Dynamics: Introduction to Kane's Method

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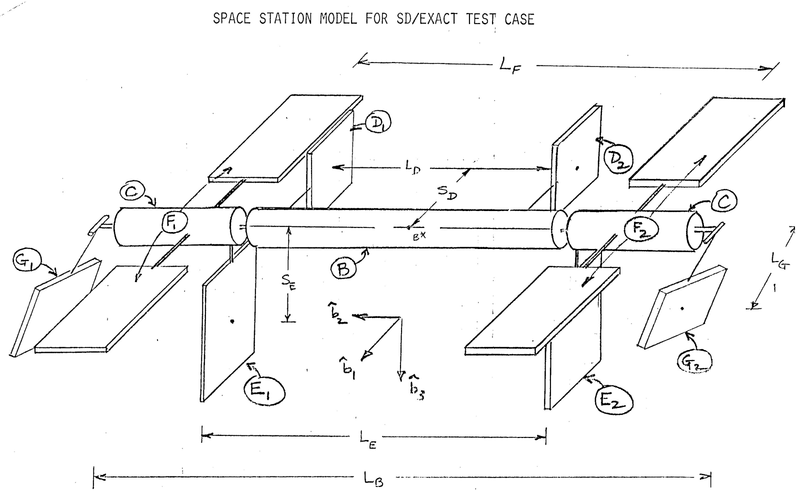


Derive dynamical equations of motion for a system of rigid bodies attached to one another

Possible Approaches

- momentum principles
- Newton-Euler method
- D'Alembert's principle
- Lagrange's equations
- Hamilton's canonical equations
- Boltzmann-Hamel equations
- Gibbs' equations
- **Kane's method**
  - Kane's equations have the simplest form and are derived with the least amount of labor[1]

[1] Kane, T. R., and Levinson, D. A., "Formulation of Equations of Motion for Complex Spacecraft," *Journal of Guidance and Control*, Vol. 3, No. 2, 1980, pp. 99–112.

Outline

Newton's Second Law

System S is made of ν particles P_i, each of mass m_i $(i = 1, \ldots, \nu)$, moving in a Newtonian reference frame N.

$$\mathbf{F}_1 = m_1 \, {}^N\mathbf{a}^{P_1} \tag{1}$$

$$\mathbf{F}_2 = m_2 \, {}^N\mathbf{a}^{P_2} \tag{2}$$

$$\ldots$$

$$\mathbf{F}_\nu = m_\nu \, {}^N\mathbf{a}^{P_\nu} \tag{3}$$

or, a single vector equation

$$\sum_{i=1}^{\nu} \left(\mathbf{F}_i - m_i \, {}^N\mathbf{a}^{P_i}\right) = \mathbf{0} \tag{4}$$

from which one can obtain a scalar equation

$$\sum_{i=1}^{\nu} \left(\mathbf{F}_i - m_i \, {}^N\mathbf{a}^{P_i}\right) \cdot \mathbf{v} = \mathbf{0} \cdot \mathbf{v} = 0 \tag{5}$$

where $\mathbf{v}$ is *any* vector

Basic Statement of Kane's Method

For a holonomic system possessing n degrees of freedom in frame N

$$\sum_{i=1}^{\nu} \left(\mathbf{F}_i - m_i \, {}^N\mathbf{a}^{P_i}\right) \cdot {}^N\mathbf{v}_r^{P_i} = 0 \qquad (r = 1, \ldots, n) \tag{6}$$

where ${}^N\mathbf{v}_r^{P_i}$ is called the rth *holonomic partial velocity* of particle P_i in N. (More about how to find partial velocities later.)

Generalized Forces

Kane calls F_r the rth *generalized active force* for S in N, and defines it as

$$F_r \triangleq \sum_{i=1}^{\nu} \mathbf{F}_i \cdot {}^N\mathbf{v}_r^{P_i} \qquad (r = 1, \ldots, n) \tag{7}$$

$F_r^\star$ is the rth *generalized inertia force* for S in N, defined as

$$F_r^\star \triangleq \sum_{i=1}^{\nu} -m_i \, {}^N\mathbf{a}^{P_i} \cdot {}^N\mathbf{v}_r^{P_i} \qquad (r = 1, \ldots, n) \tag{8}$$

Kane's dynamical equations of motion:

$$F_r + F_r^\star = 0 \qquad (r = 1, \ldots, n) \tag{9}$$

Advantages of Generalized Forces

- Generalized active forces
  - Constraint forces do not appear in Kane's equations of motion
    - Forces exerted on particles across smooth surfaces
    - Contact forces exerted by two bodies rolling on each other
  - Constraint forces do appear when using Newton-Euler or D'Alembert's method; extra work to eliminate them
  - If constraint forces are of interest, Kane shows how to bring them into evidence
- Generalized inertia forces
  - Forming Kane's generalized inertia forces is much easier than
    - Forming the system kinetic energy and then differentiating it (Lagrange's Eqs.)
    - Forming the Gibbs function and then differentiating it (Gibbs' method)

Motion Variables

When the configuration in N of a system S can be described with n *generalized coordinates* q_r, one can define n *motion variables* u_r as linear combinations of the time derivatives of q_r,

$$u_r \triangleq \sum_{s=1}^{n} Y_{rs}\dot{q}_s + Z_r \qquad (r = 1, \ldots, n) \tag{10}$$

where Y_{rs} and Z_r $(r, s = 1, \ldots, n)$ are functions of $q_1, \ldots, q_n$ and the time t. Must be able to solve Eqs. (10) uniquely for $\dot{q}_1, \ldots, \dot{q}_n$.

One of the chief disadvantages of using Lagrange's equations is that state variables cannot be u's and must be $\dot{q}$'s.

Partial Velocities, Partial Angular Velocities

The velocity in any reference frame A of a particle P belonging to S can be expressed uniquely in terms of motion variables and *partial velocities* ${}^A\mathbf{v}_r^P$,

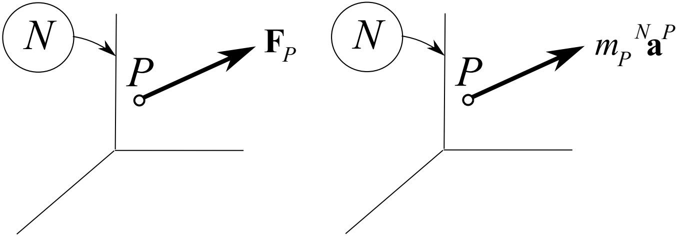
$$ {}^A\mathbf{v}^P = \sum_{r=1}^{n} {}^A\mathbf{v}_r^P u_r + {}^A\mathbf{v}_t^P \tag{11} $$

The angular velocity in any reference frame A of a rigid body B belonging to S can be expressed uniquely in terms of motion variables and *partial angular velocities* ${}^A\boldsymbol{\omega}_r^B$,

$$ {}^A\boldsymbol{\omega}^B = \sum_{r=1}^{n} {}^A\boldsymbol{\omega}_r^B u_r + {}^A\boldsymbol{\omega}_t^B \tag{12} $$

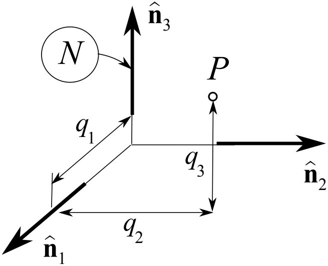
where ${}^A\mathbf{v}_r^P$, ${}^A\boldsymbol{\omega}_r^B$ $(r = 1, \ldots, n)$, ${}^A\mathbf{v}_t^P$, and ${}^A\boldsymbol{\omega}_t^B$ are functions of $q_1, \ldots, q_n$ and the time t.

Example, Single Particle



$$ {}^N\mathbf{v}_r^P \cdot \left(\mathbf{F}_P - m_P\, {}^N\mathbf{a}^P\right) = 0 \qquad (r = 1, 2, 3) \tag{13} $$

Motion Variables, Partial Velocities



The velocity of P in N

$$
\begin{aligned}
{}^N\mathbf{v}^P &= \dot{q}_1\hat{\mathbf{n}}_1 + \dot{q}_2\hat{\mathbf{n}}_2 + \dot{q}_3\hat{\mathbf{n}}_3 \\
&\overset{\triangle}{=} u_1\hat{\mathbf{n}}_1 + u_2\hat{\mathbf{n}}_2 + u_3\hat{\mathbf{n}}_3
\end{aligned} \tag{14}
$$

- Motion variables u_r are time derivatives of generalized coordinates q_r $(r = 1, 2, 3)$
- Partial velocities are simply the vector coefficients of the motion variables in the expression for ${}^N\mathbf{v}^P$; that is, ${}^N\mathbf{v}_r^P = \hat{\mathbf{n}}_r$ $(r = 1, 2, 3)$

Contribution of a Rigid Body to Generalized Active Forces

Let the set of contact forces and distance forces acting on a rigid body B be equivalent to a single force $\mathbf{F}_B$ applied at the mass center, $B^\star$, together with a couple whose torque is $\mathbf{T}_B$.

The contribution of B to F_r is given by

$$(F_r)_B = {}^N\mathbf{v}_r^{B^\star} \cdot \mathbf{F}_B + {}^N\boldsymbol{\omega}_r^B \cdot \mathbf{T}_B \qquad (r = 1, \ldots, n) \tag{15}$$

where ${}^N\mathbf{v}_r^{B^\star}$ is the rth partial velocity of $B^\star$ in N, and ${}^N\boldsymbol{\omega}_r^B$ is the rth partial angular velocity of B in N.

Contribution of a Rigid Body to Generalized Inertia Forces

The contribution of B to $F_r^\star$ is

$$(F_r^\star)_B = {}^N\mathbf{v}_r^{B^\star} \cdot \mathbf{R}^\star + {}^N\boldsymbol{\omega}_r^B \cdot \mathbf{T}^\star \qquad (r = 1, \ldots, n) \tag{16}$$

Inertia force for B in N:

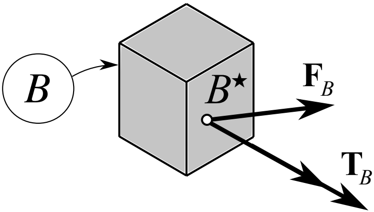
$$\mathbf{R}^\star \triangleq -m_B\, {}^N\mathbf{a}^{B^\star} \tag{17}$$

where m_B is the mass of B, and ${}^N\mathbf{a}^{B^\star}$ is the acceleration in frame N of the mass center of B.
Inertia torque for B in N:

$$\mathbf{T}^\star \triangleq -\left(\underline{\mathbf{I}} \cdot {}^N\boldsymbol{\alpha}^B + {}^N\boldsymbol{\omega}^B \times \underline{\mathbf{I}} \cdot {}^N\boldsymbol{\omega}^B\right) = -\frac{{}^N d\, {}^N\mathbf{H}^{B/B^\star}}{dt} \tag{18}$$

where $\underline{\mathbf{I}}$ is the inertia dyadic of B for $B^\star$, ${}^N\boldsymbol{\omega}^B$ is the angular velocity of B in N, and ${}^N\boldsymbol{\alpha}^B$ is the angular acceleration of B in N.

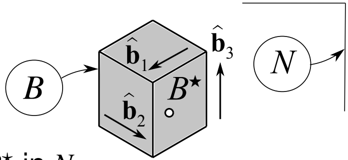
Example, Rigid Body



Substitution from Eqs. (15)–(18) into Eqs. (9) yields

$$
{}^N\mathbf{v}_r^{B^\star} \cdot \left(\mathbf{F}_B - m_B\, {}^N\mathbf{a}^{B^\star}\right) + {}^N\boldsymbol{\omega}_r^B \cdot \left(\mathbf{T}_B - \frac{{}^N d\, {}^N\mathbf{H}^{B/B^\star}}{dt}\right) = 0 \qquad (r = 1, \ldots, 6) \tag{19}
$$

Partial Velocities, Partial Angular Velocities



The velocity of $B^\star$ in N

$$ {}^N\mathbf{v}^{B^\star} \triangleq u_1\hat{\mathbf{n}}_1 + u_2\hat{\mathbf{n}}_2 + u_3\hat{\mathbf{n}}_3 \tag{20} $$

The angular velocity of B in N

$$ {}^N\boldsymbol{\omega}^B = \omega_1\hat{\mathbf{b}}_1 + \omega_2\hat{\mathbf{b}}_2 + \omega_3\hat{\mathbf{b}}_3 \triangleq u_4\hat{\mathbf{b}}_1 + u_5\hat{\mathbf{b}}_2 + u_6\hat{\mathbf{b}}_3 \tag{21} $$

- Motion variables u_4, u_5, u_6, are *linear combinations* of the time derivatives of generalized coordinates
- Partial angular velocities are simply the vector coefficients of the motion variables in the expression for ${}^N\boldsymbol{\omega}^B$; that is, ${}^N\boldsymbol{\omega}_r^B = \mathbf{0}$ $(r = 1, 2, 3)$, ${}^N\boldsymbol{\omega}_r^B = \hat{\mathbf{b}}_{r-3}$ $(r = 4, 5, 6)$

Equations of Motion

$r =$	1	2	3	4	5	6
${}^N\mathbf{v}_r^{B^\star}$	$\hat{\mathbf{n}}_1$	$\hat{\mathbf{n}}_2$	$\hat{\mathbf{n}}_3$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$
${}^N\boldsymbol{\omega}_r^B$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\hat{\mathbf{b}}_1$	$\hat{\mathbf{b}}_2$	$\hat{\mathbf{b}}_3$

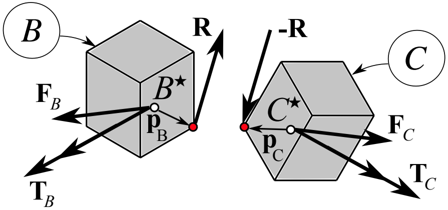
Dynamical equations of motion:

$$\hat{\mathbf{n}}_r \cdot \left(\mathbf{F}_B - m_B\, {}^N\mathbf{a}^{B^\star}\right) = 0 \qquad (r = 1, 2, 3) \tag{22}$$

$$\hat{\mathbf{b}}_{r-3} \cdot \left(\mathbf{T}_B - \frac{{}^N d\, {}^N\mathbf{H}^{B/B^\star}}{dt}\right) = 0 \qquad (r = 4, 5, 6) \tag{23}$$

Equations (23) immediately lead to Euler's rotational equations of motion; Lagrange's approach yields equations that are much more complex

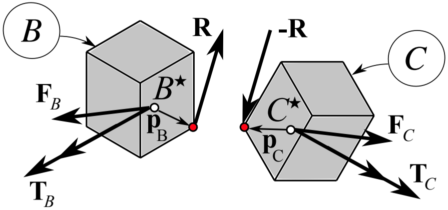
Example, Smooth Ball-and-Socket Joint



$$
\begin{aligned}
&{}^N\mathbf{v}_r^{B^\star} \cdot \left(\mathbf{F}_B + \mathbf{R} - m_B\, {}^N\mathbf{a}^{B^\star}\right) \\
&+ {}^N\boldsymbol{\omega}_r^B \cdot \left(\mathbf{T}_B + \mathbf{p}_B \times \mathbf{R} - \frac{{}^N d\, {}^N\mathbf{H}^{B/B^\star}}{dt}\right) \\
&+ {}^N\mathbf{v}_r^{C^\star} \cdot \left(\mathbf{F}_C - \mathbf{R} - m_C\, {}^N\mathbf{a}^{C^\star}\right) \\
&+ {}^N\boldsymbol{\omega}_r^C \cdot \left(\mathbf{T}_C - \mathbf{p}_C \times \mathbf{R} - \frac{{}^N d\, {}^N\mathbf{H}^{C/C^\star}}{dt}\right) = 0 \quad (r = 1, \dots, 9)
\end{aligned}
\tag{24}
$$

It can be shown that the constraint force **R** does not contribute to the equations of motion

Noncontributing Force



$$
{}^N\mathbf{v}^{C^\star} = {}^N\mathbf{v}^{B^\star} + {}^N\boldsymbol{\omega}^B \times \mathbf{p}_B + {}^N\boldsymbol{\omega}^C \times (-\mathbf{p}_C)
$$

$$
{}^N\mathbf{v}_r^{C^\star} = {}^N\mathbf{v}_r^{B^\star} + {}^N\boldsymbol{\omega}_r^B \times \mathbf{p}_B - {}^N\boldsymbol{\omega}_r^C \times \mathbf{p}_C
$$

$$
\begin{aligned}
{}^N\mathbf{v}_r^{C^\star} \cdot (-\mathbf{R}) &= -\,{}^N\mathbf{v}_r^{B^\star} \cdot \mathbf{R} - {}^N\boldsymbol{\omega}_r^B \times \mathbf{p}_B \cdot \mathbf{R} + {}^N\boldsymbol{\omega}_r^C \times \mathbf{p}_C \cdot \mathbf{R} \\
&= -\,{}^N\mathbf{v}_r^{B^\star} \cdot \mathbf{R} - {}^N\boldsymbol{\omega}_r^B \cdot \mathbf{p}_B \times \mathbf{R} + {}^N\boldsymbol{\omega}_r^C \cdot \mathbf{p}_C \times \mathbf{R}
\end{aligned}
$$

These terms cancel the other terms involving $\mathbf{R}$ in Eqs. (24), showing that $\mathbf{R}$ ultimately does not appear in Kane's equations of motion, as is claimed on slide 8.

Partial Velocities, Partial Angular Velocities

One possible set of motion variables

$$ {}^N\mathbf{v}^{B^\star} = u_1\hat{\mathbf{n}}_1 + u_2\hat{\mathbf{n}}_2 + u_3\hat{\mathbf{n}}_3 \tag{25} $$

$$ {}^N\boldsymbol{\omega}^{B} = u_4\hat{\mathbf{b}}_1 + u_5\hat{\mathbf{b}}_2 + u_6\hat{\mathbf{b}}_3 \tag{26} $$

$$ {}^N\boldsymbol{\omega}^{C} = u_7\hat{\mathbf{c}}_1 + u_8\hat{\mathbf{c}}_2 + u_9\hat{\mathbf{c}}_3 \tag{27} $$

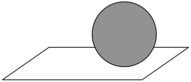
$r =$	1	2	3	4	5	6	7	8	9
${}^N\mathbf{v}_r^{B^\star}$	$\hat{\mathbf{n}}_1$	$\hat{\mathbf{n}}_2$	$\hat{\mathbf{n}}_3$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$
${}^N\boldsymbol{\omega}_r^{B}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\hat{\mathbf{b}}_1$	$\hat{\mathbf{b}}_2$	$\hat{\mathbf{b}}_3$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$
${}^N\boldsymbol{\omega}_r^{C}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\hat{\mathbf{c}}_1$	$\hat{\mathbf{c}}_2$	$\hat{\mathbf{c}}_3$

$$ {}^N\mathbf{v}_r^{C^\star} = \begin{cases} \hat{\mathbf{n}}_r & (r = 1,2,3) \\ \hat{\mathbf{b}}_{r-3} \times \mathbf{p}_B & (r = 4,5,6) \\ -\hat{\mathbf{c}}_{r-6} \times \mathbf{p}_C & (r = 7,8,9) \end{cases} $$

Nonholonomic Systems

A nonholonomic system is one that is subject to motion constraints

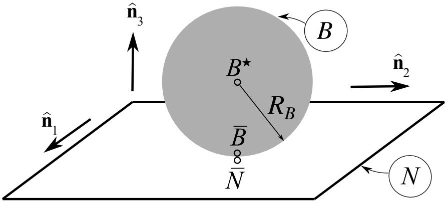
Examples of motion constraints

- Rolling (absence of slipping) 
- Sharp-edged blade  $\quad {}^N\mathbf{v}^P \cdot \hat{\mathbf{b}}_2 = 0$

With Lagrange's method, must introduce and subsequently eliminate multipliers associated with unknown constraint forces

With Kane's method, one accounts for the motion constraints when forming velocities of points, and angular velocities of rigid bodies: don't need to introduce multipliers

Example, Rolling Sphere



$$^N\boldsymbol{\omega}^B = u_1\hat{\mathbf{n}}_1 + u_2\hat{\mathbf{n}}_2 + u_3\hat{\mathbf{n}}_3 \tag{28}$$

$$^N\mathbf{v}^{B^\star} = u_4\hat{\mathbf{n}}_1 + u_5\hat{\mathbf{n}}_2 \tag{29}$$

Configuration constraint: $B^\star$ must remain a constant distance above horizontal surface, so $^N\mathbf{v}^{B^\star} \cdot \hat{\mathbf{n}}_3 = 0$.

Motion constraint: for rolling to take place, $^N\mathbf{v}^{\overline{B}} = {}^N\mathbf{v}^{\overline{N}} = \mathbf{0}$

$$^N\mathbf{v}^{\overline{B}} = {}^N\mathbf{v}^{B^\star} + {}^N\boldsymbol{\omega}^B \times (-R_B\hat{\mathbf{n}}_3) = (u_4 - R_B u_2)\hat{\mathbf{n}}_1 + (u_5 + R_B u_1)\hat{\mathbf{n}}_2 \tag{30}$$

Nonholonomic constraint equations

$$u_4 = R_B u_2 \qquad u_5 = -R_B u_1 \tag{31}$$

Partial Velocities, Partial Angular Velocities

The rolling sphere has three degrees of freedom in N. Use the nonholonomic constraint equations to rewrite ${}^N\boldsymbol{\omega}^B$ and ${}^N\mathbf{v}^{B^\star}$

$$ {}^N\boldsymbol{\omega}^B = u_1\hat{\mathbf{n}}_1 + u_2\hat{\mathbf{n}}_2 + u_3\hat{\mathbf{n}}_3 \tag{32}$$

$$ {}^N\mathbf{v}^{B^\star} = R_B(u_2\hat{\mathbf{n}}_1 - u_1\hat{\mathbf{n}}_2) \tag{33}$$

Nonholonomic partial angular velocities, ${}^N\tilde{\boldsymbol{\omega}}_r^B$, and *nonholonomic partial velocities*, ${}^N\tilde{\mathbf{v}}_r^{B^\star}$:

$r =$	1	2	3
${}^N\tilde{\boldsymbol{\omega}}_r^B$	$\hat{\mathbf{n}}_1$	$\hat{\mathbf{n}}_2$	$\hat{\mathbf{n}}_3$
${}^N\tilde{\mathbf{v}}_r^{B^\star}$	$-R_B\hat{\mathbf{n}}_2$	$R_B\hat{\mathbf{n}}_1$	$\mathbf{0}$

Equations of Motion

Dynamical equations of motion

$$
{}^N\tilde{\mathbf{v}}_r^{B^\star} \cdot \left(\mathbf{F} - m_B g \hat{\mathbf{n}}_3 - m_B\,{}^N\mathbf{a}^{B^\star}\right) \tag{34}
$$

$$
+\,{}^N\tilde{\boldsymbol{\omega}}_r^{B} \cdot \left(-R_B\hat{\mathbf{n}}_3 \times \mathbf{F} - \frac{{}^N d\,{}^N\mathbf{H}^{B/B^\star}}{dt}\right) = 0 \quad (r = 1,2,3)
$$

where $\mathbf{F}$ is the contact force applied to B at $\overline{B}$. Note that with ${}^N\mathbf{v}^{\overline{B}} = \mathbf{0}$, Eq. (30) yields

$$
{}^N\tilde{\mathbf{v}}_r^{B^\star} = {}^N\tilde{\boldsymbol{\omega}}_r^{B} \times R_B\hat{\mathbf{n}}_3 \qquad (r = 1,2,3) \tag{35}
$$

Hence

$$
\begin{aligned}
&{}^N\tilde{\mathbf{v}}_r^{B^\star} \cdot \mathbf{F} + {}^N\tilde{\boldsymbol{\omega}}_r^{B} \cdot (-R_B\hat{\mathbf{n}}_3 \times \mathbf{F}) \\
&= ({}^N\tilde{\boldsymbol{\omega}}_r^{B} \times R_B\hat{\mathbf{n}}_3) \cdot \mathbf{F} - {}^N\tilde{\boldsymbol{\omega}}_r^{B} \cdot (R_B\hat{\mathbf{n}}_3 \times \mathbf{F}) \\
&= {}^N\tilde{\boldsymbol{\omega}}_r^{B} \cdot (R_B\hat{\mathbf{n}}_3 \times \mathbf{F}) - {}^N\tilde{\boldsymbol{\omega}}_r^{B} \cdot (R_B\hat{\mathbf{n}}_3 \times \mathbf{F}) = 0 \qquad (r = 1,2,3)
\end{aligned} \tag{36}
$$

and $\mathbf{F}$ ultimately does not appear in Kane's equations of motion, as is claimed on slide 8.