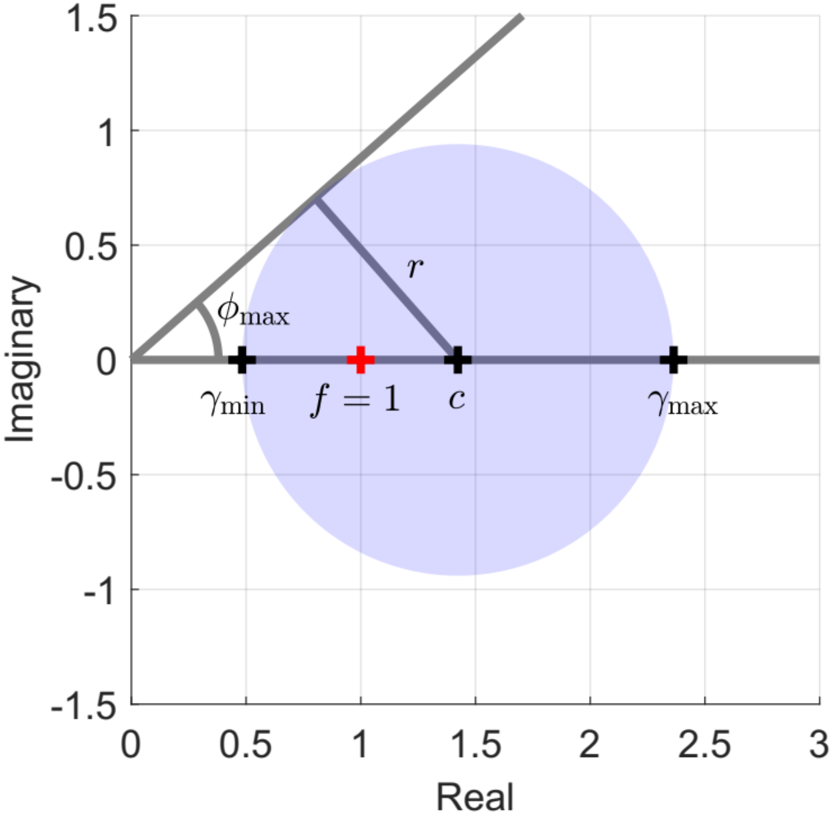


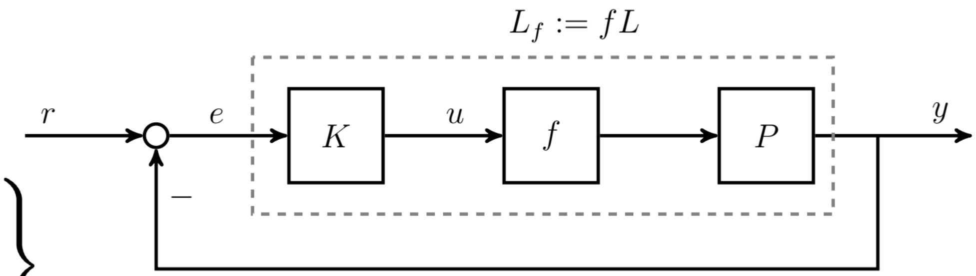
- Disk of gain and phase variations around *f=1.*
- Disk grows with increasing α
- Disk is symmetric in dB:

  $\gamma_{min}=1/\gamma_{max}$
- Ex: $\gamma_{max}=2$ *and* $\gamma_{min}=0.5$ *corresponds to +/- 6dB*



Modeling Gain and Phase Variations

Symmetric Disks



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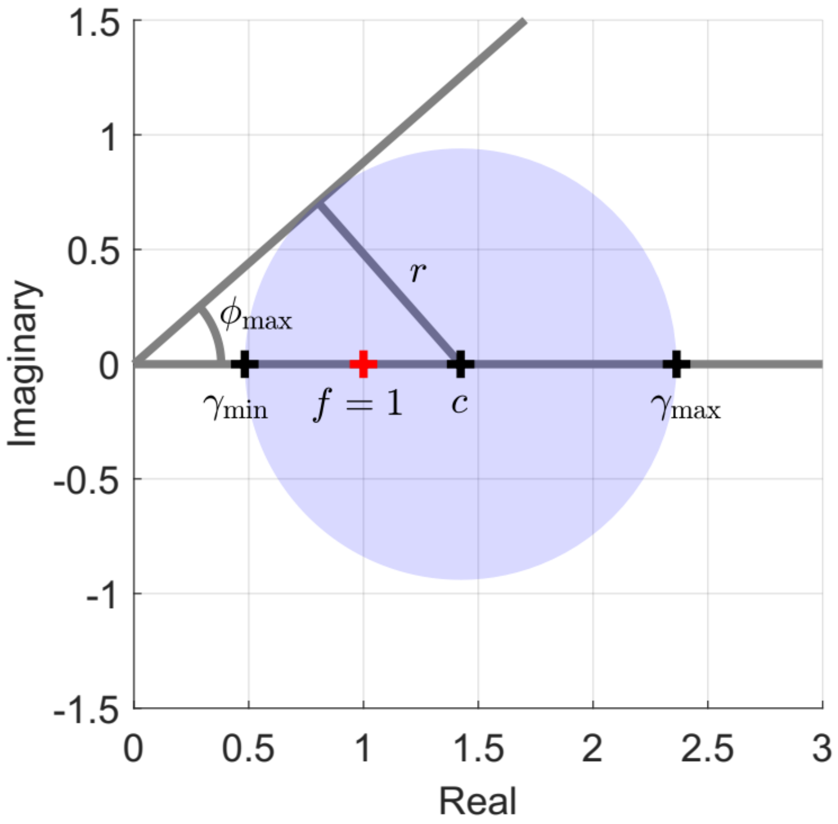
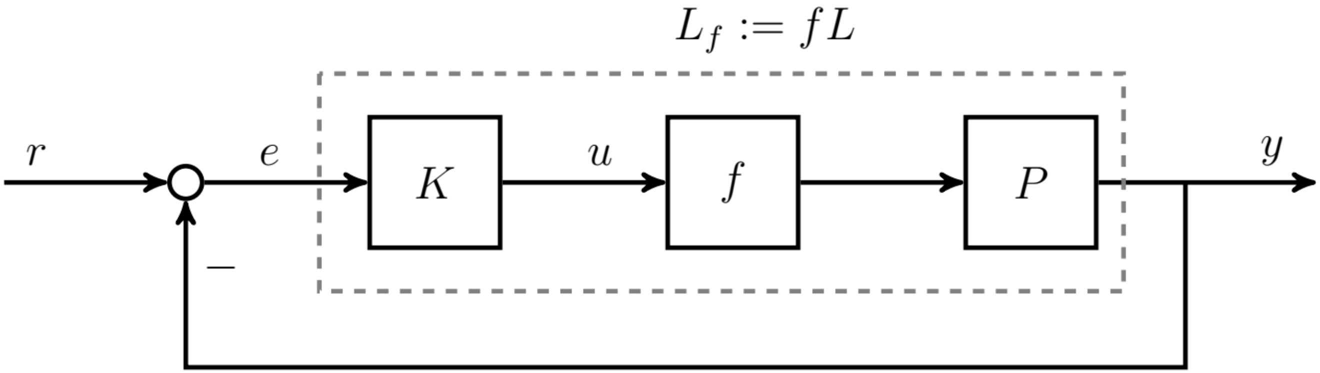
- Disk of gain and phase variations around *f=1.*
- Disk grows with increasing α
- Disk is symmetric in dB:
- We can relate the symmetric disk to classical gain and phase variations.



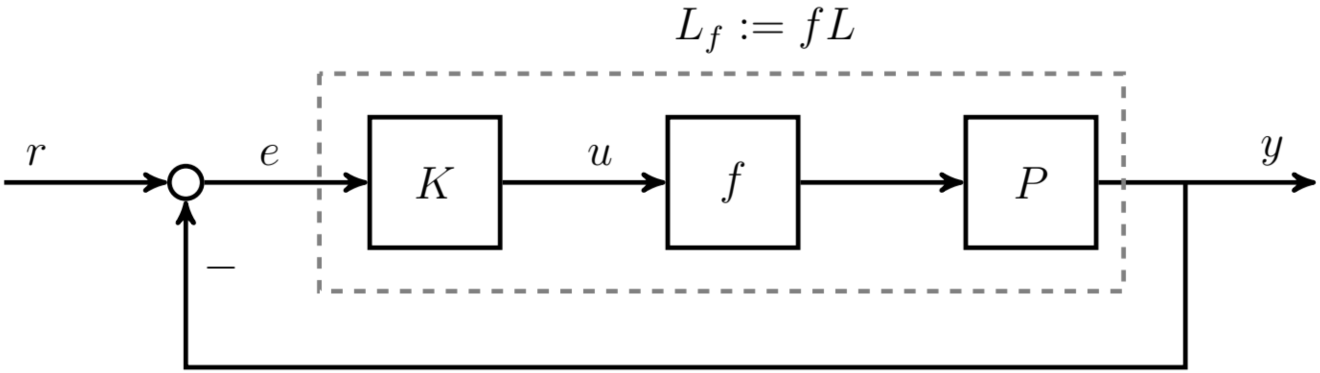
Symmetric Disk Margins

- **Definition:** The <u>symmetric disk margin</u> α_{max} is the largest value of α such that closed-loop with *is* stable for all complex perturbations $f \in D(\alpha)$.



Symmetric Disk Margins

- **Definition:** The symmetric disk margin α_{max} is the largest value of α such that closed-loop with *is* stable for all complex perturbations $f \in D(\alpha)$.
- **Notation:**
  - Nominal Sensitivity Transfer Function: $S = \frac{1}{1+PK}$
  - H_∞ Norm: $\|S\|_\infty = \max_\omega |S(j\omega)|$



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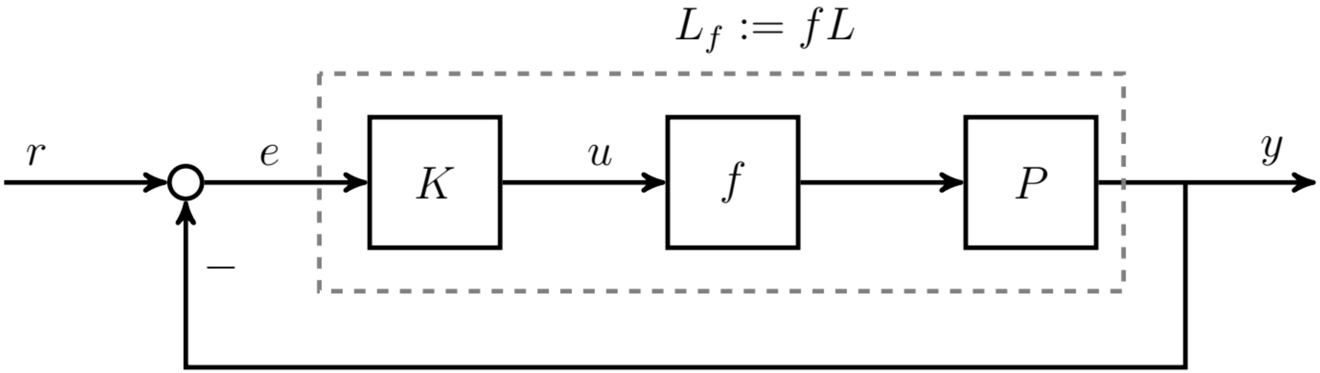
- **Result:** Assume the feedback loop is nominally stable with $f=1$. The symmetric disk margin is:

$$\alpha_{\{max\}} = \frac{1}{||S(j\omega)-0.5||_\infty}$$



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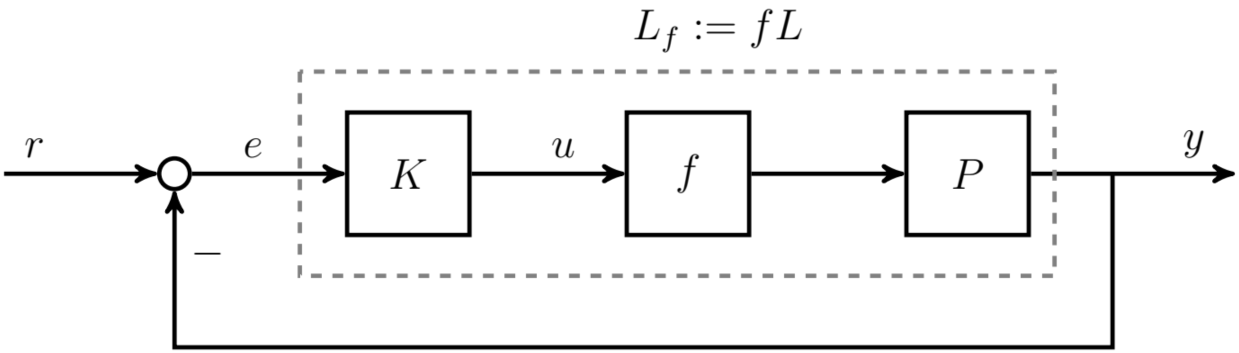
- **Proof:** Perturbation $f \in D(\alpha)$ causes pole at $j\omega_0$ if

$$1+fL(j\omega_0) = 0$$



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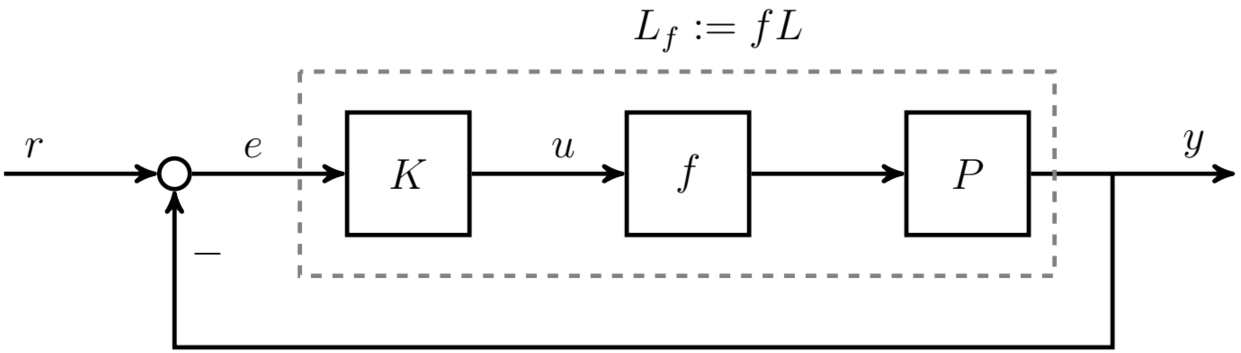
Substitute $f = \frac{1+0.5\,\delta}{1-0.5\,\delta}$ and solve for $\delta = \frac{1}{S(j\omega_0)-0.5}$



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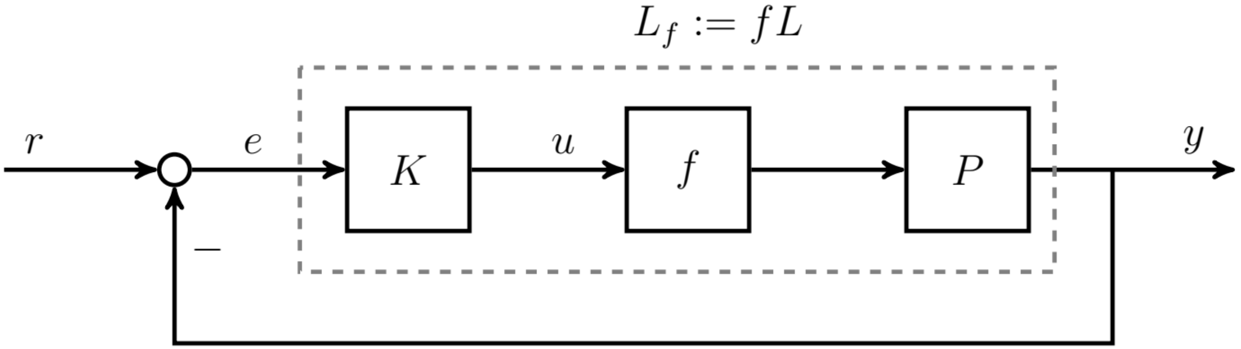
Substitute $f = \frac{1+0.5\,\delta}{1-0.5\,\delta}$ and solve for $\delta = \frac{1}{S(j\omega_0)-0.5}$

Smallest δ causing instability ↔ Largest $|S(j\omega_0) - 0.5|$



Symmetric Disk Margins

- **Result:** Assume the feedback loop is nominally stable with $f=1$. The symmetric disk margin is:

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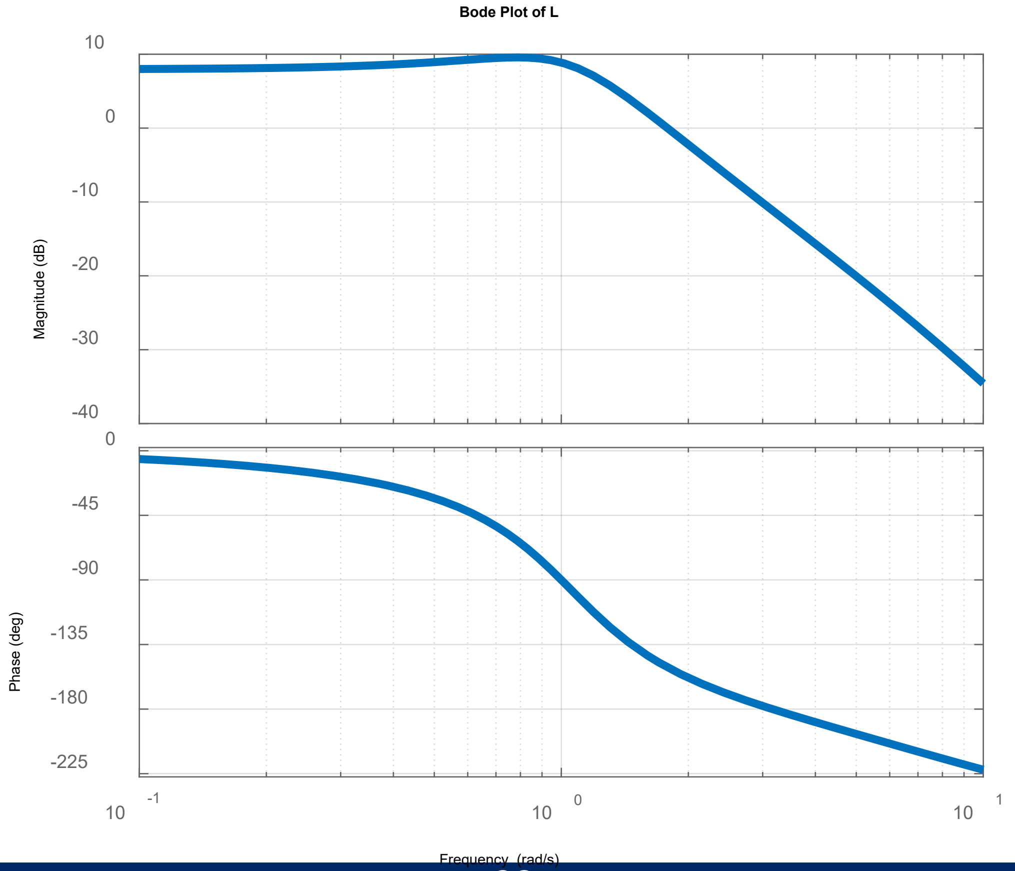
- **Comments:**
  - There is a complex destabilizing perturbation f_0 on the edge of the disk $D(\alpha_{max})$.
  - A state-space system $F_0(s)$ can be constructed that destabilizes the system and remains in the disk $D(\alpha_{max})$.
  - The destabilizing perturbation $F_0(s)$ can be studied further in high fidelity simulations.

Example

- Plant, Controller, Loop

$$P(s) = \frac{1}{s^3 + 10s^2 + 10s + 10}, \quad K(s) = 25, \quad L(s) = \frac{25}{s^3 + 10s^2 + 10s + 10}$$



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- Classical Margins (`allmargin`)
  - Gain Margin = 3.6 at ω = 3.16 rad/sec
  - Phase Margin = 29.1^o at ω = 1.78 rad/sec

- Symmetric Disk Margin (`diskmargin`)
  - α_{max} = 0.4581 at ω = 1.96 rad/sec
  - Implies gain margins of at least [0.63,1.59] and phase margin of at least 25.8^o.

Sample Matlab Code for Reference

Classical

```
>> L = tf(25,[1 10 10 10]);

>> allmargin(L)
ans =
  struct with fields:
     GainMargin: 3.6000
    GMFrequency: 3.1623
    PhaseMargin: 29.1104
    PMFrequency: 1.7844
    DelayMargin: 0.2847
    DMFrequency: 1.7844
         Stable: 1
```

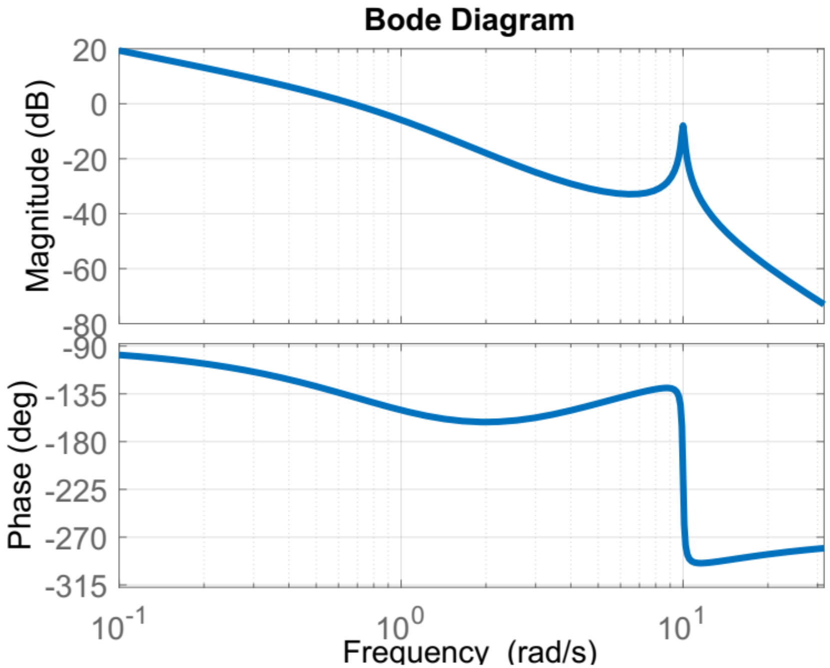
Disk Margins

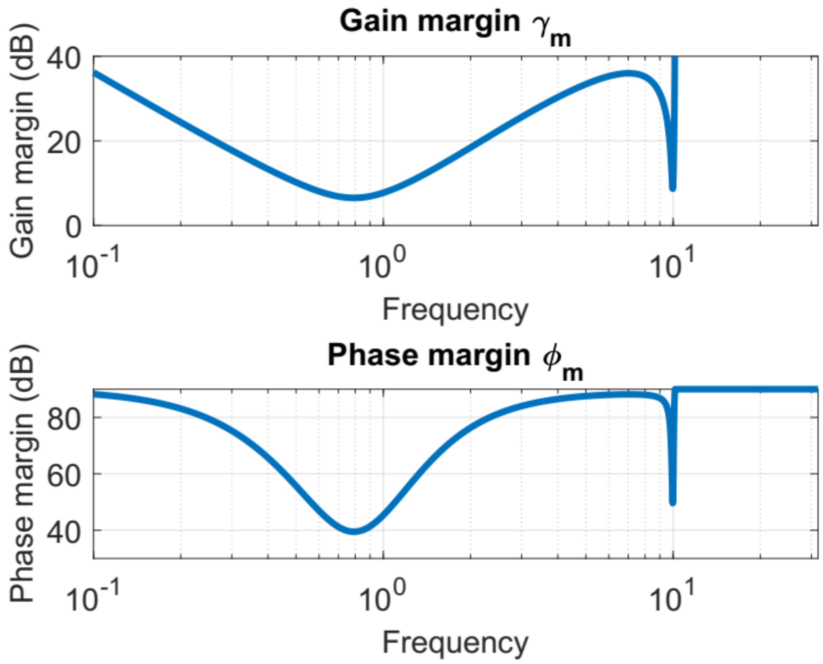
```
>> L = tf(25,[1 10 10 10]);
>> S = feedback(1,L);
>> alpha_max = 1/norm(S-0.5,inf)

alpha_max =
    0.4581


>> diskmargin(L)
ans =
  struct with fields:
           GainMargin: [0.6273 1.5942]
          PhaseMargin: [-25.8017 25.8017]
           DiskMargin: 0.4581
           LowerBound: 0.4581
           UpperBound: 0.4581
            Frequency: 1.9550
  WorstPerturbation: [1×1 ss]
```

Destabilizing state-space perturbation

Frequency-Dependent Disk Margins

$$L(s) = \frac{6.25(s+3)(s+5)}{s(s+1)^2(s^2+0.18s+100)}$$





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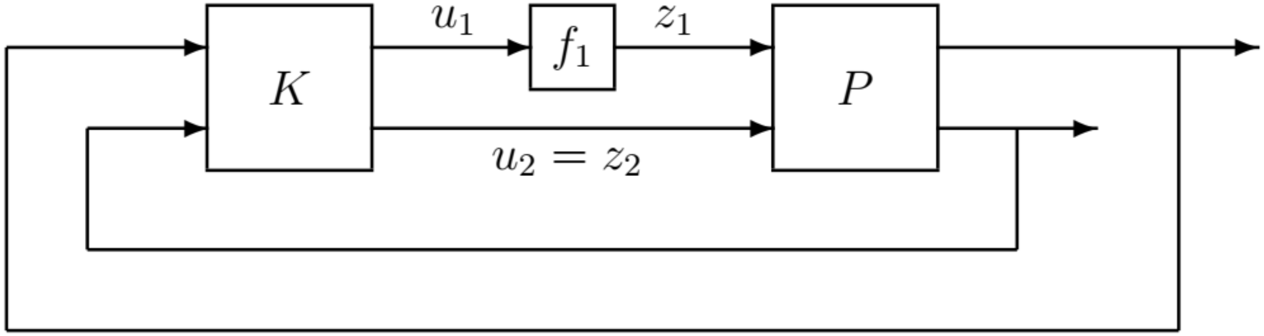
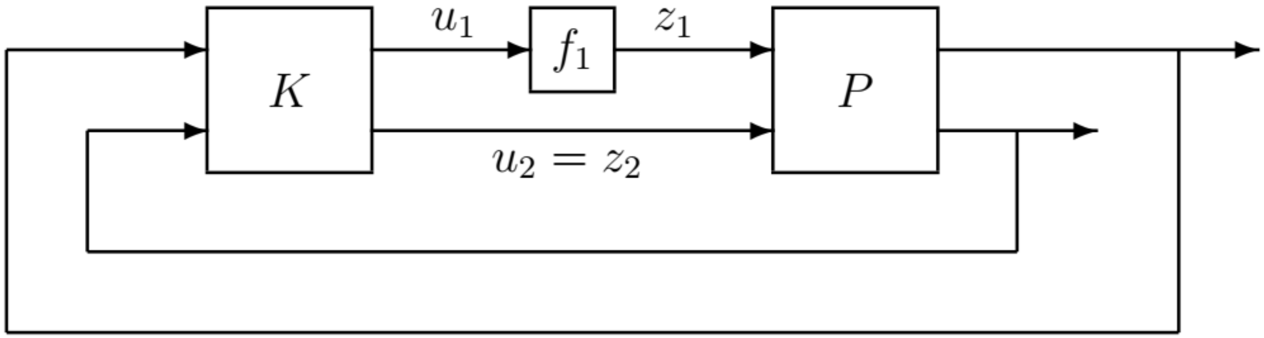
Loop-at-a-Time Margins

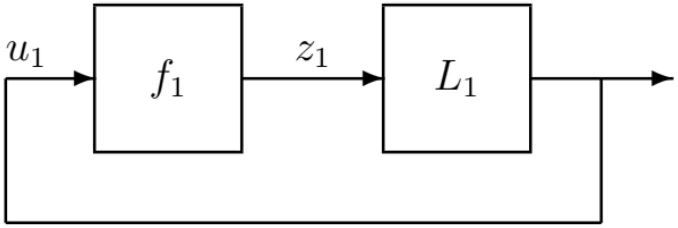
- Introduce perturbation in one loop with other loops closed.



Loop-at-a-Time Margins

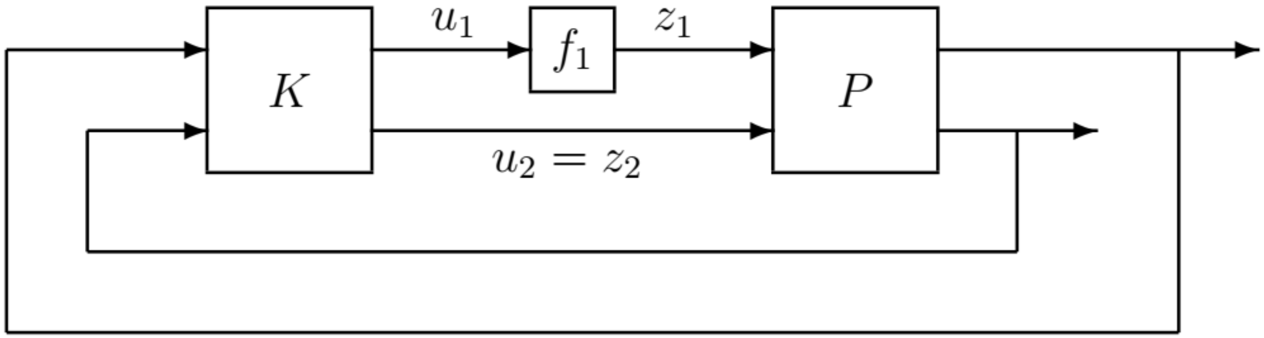
- Introduce perturbation in one loop with other loops closed.
- Convert to equivalent SISO robustness margin loop.

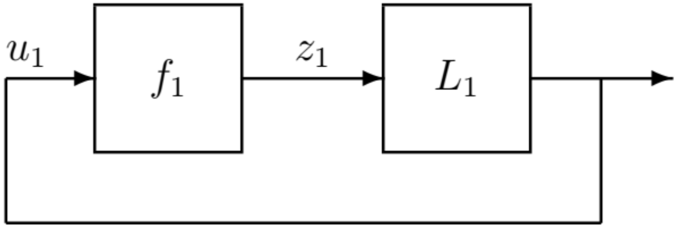




Loop-at-a-Time Margins

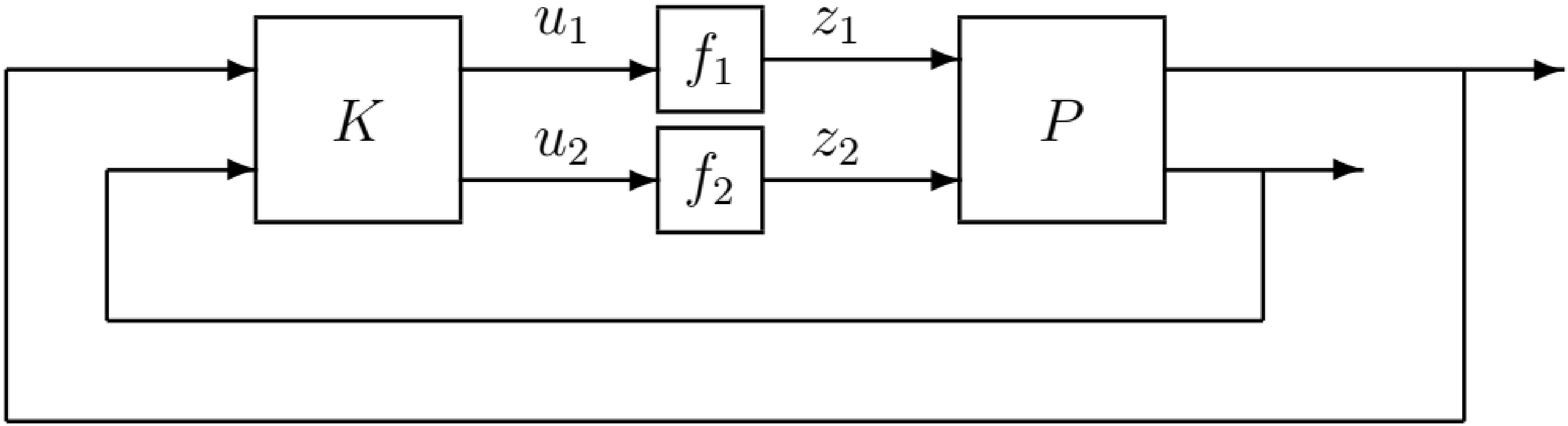
- Introduce perturbation in one loop with other loops closed.
- Convert to equivalent SISO robustness margin loop.
- This analysis does not account for cross-coupling in the loops can be overly optimistic.





Multi-Loop Disk Margins

- Introduce symmetric disk margin perturbations in all loops (at input, at output, or both).
- The perturbations are allowed to vary independently.
- Compute largest α for which stability is maintained.
- The margin is computed using more general theory (structured singular value, mu μ)



Example: Spinning Satellite

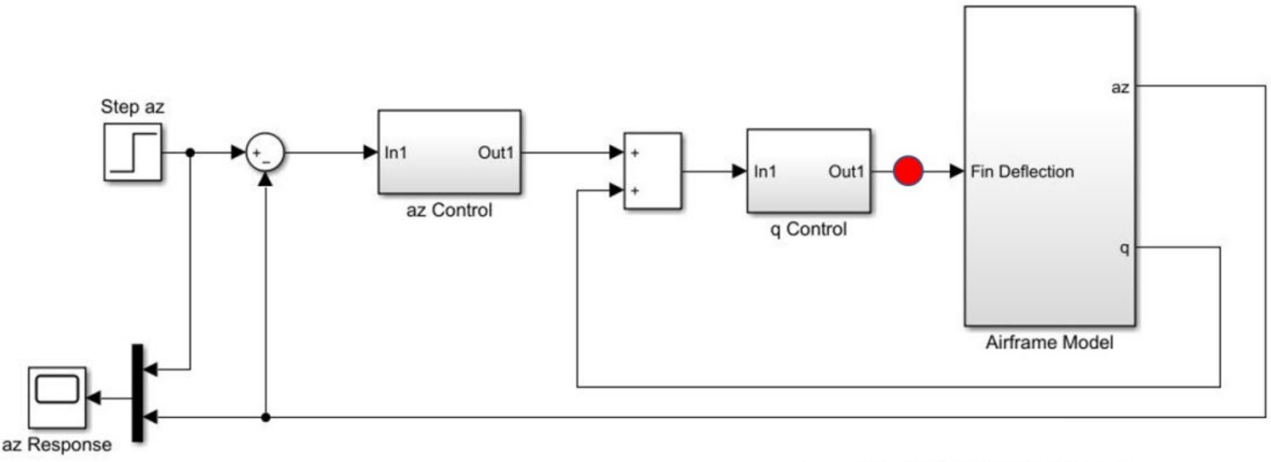
- Plant and controller with *a=10*:

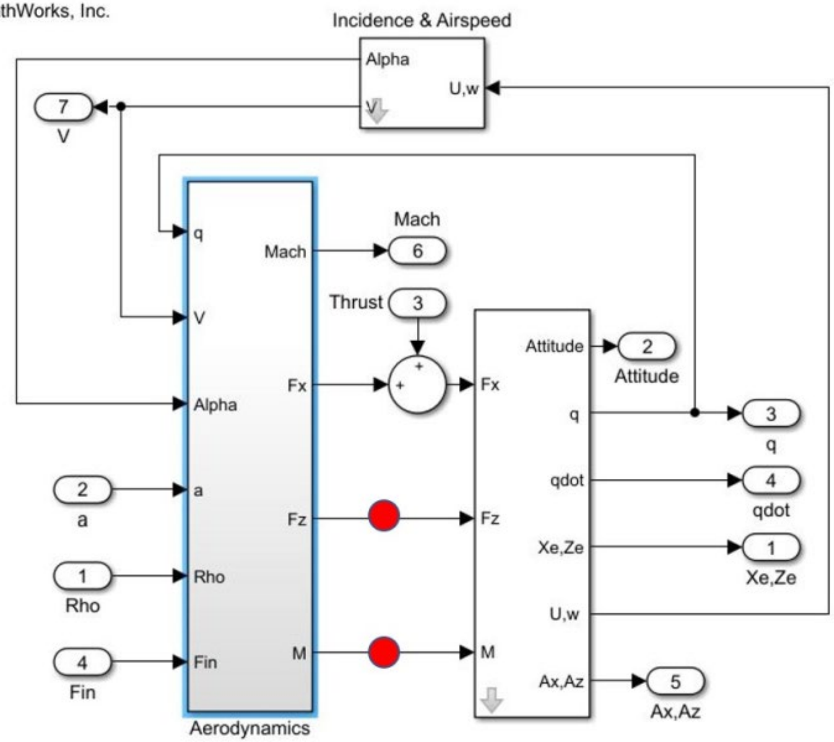
$$P := \frac{1}{s^2+a^2}\begin{bmatrix} s-a^2 & a(s+1) \\ -a(s+1) & s-a^2 \end{bmatrix} \text{ and } K := -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- Classical Loop-at-a-Time Margins
  - Loop 1: Gain Margins [0,inf], Phase Margin =90^o
  - Loop 2: Gain Margins [0,inf], Phase Margin =90^o
- Multi-loop Disk Margins
  - Input: α_{max} = 0.0997
  - Input and Output: α_{max} = 0.0498
- System can be destabilized by small perturbations at the input: f_1 = 0.9 and f_2 = 1.1.

Example Ref: Doyle, Robustness of multiloop linear feedback systems, CDC, 1978.

Disk Margins in Simulink





Advanced Robustness Analysis

Move beyond classical SISO stability (gain/phase) margins

1. Multi-loop (MIMO) systems with multiple uncertainties
2. More detailed uncertainty descriptions including

Structured Singular Value (μ)

- Parametric,
- Non-parametric (dynamic)
- Nonlinearities, e.g. saturation

Integral Quadratic Constraints (IQCs)

3. Consider both robust stability and robust performance

Developments go back to the Lur'e problem (40's) with key contributions in the 80's and 90's:

- μ: Safonov, Stein, Doyle, Packard, …
- IQCs: Yakubovich, Megretski, Rantzer, …

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Generalized Robustness Analysis

1. Need to assess margins for time-varying systems

Wind Turbine
Periodic /
Parameter-Varying

Flexible Aircraft
Parameter-Varying

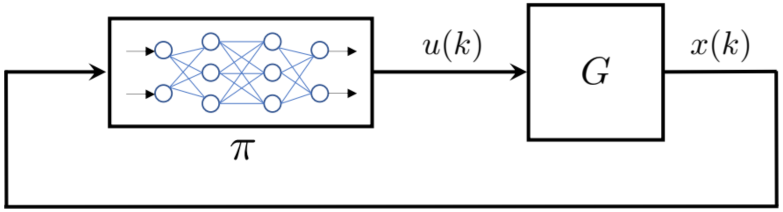
Vega Launcher
Time-Varying
(Source: ESA)

Robotics
Time-Varying
(Source: ReWalk)

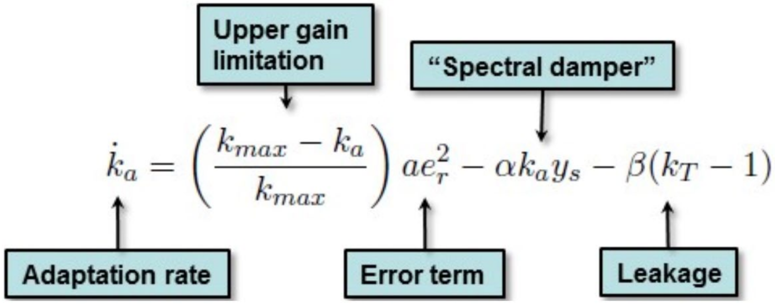
Generalized Robustness Analysis

1. Need to assess margins for time-varying systems

2. Need to assess margins for non-traditional controllers



Neural Networks
(Figure: Yin, Seiler, Arcak, arXiv 2020)


Upper gain limitation

"Spectral damper"

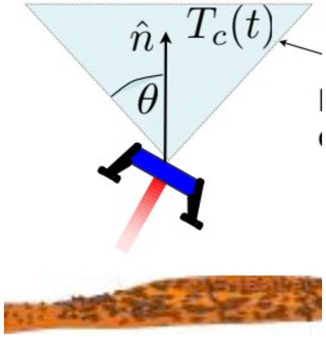
$$\dot{k}_a = \left(\frac{k_{max}-k_a}{k_{max}}\right) a e_r^2 - \alpha k_a y_s - \beta(k_T - 1)$$

Adaptation rate

Error term

Leakage

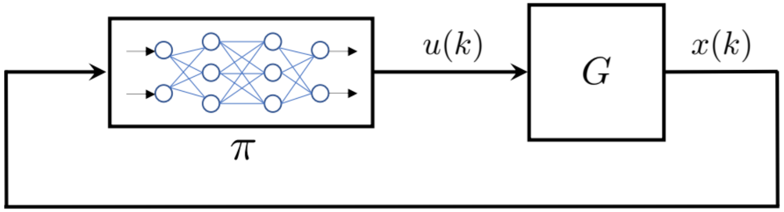
Adaptive Control
(Figure: VanZwieten, Hannan, Wall, ESA GNC 2017 with additional work by Balas + Navarro-Tapia, Marcos, & Bennani)



Optimization-based Control
(Figure: Açıkmeşe, 2020)

Generalized Robustness Analysis

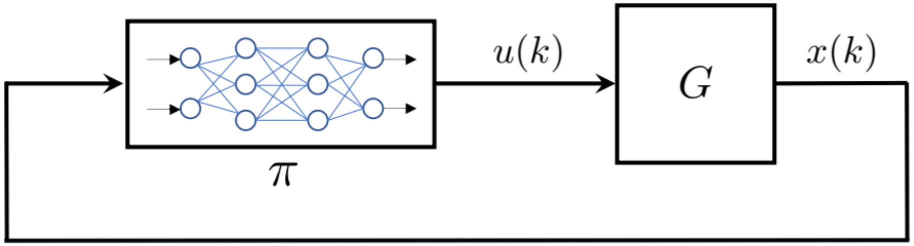
1. Need to assess margins for time-varying systems
   - See paper by Seiler, Moore, Meissen, Arcak and Packard, for details on robustness of time-varying systems.
2. Need to assess margins for non-traditional controllers
   - We'll focus on results for neural networks from Yin, Seiler, Arcak, arXiv, 2020.



Neural Networks
(Figure: Yin, Seiler, Arcak, arXiv 2020)

Problem Formulation

- Plant *G* is LTI & Neural Network π is a static, state-feedback.



- Neural-network has ℓ-layers:

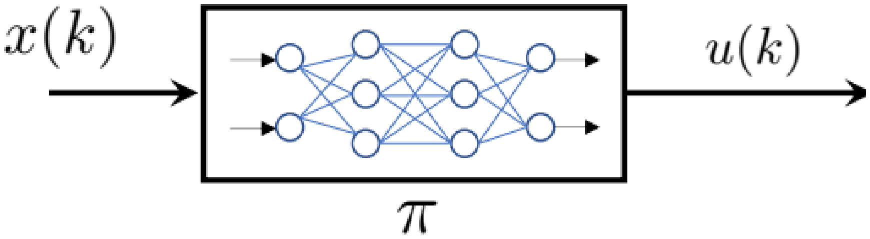
$$\begin{aligned} w^0(k) &= x(k), \\ w^i(k) &= \phi^i\left(W^i w^{i-1}(k) + b^i\right), \; i = 1, \dots, \ell, \\ u(k) &= W^{\ell+1} w^\ell(k) + b^{\ell+1}, \end{aligned}$$

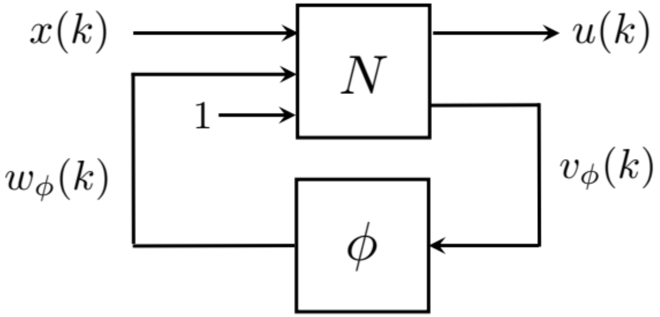
where W^i, b^i, *and* ϕ^i are the weights, biases, & activation functions.

Goal: Compute an estimate of the region of attraction (ROA) of initial conditions that converge back to the equilibrium point.

Approach:

1. Isolate the nonlinear activation functions



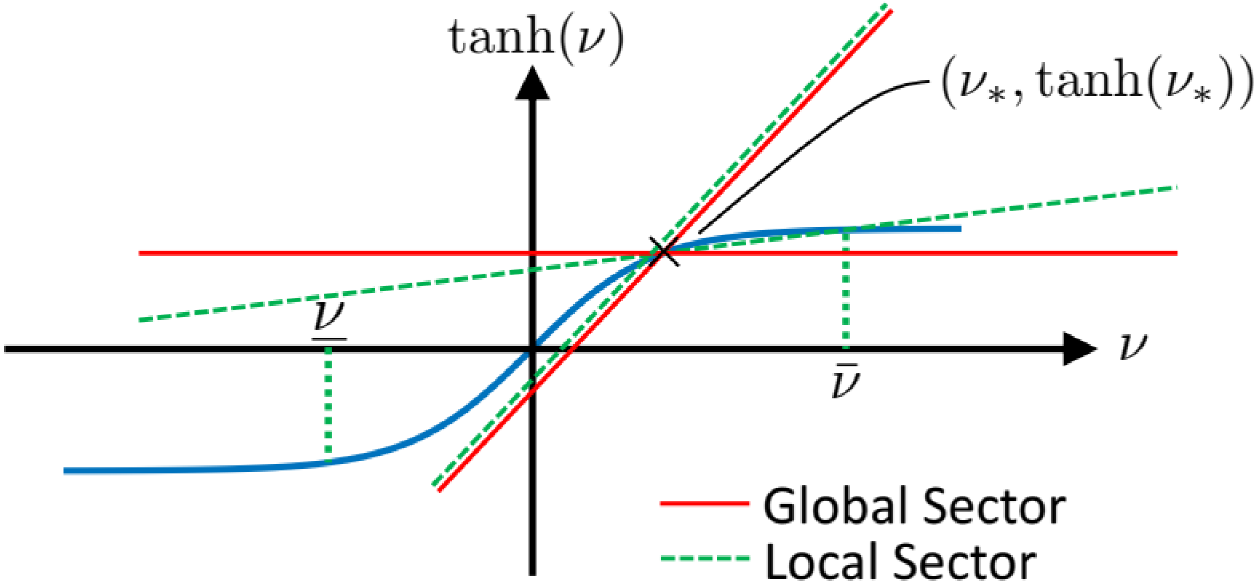


$$N := \left[\begin{array}{c|cccc|c} 0 & 0 & 0 & \cdots & W^{\ell+1} & b^{\ell+1} \\ \hline W^1 & 0 & \cdots & 0 & 0 & b^1 \\ 0 & W^2 & \cdots & 0 & 0 & b^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & W^\ell & 0 & b^\ell \end{array}\right]$$

Approach:

1. Isolate the nonlinear activation functions

2. Express local quadratic constraints on the activation function.



Approach:

1. Isolate the nonlinear activation functions

2. Express local quadratic constraints on the activation function.

3. Use Lyapunov theory, local quadratic constraints, and convex optimization to estimate the region of attraction.
    - Lyapunov condition also proves local region assumption used to derive quadratic constraints is valid.

Comments:

- The framework can be extended to handle nonlinearities and uncertainties in the plant *G*.
- This extension can be used to compute disk margins for neural network-based controllers.

Inverted Pendulum

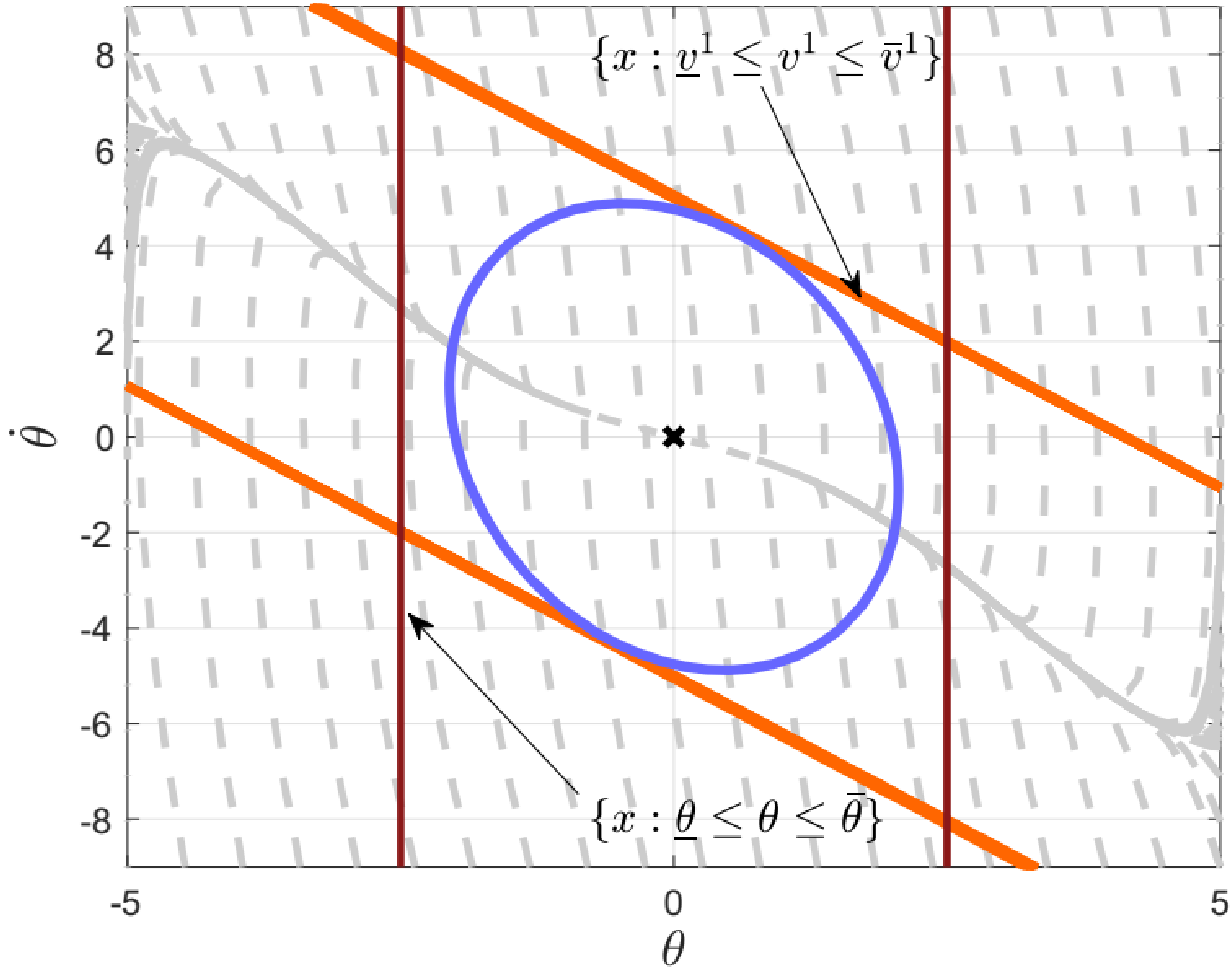
- Equations of Motion with angle θ (rad):

$$\ddot{\theta}(t) = \frac{mgl\sin(\theta(t)) - \mu\dot{\theta}(t) + u(t)}{ml^2},$$

  - mass m=0.15kg, length l = 0.5m, friction μ=0.5 Nms/rad.
  - Dynamics discretized with dt=0.02s.
  - Trigonometric terms also bounded with sector constraints

- Neural network designed via reinforcement learning
  - 2 Layers
  - 32 neurons in each layer
  - tanh as the activation function
  - All biases set to zero

Inverted Pendulum



$\{x : \underline{v}^1 \leq v^1 \leq \bar{v}^1\}$

$\{x : \underline{\theta} \leq \theta \leq \bar{\theta}\}$

Lateral Vehicle Control

- Equations of Motion with perp. distance to lane edge e (m) and e_θ is the angle between the car and lane (rad):

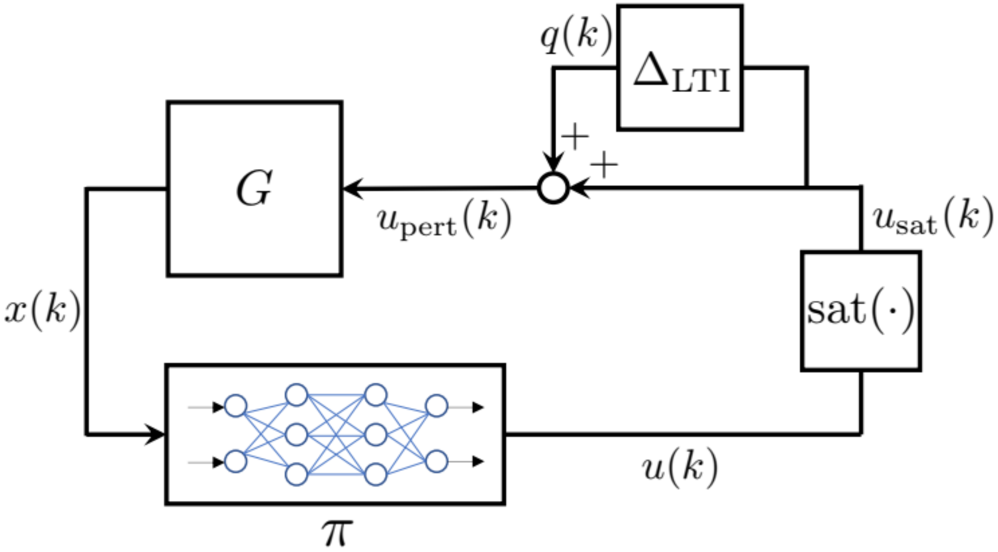
$$
\begin{bmatrix} \dot{e} \\ \ddot{e} \\ \dot{e}_\theta \\ \ddot{e}_\theta \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & \frac{C_{\alpha f}+C_{\alpha r}}{mU} & -\frac{C_{\alpha f}+C_{\alpha r}}{m} & \frac{aC_{\alpha f}-bC_{\alpha r}}{mU} \\ 0 & 0 & 0 & 1 \\ 0 & \frac{aC_{\alpha f}-bC_{\alpha r}}{I_z U} & -\frac{aC_{\alpha f}-bC_{\alpha r}}{I_z} & \frac{a^2C_{\alpha f}+b^2C_{\alpha r}}{I_z U} \end{bmatrix} \begin{bmatrix} e \\ \dot{e} \\ e_\theta \\ \dot{e}_\theta \end{bmatrix}
$$

$$
+ \begin{bmatrix} 0 \\ -\frac{C_{\alpha f}}{m} \\ 0 \\ -\frac{aC_{\alpha f}}{I_z} \end{bmatrix} u + \begin{bmatrix} 0 \\ \frac{aC_{\alpha f}-bC_{\alpha r}}{m} - U^2 \\ 0 \\ \frac{a^2C_{\alpha f}+b^2C_{\alpha r}}{I_z} \end{bmatrix} c \qquad (35)
$$

  - Parameters given the paper.
  - Dynamics discretized with dt=0.02s.

Lateral Vehicle Control

- Equations of Motion with perp. distance to lane edge e (m) and e_θ is the angle between the car and lane (rad):
  - Parameters given the paper.
  - Dynamics discretized with dt=0.02s.
  - Saturation and unmodeled dynamics included in analysis.



Lateral Vehicle Control

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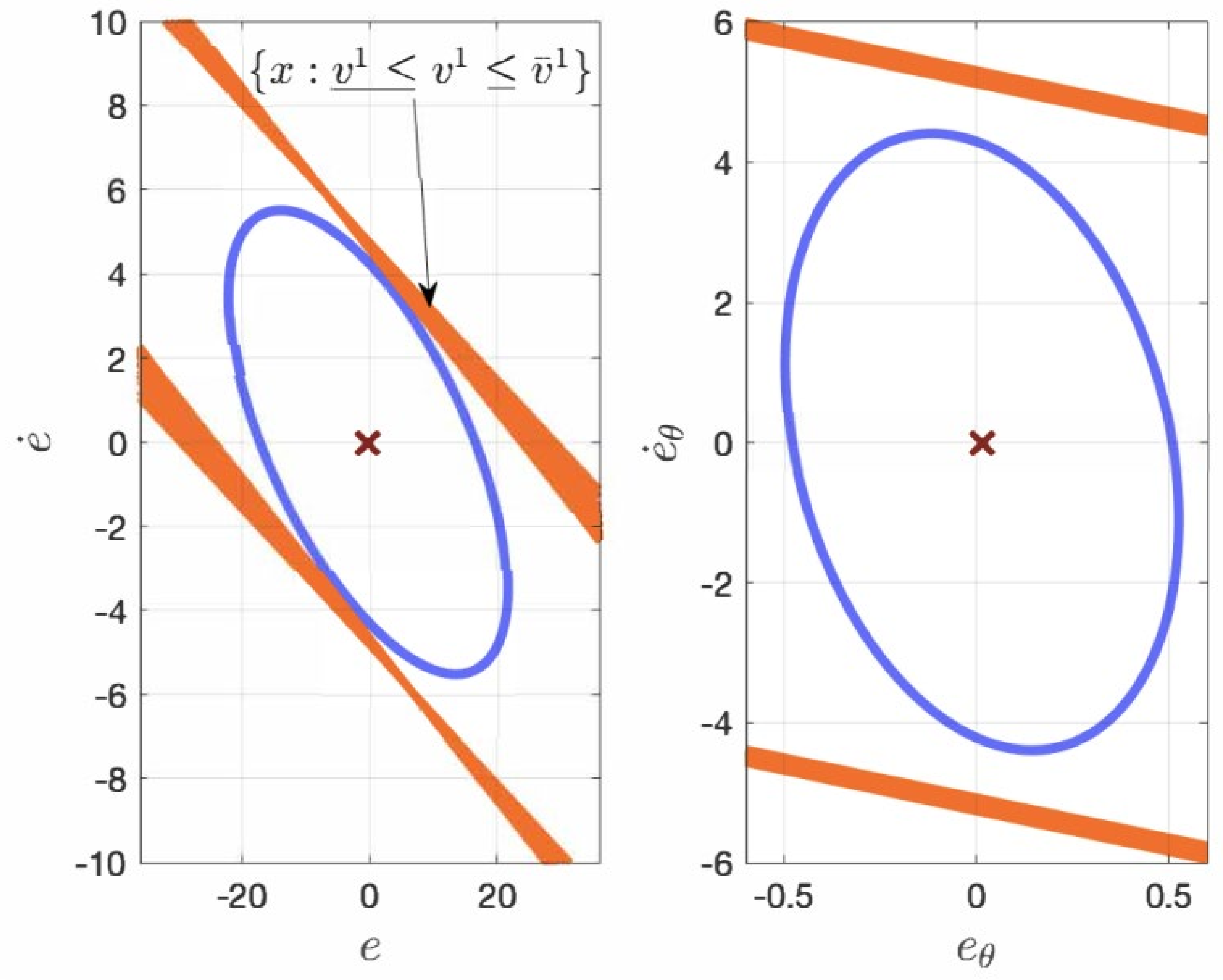
Lateral Vehicle Control



$\{x : \underline{v}^1 \le v^1 \le \bar{v}^1\}$

Conclusions

- Disk margins are a natural extension to classical margins
  - Allow simultaneous gain and phase variations
  - Extensions to multi-loop margins
  - Part of more general frameworks (structure singular value / mu, IQCs) for robustness analysis.

- Stability Margins for Non-Classical Controllers
  - The more general tools (IQCs) can be used to assess robustness of linear time-varying systems and neural-network controllers

- Future Work: Additional tools to assess robustness of adaptive and optimization-based controllers

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- ONR
  - ONR BRC: “Finite-Horizon Robustness: Moving Beyond Traditional Stability Analysis.” Tech Monitor: B. Holm-Hansen.
- NASA
  - NRA NNX14AL36A: "Lightweight Adaptive Aeroelastic Wing for Enhanced Performance Across the Flight Envelope," Tech. Monitor: J. Ouelette.
  - NRA NNX12AM55A: “Analytical Validation Tools for Safety Critical Systems Under Loss-of-Control Conditions.” Tech. Monitor: C. Belcastro.
- Eolos Consortium and Saint Anthony Falls Laboratory
  - http://www.eolos.umn.edu/ & http://www.safl.umn.edu/