NESC ACADEMY WEBCAST

WELCOME…

Neil Dennehy

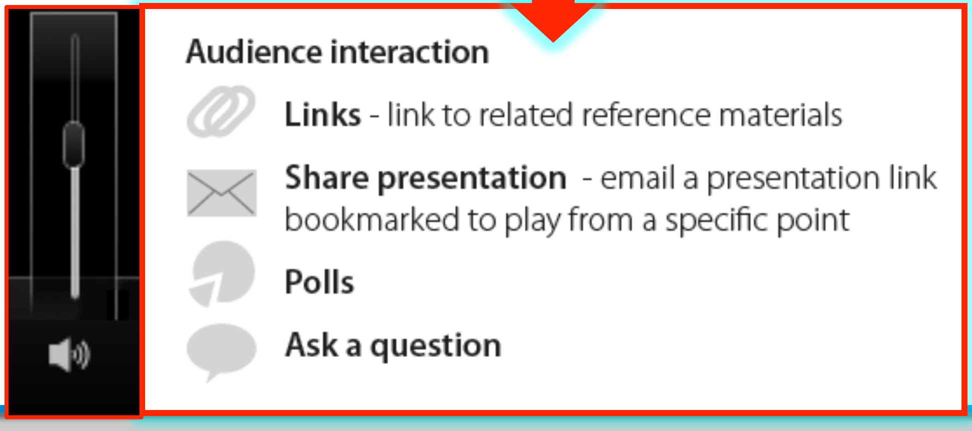
cornelius.j.dennehy@nasa.gov

Fundamentals of Linear Stability

by

Ken Lebsock

NESC ACADEMY WEBCAST



Fundamentals of Linear Stability

3 Thank you Neil, I'm pleased to be here.

Webcast Presentation by: Kenneth Lebsock Ph.D.

NASA Engineering & Safety Center (NESC)
GN&C Technical Discipline Team (TDT) Deputy

Introduction

1) Prequel
2) Stability of Aerospace Systems
3) Quantifying Stability

Prequel

- Roughly speaking, a linear system is said to be stable when it returns to its steady-state condition after a disturbance and unstable if it diverges after a disturbance. Stability is a System Property.

- This presentation deals with only the first of the two main categories of commonly used approaches to analyze the stability of a linear system:
  1) Frequency Analysis Methods based on the Nyquist Criterion and displayed graphically with Nyquist Plots, Nichols Charts, or Bode Gain and Phase Plots
  2) Pole-Zero-Configuration Methods based on the Routh-Hurwitz Criterion and displayed graphically in Root Locus Plots

- The importance of Frequency Analysis Methods lies in the fact that they can be used to determine the relative degree of system stability by producing the so-called phase and gain stability margins.
  – These stability margins are needed for frequency domain controller design techniques.

Stability of Aerospace Systems

- Stability is of concern for any control system but it is especially important in the attitude control of aerospace vehicles which may be inherently unstable in the absence of active control (e.g. NASA X-29).
- Gain margin and phase margin represent the tolerance of a control loop to perturbations in loop gain and phase delay.

- Stability requirements specified as gain and phase margins are convenient because they can be measured in a closed loop control system by artificially introducing variations in the loop gain or the phase delay until instability is observed.
- These stability margins can be found analytically by examination of the Open loop ferFunctionfer function Bode gain and phase plots as functions of frequency, the Nichols chart gain and phase cross plot, or the Nyquist plot of Real vs. Imaginary parts.

Quantifying Stability

- In this presentation of control loop design in the frequency domain, different stability metrics will be discussed:
  - Gain Margin and Phase Margin
  - Stability Margin

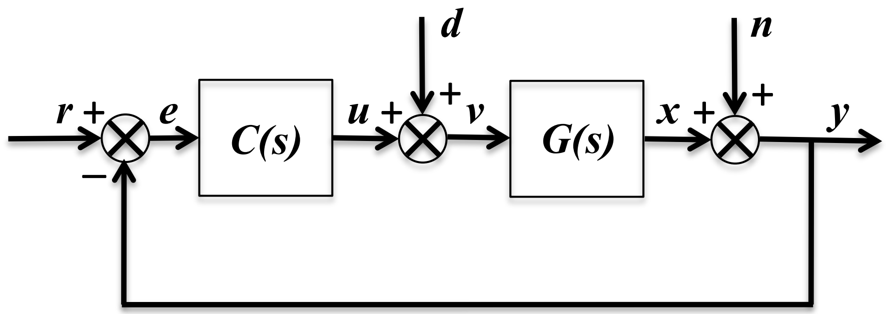
- Three examples of determining Gain and Phase Margins from the Open Loop Transfer Function, $L(j\omega)$, will be demonstrated:
  - Bode plots of magnitude and phase of $L(j\omega)$ vs. frequency
  - Nyquist plot of the real and imaginary parts of $L(j\omega)$
  - Nichols cross-plot of magnitude vs. phase of $L(j\omega)$

- The more compact and precise Stability Margin is found by two methods:
  - Graphically from the Nyquist plot of the real and imaginary parts of $L(j\omega)$
  - Analytically from the maximum value of the Sensitivity Function $S(j\omega)$

Transfer Functions

1) Open-Loop Transfer Function
2) Complementary Sensitivity Transfer Function
3) Sensitivity Function

Open-Loop Transfer Function



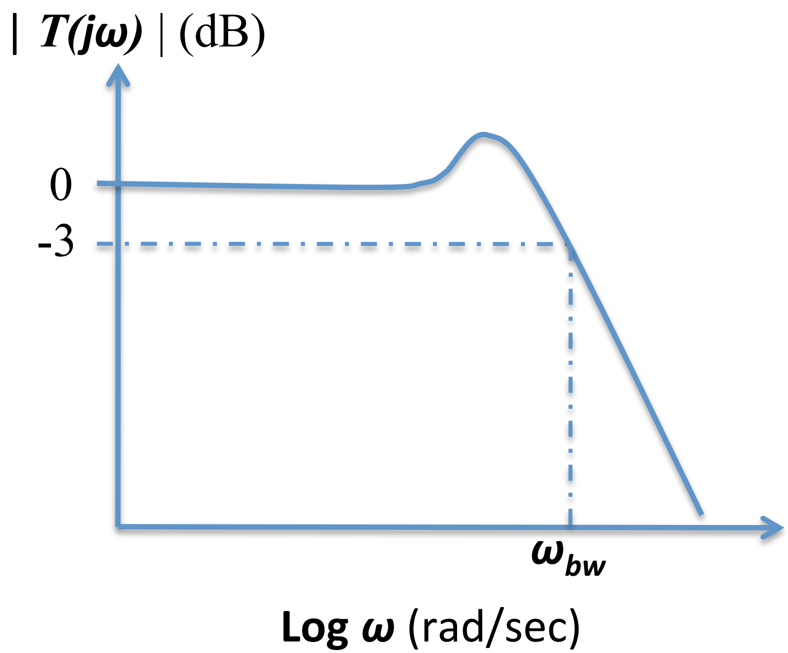
Basic LTI SISO Feedback Control Loop

- The Open Loop Transfer Function $L(s) = G(s)\ C(s)$ where $G(s)$ is the Transfer Function of the Plant Dynamics and $C(s)$ is the Transfer Function of the Controller.

- The symbol s is the Laplace variable. In the frequency domain analysis we use the substitution: $s \Rightarrow j\omega$ where j is √-1, and ω is the frequency in rad/sec.

- The Open Loop Transfer Function describes how the system output x would respond to an input r if the feedback loop were not closed.

Complementary Sensitivity Transfer Function



• The Complementary Sensitivity Function is the Closed Loop Transfer Function:

$$T(s) = \frac{L(s)}{1+L(s)} = \frac{x(s)}{r(s)}$$

• ***T(s)*** describes how the system output x responds to an input r when the feedback loop is closed.

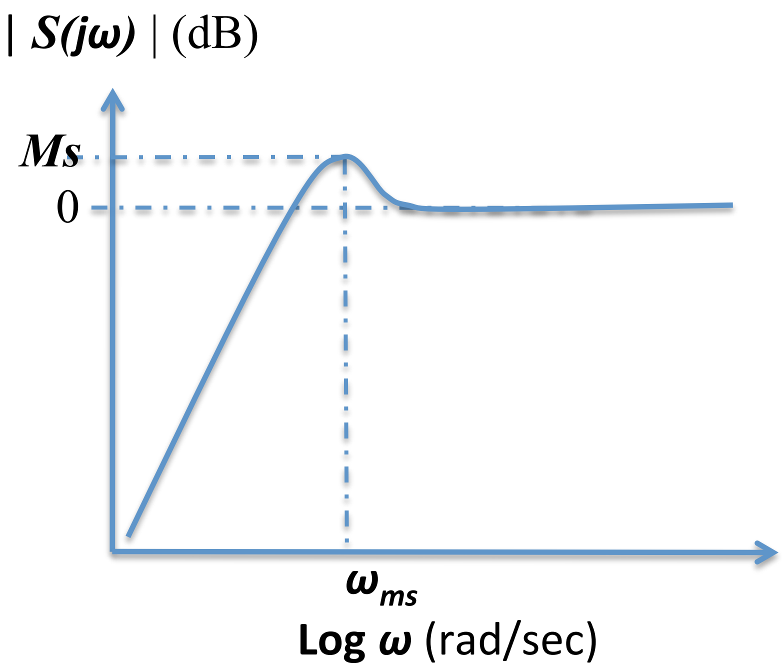
• The Closed Loop becomes unstable as:

$$L(s) \Rightarrow -1$$

• The ideal Closed Loop Transfer Function has unity gain and zero phase shift at low frequencies so that the reference input r is tracked perfectly. Resonant peaking , just under the closed loop 3dB bandwidth, is undesirable but often inevitable.

• We desire that the Closed Loop Transfer Function rolls off at frequencies that are higher than the system bandwidth in order to filter out delitrious sensor noise.

Sensitivity Function



• The Sensitivity Function:

$$S(s) = \frac{1}{1+L(s)} = \frac{T(s)}{L(s)}$$

is the ratio of the output of the closed loop system to the output of the open loop system. It describes the effect of feedback on the output.

• The Sensitivity and Complementary Sensitivity Functions are constrained at all frequencies by: $S(j\omega) + T(j\omega) = 1$

• Plant disturbances with frequencies such that **$|S(j\omega)| < 1$** (i.e. 0 dB) are attenuated.

• Disturbances at frequencies where **$|S(j\omega)| > 1$** are amplified by feedback.

• The maximum Sensitivity, $Ms = \|S(j\omega)\|_\infty$, occurring at the frequency ω_{ms} , is thus a measure of the largest amplification of the low frequency Plant disturbances.

Bode Plots and Stability

1) Gain and Phase Margins
2) Calculation of Stability Margins from $L(j\omega)$
3) Stability Margins from Bode Plots
4) Stability Margins from Bode Plots (cont.)

Gain and Phase Margins

- Gain margin and phase margin are two independent measures of relative stability. They measure how "close" a system is to crossing the boundary between stability and instability.

- Gain margin is the amount of change in the value of the gain of the Open Loop Transfer Function $L(j\omega)$, from its present value, to that value that will make the Bode magnitude pass through the 0 dB at the same frequency where the phase is -180 degrees.

- Phase margin is the amount of pure phase shift (no change in magnitude) that will make the phase shift of $L(j\omega)$ equal to -180 degrees at the same frequency where the magnitude is 0 dB (1 in absolute value).

- The Gain and Phase margins may be found from:
  - Bode plots of magnitude and phase of $L(j\omega)$ vs. frequency
  - Nyquist plot of the real and imaginary parts of $L(j\omega)$
  - Nichols cross plot of magnitude vs. phase of $L(j\omega)$

Calculation of Stability Margins from $L(j\omega)$

- Simple algorithms are used to calculate the gain and phase margins directly from the Open Loop Transfer Function.

- Iterate to find the frequency, ω_{gm} , at which Arg[$L(j\omega)$] = -180°
  - The gain margin is - Abs[$L(j\omega_{gm})$]

- Iterate to find the frequency, ω_{pm} , at which Abs[$L(j\omega)$] = 1 (i.e. 0 dB)
  - The phase margin is Arg[$L(j\omega_{pm})$] + 180°

- The Bode magnitude plot should be examined carefully to determine if there are multiple crossings of the 0 dB line.

- Also check if resonant peaks occur, especially in the frequency range: $\omega_{pm} < \omega < \omega_{gm}$.

Stability Margins from Bode Plots

- Illustrative Example
- Open Loop Transfer Function:

$$L(s) = \frac{7.6\,(s^2 + 0.1s + 0.55)}{s\,(s+1)(s+20)(s^2 + 0.06s + 0.5)}$$

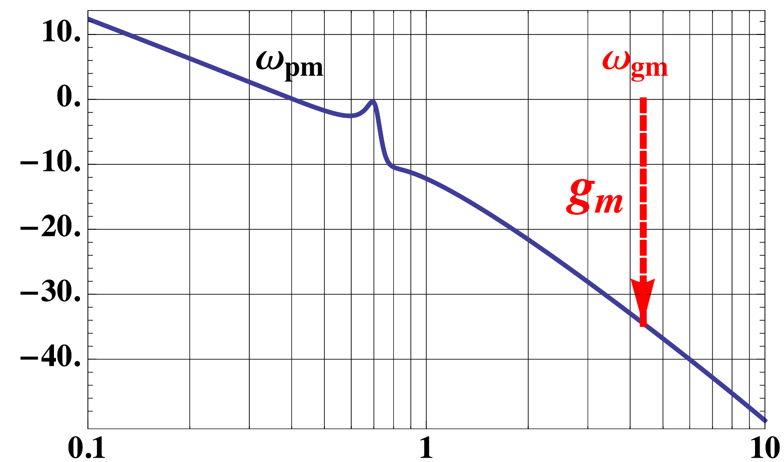
- The analytic calculation of the Gain and Phase Margins indicates that the System is extremely stable.

  – At the -180° Phase Cross-Over Frequency ω_{gm} = 4.37 rad/sec, g_m = 34.5 dB.

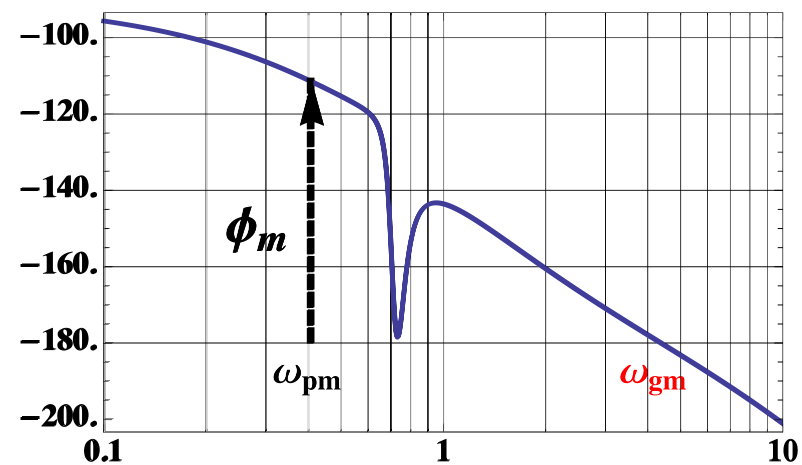
  – At the 0 dB Gain Cross-Over Frequency, ω_{pm} = 0.41 rad/sec, ϕ_m = 68.6°.

- The control loop easily meets a stability requirement of $g_m = 6$ dB $\Phi_m = 30°$.

Magnitude (dB) vs Freq (rad/sec)

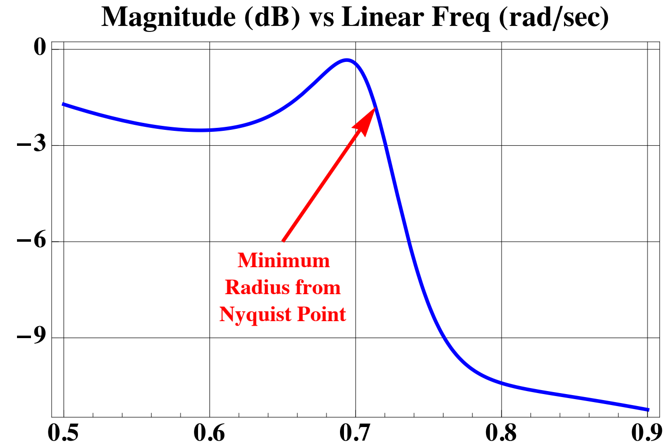


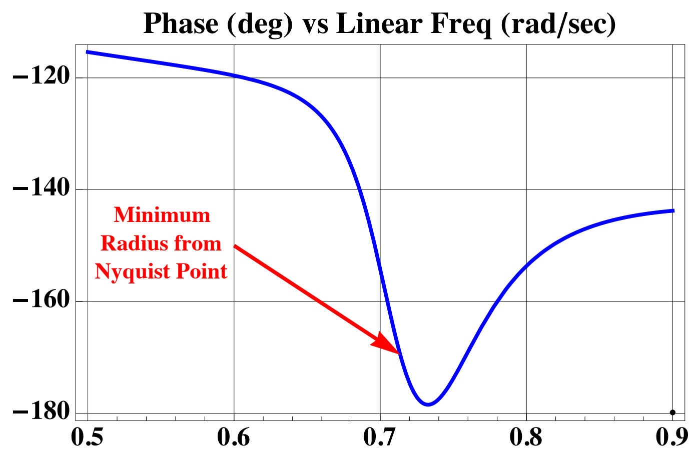
Phase (deg) vs Freq (rad/sec)



Stability Margins from Bode Plots (cont.)

- The tenuous stability of the system is obvious in a blowup of the Bode plots in the frequency range 0.5 – 0.9 rad/sec.
- The magnitude has a local maximum at -0.33 dB at ω = 0.69 rad/sec.
- The phase angle has a local minimum of -178.5 deg at ω = 0.73 rad/sec.
- Neither of these two extreme values uniquely defines a precise measure for the stability of the system because they do not occur at the same frequency.
- The worst case stability occurs at a frequency somewhere between these two cases where $L(j\omega)$ approaches the closest to -1.





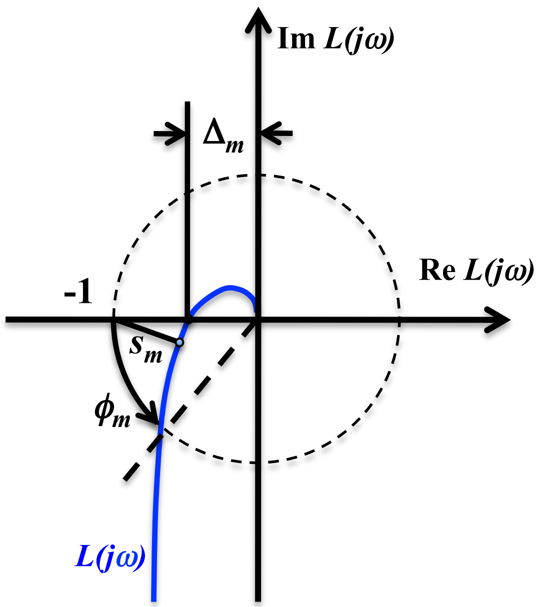
Nyquist Plots and Stability

1) Stability Margins on the Nyquist Plot
2) Blowup of Nyquist Plot for the illustrative Example
3) Gain & Phase Margins from Nyquist Plot for Example
4) Gain and Phase Margins Bounds from the Sensitivity Function
5) Stability Margin from Maximum Sensitivity for Example
6) Calculation of Stability Margin from $\boldsymbol{S(j\omega)}$

Stability Margins on the Nyquist Plot

- Recall that Ms, the maximum magnitude of $S(j\omega)$, is also the minimum of $|1+L(j\omega)|$.
- s_m is the shortest distance from the Nyquist curve, $L(j\omega)$, to $(-1, 0\,j)$, the critical point, i.e. $s_m = \text{Min}|1+L(j\omega)|$.
- An alternative way to express margins is by a single number, s_m , the Stability Margin.
- Therefore the maximum Sensitivity is also a measure of minimum stability since $Ms=1/s_m$.
- The Sensitivity maximum is a more compact indicator of stability than a pair of gain and phase margins.

Nyquist Plot (for a generic $L(j\omega)$)

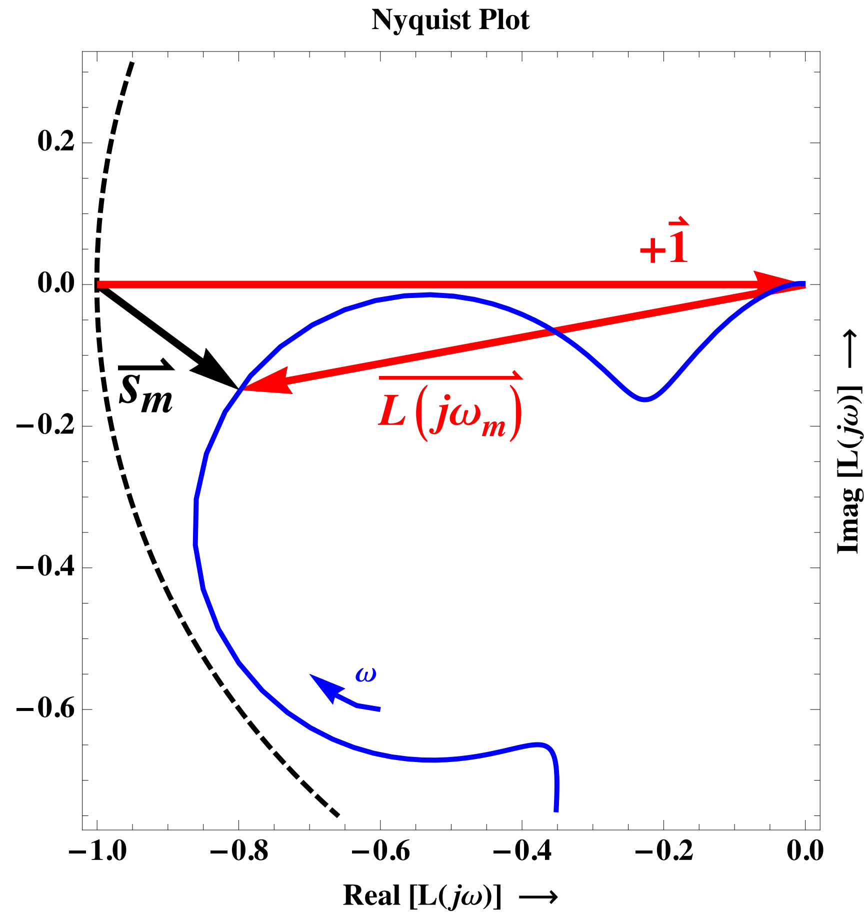


- $g_m = 20\text{Log}_{10}[1/\Delta_m]$ = Gain Margin
- ϕ_m = Phase Margin
- $s_m = 1/Ms$ = Stability Margin

Blowup of Nyquist Plot for Example $L(j\omega)$

$$L(s)=\frac{7.6(s^2+0.1s+0.55)}{s(s+1)(s+20)(s^2+0.06s+0.5)}$$

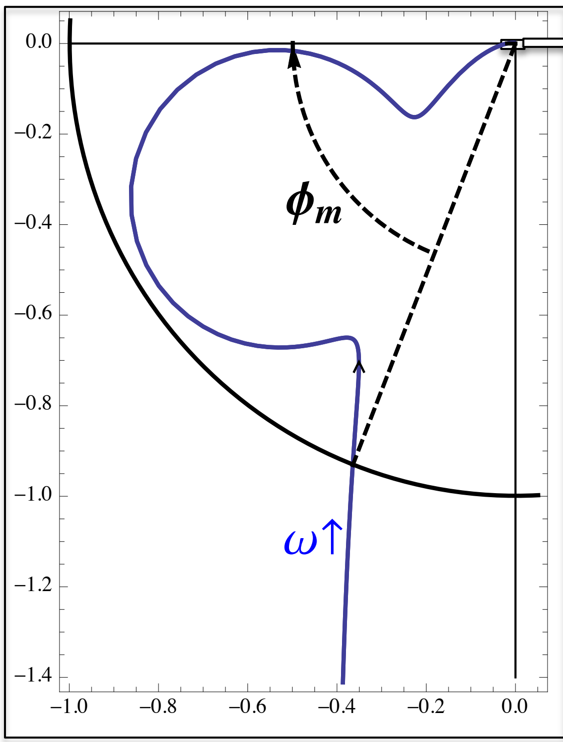
- *Ms,* the maximum value of the magnitude of the Sensitivity occurs at some frequency ω_m, i.e. $Ms = |S(j\omega_m)| = 1/|1+L(j\omega_m)|$.
- The denominator, $|1+L(j\omega_m)|$, reaches its' minimum value at the same frequency, ω_m.
- The magnitude of the position vector from the Nyquist point, *(-1, 0j)*, to the $L(j\omega)$ locus reaches a minimum value, at $\omega = \omega_m$:
  $s_m = |1+L(j\omega_m)|$
- Therefore the closest approach of the Open Loop Transfer Function, $L(j\omega)$, to the Nyquist point, *(-1, 0j)*, is:
  $s_m = 1 / Ms$

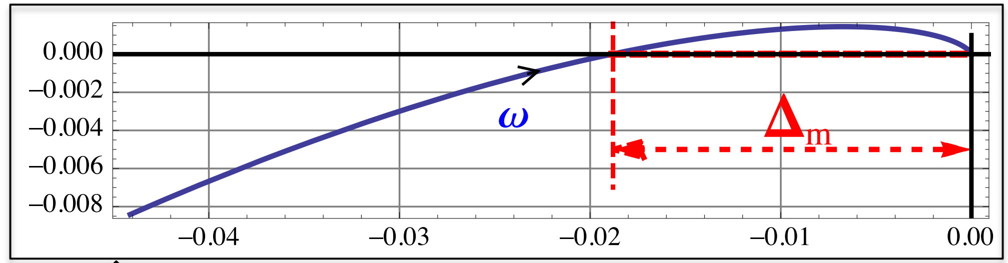


Gain & Phase Margins from Nyquist Plot for Example

Phase Margin is found from the Intersection of $L(j\omega)$ with the Unit Circle on the Nyquist Plot

- ϕ_m = 68.6 deg





Expanded Region near Nyquist Plot Origin

The Gain Margin is found at the Intersection of $L(j\omega)$ with the Negative Real Axis using an Expanded Scale around the Origin of the Nyquist Plot

- $g_m = 20\text{Log}_{10}[1/\Delta_m]$ = 34.5 dB

Gain and Phase Margins Bounds from the Sensitivity Function

- The relationships between the maximum Sensitivity, ***Ms***, and the lower bounds of the gain and phase margins are given by the following inequalities:

$$g_m \geq 20\text{Log}_{10}\left[\frac{Ms}{Ms-1}\right](\text{dB}) \quad \text{and} \quad \phi_m \geq 2\text{ArcSin}\left[\frac{1}{2Ms}\right] \geq \frac{1}{Ms}(\text{rad})$$

- Typical specifications for maximum Sensitivity magnitude are in the range of 1.33 to 2 which corresponds to gain and phase margins of (12 dB and 45°) to (6 dB and 30°).

- These inequalities are useful even for poorly behaved Open Loop Transfer Functions.

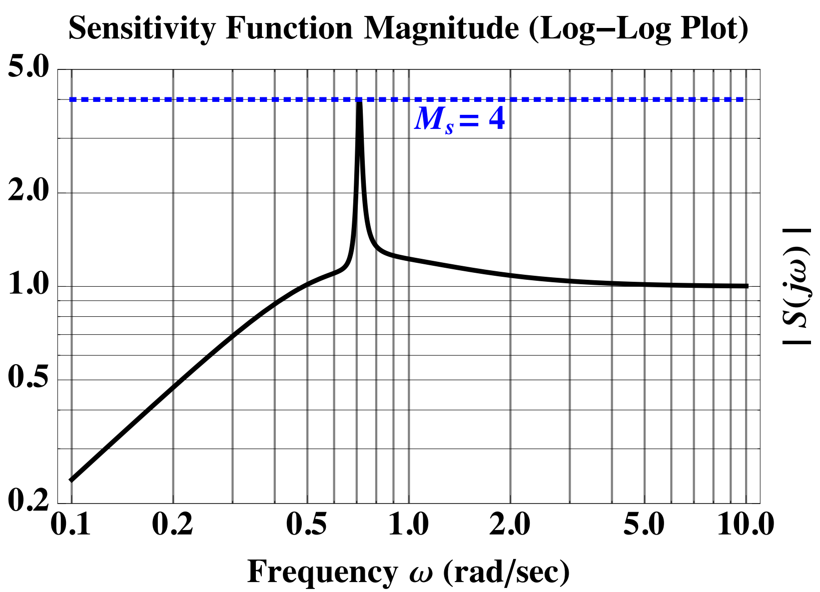
- Be very careful if the lower bounds of the gain and phase margins found from these inequalities are relatively small compared to the gain and phase margins found from the Bode plots.

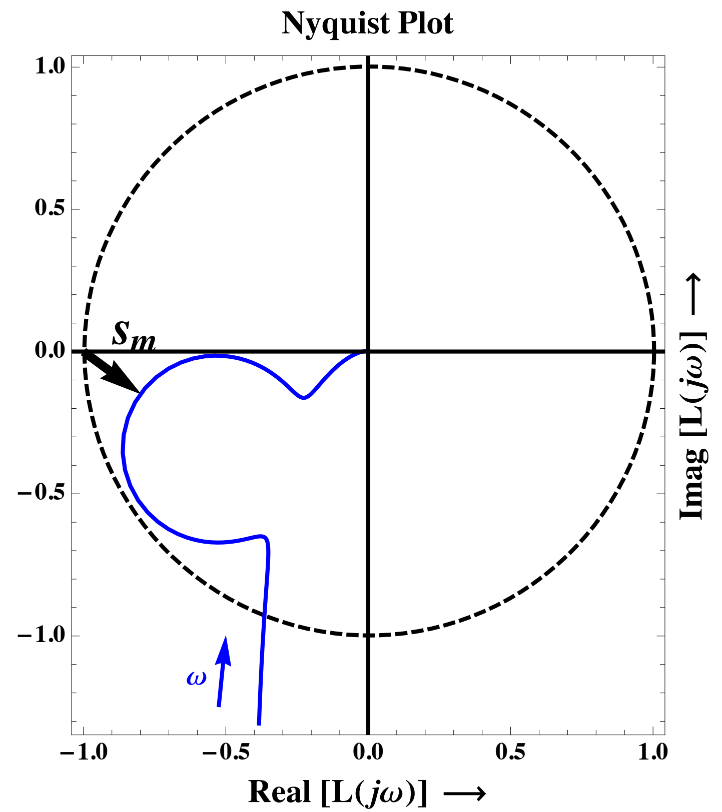
Stability Margin from Maximum Sensitivity for Example

- Example

$$L(s)=\frac{7.6(s^2+0.1s+0.55)}{s(s+1)(s+20)(s^2+0.06s+0.5)}$$

- The magnitude of the Sensitivity is at its maximum value of Ms = 4.0 when the frequency is ω_m= 0.714 rad/sec





$$s_m = 1/Ms = 0.25$$

$$g_m \geq 2.50 \text{ dB} \quad \text{and} \quad \Phi_m \geq 14.4°$$

Calculation of Stability Margin from $S(j\omega)$

- A simple algorithm is used to calculate the Stability Margin, s_m, directly from the Sensitivity Transfer Function.
  - Iterate on frequency to find the maximum of the absolute value of the Sensitivity function, $Ms = \mathrm{Max}[\ |S(j\omega)|\ ]$.
  - The Stability Margin is simply $s_m = 1/Ms$.
- Check the Nyquist plot to confirm that the value of s_m found by iteration is the closest approach of $L(j\omega)$ to the $(-1, 0\,j)$ point.

Nichols Plots and Stability

1) Gain & Phase Margins on Nichols Plot
2) Nichols Plot Stability Margin Contours
3) Stability Boundary on Nichols Plot
4) Blowup of Nichols Plot for Example 1

Gain & Phase Margins on Nichols Plot

- Example

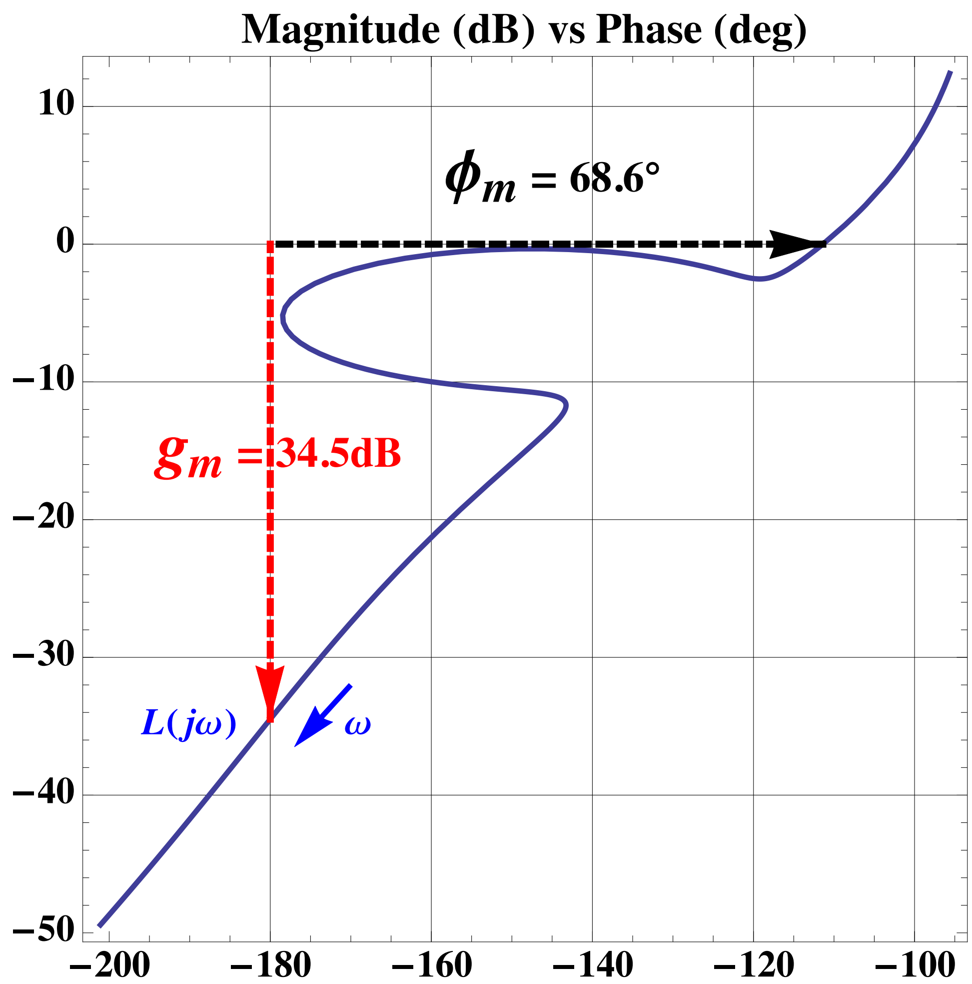
$$L(s) = \frac{7.6\,(s^2 + 0.1s + 0.55)}{s\,(s+1)(s+20)(s^2 + 0.06s + 0.5)}$$

- The Gain and Phase Margins are readily apparent on the Nichols plot, perhaps more so than on the Bode plots:

$$g_m = 34.5 \text{ dB} \quad \Phi_m = 68.6°$$

- However the loop approaching the (0 dB, -180°) point indicates that there is far less loop stability than is indicated by the gain and phase margins on the Nichols plot.

- The Nichols plot needs an overlay of equal Stability Margin, s_m, contour lines to precisely define stability.



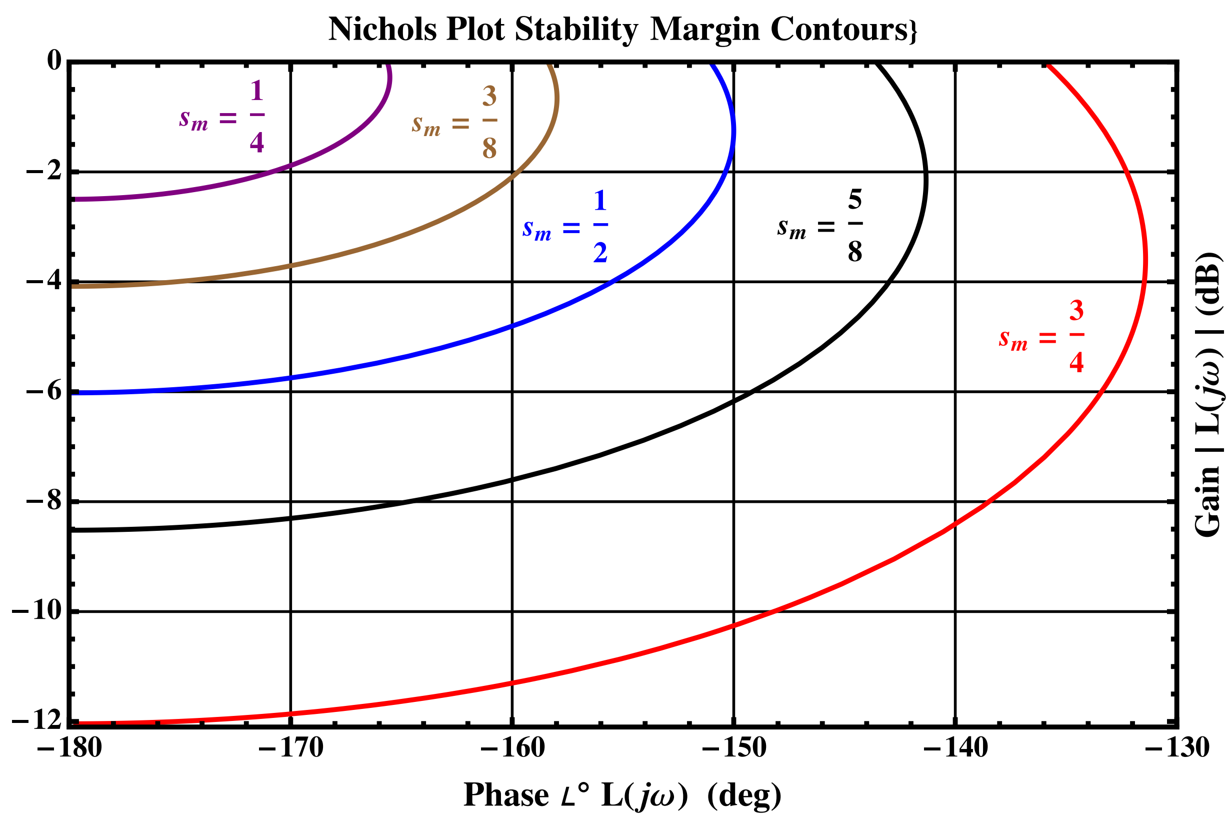
Nichols Plot Stability Margin Contours

- Realistically the Stability Margin specifications should be in the range of:

  $\frac{1}{2} \leq s_m \leq \frac{3}{4}$.

- Then the stability contour will lie between the blue and red contours.

- The locus of the open loop transfer function, $L(j\omega)$, should not pass above and to the left of the specified stability margin contour on the Nichols Plot.



Nichols Plot Contours equivalent to constant radius, s_m, about the (-1, 0 j) point of the Nyquist Stability Criterion

Stability Requirements on Nichols Plot

- Stability requirements are commonly specified as a pair of numbers for example:

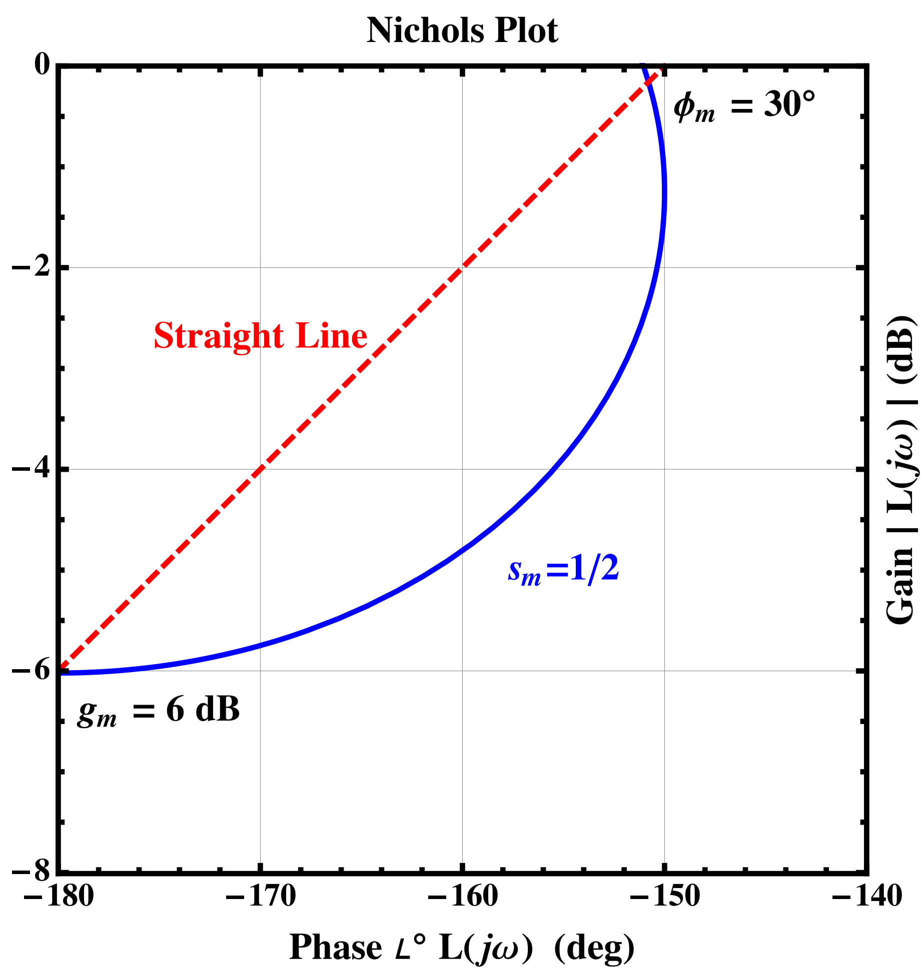
$$g_m = 6 \text{ dB} \quad \Phi_m = 30°.$$

- The actual margins are easily read on a Nichols plot as the points where the ***L(jω)*** locus crosses the gain and phase axes.

- Often a straight line is naively drawn between these two points to define the stability requirements boundary under the assumption that gain and phase errors combine linearly.

- This straight line assumption is not correct.

- The stability requirements boundary on the Nichols plot should be a Stability Margin contour similar to the ones shown on the preceding chart.



Blowup of Nichols Plot for Example 1

- Example 1

$$L(s) = \frac{7.6\,(s^2 + 0.1s + 0.55)}{s\,(s+1)\,(s+20)\,(s^2 + 0.06s + 0.5)}$$

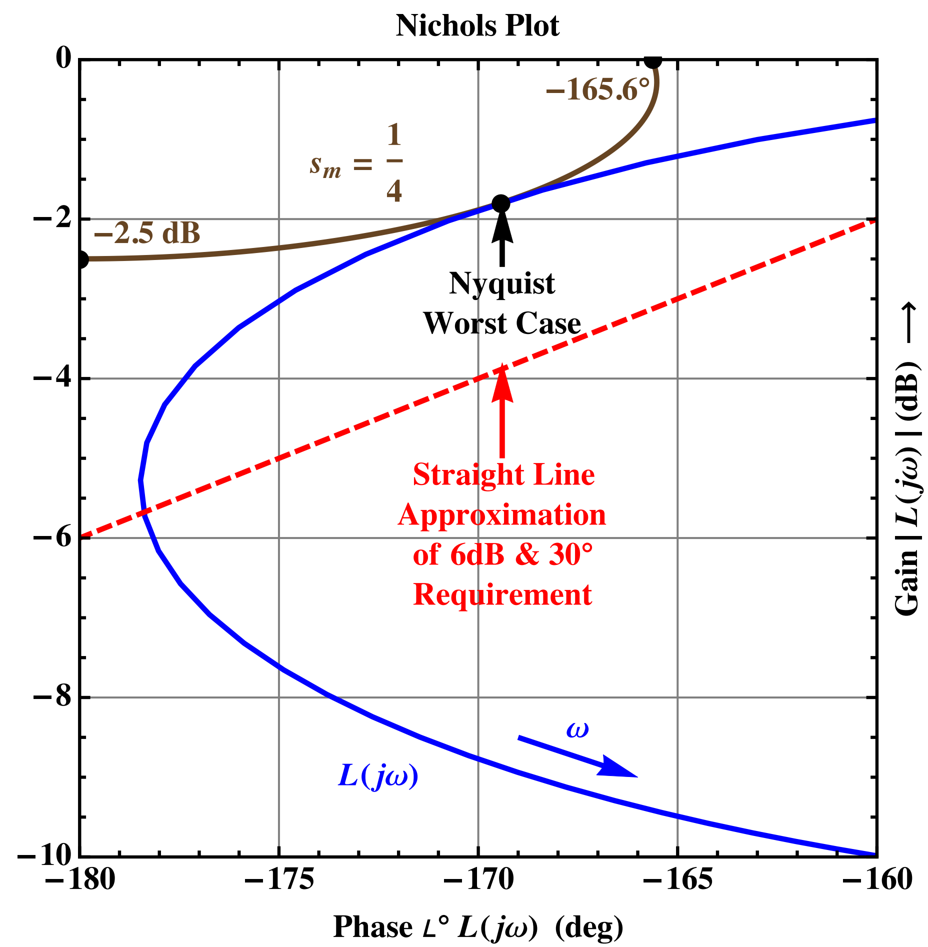
- The closest approach to the unstable (-1, 0 j) point of the Nyquist Criterion occurs at:

  ω_m = 0.714 rad/sec

  yielding the simultaneous changes of Gain and Phase:

  $$\Delta g = 1.8\ \text{dB}$$
  $$\Delta\phi = 10.6\ \text{deg}$$

- This appears on the Nichols plot as the point of tangency between the Open Loop Transfer Function and the Nyquist Stability Margin $s_m = 1/4$ contour curve.



Summary and Conclusion

1) Stability Summary
2) Loop Design Tradeoffs
3) Stability Margin Relationships
4) References

Stability Summary

- Stability requirements are often specified in terms of gain margin and phase margin. These are independent margins describing how much either the gain or phase alone can be varied before the system becomes unstable. These two margins are represented as separate independent points on the Nichols plot.

- However the true stability objective is to specify how "close" a system is to becoming unstable for any combination of gain and phase. This is compactly and precisely measured by the Stability Margin which describes the closest approach of the Open Loop Transfer Function to the $(-1, 0\,j)$ point on the complex plane in a Nyquist plot.

- Stability margin contours can be drawn on the Nichols plot to show the Nyquist boundary of different combinations of gain and phase deviations that can be tolerated with a certain level of robustness to instability.

- The gain and phase margins, as well as the Nyquist Stability Margin contour connecting them, are defined in the frequency domain. The margins and the contour can be verified in time domain simulations by injecting different combinations of gain and phase perturbations from nominal.

Loop Design Tradeoffs

- Loop design can be viewed as simultaneously tuning the Sensitivity Transfer Function to achieve disturbance rejection and loop stability goals and tuning the Complementary Sensitivity Transfer Function to achieve performance and noise rejection goals.

- $\boldsymbol{S(j\omega)}$ Sensitivity Transfer Function (Disturbance Rejection Transfer Function)
  - Rolling off $\boldsymbol{S(j\omega)}$ suppresses low frequency load disturbances.
  - However decreasing $S(j\omega)$ at low frequencies increases its maximum value, Ms, which reduces stability. This is the result of the Bode Integral being Constant.

$$\int_{0}^{\infty} \mathrm{Ln}\,|\,S(j\omega)\,|\,d\omega \triangleq \mathrm{Constant}$$

- $\boldsymbol{T(j\omega)}$ Complementary Sensitivity Transfer Function (Closed Loop Transfer Function)
  - Tracking Performance may be improved by increasing the bandwidth of $\boldsymbol{T(j\omega)}$.
  - However limiting the bandwidth by rolling off $\boldsymbol{T(j\omega)}$ is necessary to suppress high frequency sensor noise.

Stability Margin Relationships

- One of the fundamental goals of the tradeoffs involved in loop tuning is to shape the Sensitivity Transfer Function so that its maximum value, ***Ms***, is compatible with the control loop stability requirements.

- Recall that the maximum value of ***S(jω)*** is the inverse of the Stability Margin:

$$Ms = 1/s_m \qquad s_m = 1/Ms$$

- Typical specifications for the maximum Sensitivity Magnitude and the Stability Margin are in the range of:

$$4/3 \leq Ms \leq 2 \qquad 3/4 \geq s_m \geq 1/2$$

- Respectively these corresponds to gain and phase margins of approximately:

(12 dB and 45°) to (6 dB and 30°).

References

1) Feedback Systems:
An Introduction for Scientists and Engineers
By Karl Johan Åström & Richard M. Murray
Available free from Google Books

2) Respect the Unstable
IEEE Control Systems Magazine,
Vol. 23, Num. 4, pp. 12-25, Aug. 2003.
By Gunter Stein
Available free at NEN GN&C Reading Room

THANK YOU FOR ATTENDING…

QUESTIONS ???

UPCOMING NESC GN&C WEBCAST

Tim Crain – November 2014